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Quasinormability of some spaces of holomorphic mappings

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ABSTRACT. A class of locally convex vector spaces with a special Schauder decomposition is considered. It is proved that the elements of this class, which includes some spaces naturally appearing in infinite dimensional holomorphy, are quasinormable though in general they are neither metrizable nor Schwartz spaces.

0. INTRODUCTION AND PRELIMINARIES

Let X be a Hausdorff real or complex locally convex vector space, and let $\beta(X', X)$ denote the strong topology on the dual X' of X. In 1954, Grothendieck [3] proved the following result.

Theorem 0. *For any Hausdorff l.c. vector space X, the following assertions are equivalent:*

(a) for any equicontinuous subset $\mathcal{E}$ of X' there is a neighbourhood V of O in X such that $\beta(X', X)$ induces on $\mathcal{E}$ the topology of uniform convergence over V.

(b) For any neighbourhood U of O in X there is a neighbourhood V of O in X such that for any $\alpha>0$ there is a bounded subset M in X with $V \subset M+\alpha U$.

Locally convex spaces for which these two statements hold are said to be quasinormable, and this class of spaces was studied in [3]. Most of the spaces in Functional Analysis belong to this class or have a close relationship with it. On the other hand, there are spaces (even Fréchet spaces) which are not quasinormable, the first example of such a space is due to Köthe in 1948. (See [4], 10.7).

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In §1 of this note we consider a family of l.c. vector spaces with a special Schauder decomposition, and prove that its components are all quasinormable. In §2 we apply our result to prove the quasinormability of spaces of holomorphic mappings. If E and F are complex Hausdorff l.c. vectors spaces and $U \subset E$ is an open subset, we let $H(U, F)$ denote the space of all F—valued holomorphic mappings on U, which we endow with some of its usual topologies: the compact-open, the ported and the bornological topology, denoted respectively by τ_0, τ_ω and τ_δ. In [1], Bierstedt and Meise proved that if E is a metrizable Schwartz space then $(H(U, F), \tau_\omega)$ is a Schwartz (hence quasinormable) space. In [5], Nelimarkka showed that if E is metrizable, then $(H(U, F), \tau_0)$ is again a Schwartz space. In this note, we prove that if E and F are Banach spaces, and $U \subset E$ is a balanced open subset of E then $(H(U, F), \tau)$ is quasinormable for $\tau = \tau_\omega, \tau_\delta$. Note that this result is not included in those mentioned above unless $\dim E < \infty$.

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1. QUASINORMABILITY OF SPACES WITH A NORMAL SCHAUDER DECOMPOSITION

Let X be a Hausdorff (real or complex) l.c. vector space. Let $CS(X)$ denote the family of continuous seminorms on X. We recall that a Schauder decomposition of X is a sequence $(X_n)_{n \in \mathbb{N}}$ of subspaces of X such that each $x \in X$ can be uniquely represented in the form $x = \sum_0^\infty x_n$ with $x_n \in X_n$ for $n \in \mathbb{N}$ (the series being convergent in the topology of X), and the projections $\pi_n : x = \sum x_k \to x_n$ are all continuous.

Definition 1. *X is said to have a normal decomposition if it has a Schauder decomposition $(X_n)_{n \in \mathbb{N}}$ and a fundamental system of seminorms $F(X) \subset CS(X)$ with the following properties:*

(a) For each $n \in \mathbb{N}$, any seminorm of the form $p|_{X_n}$ with $p \in F(X)$ generates the topology of X_n.

(b) There exists a sequence of positive numbers $(\delta_n)_{n \in \mathbb{N}}$ with $x = \sum_0^\infty \frac{1}{\delta_n} < \infty$ such that.

$$\hat{p}\left(\sum_0^\infty x_n\right) := \sum_0^\infty \delta_n p(x_n) \text{ for } x = \sum x_n \in X \qquad (*)$$

satisfies $\hat{p} \in CS(X)$ whenever $p \in F(X)$.

Remarks. Note that by (a), each X_n is a normed space. By changing a finite number of δ_n, we may assume that $\delta_n \geqq 1$ for all $n \in \mathbb{N}$, hence

$$p(x) \leq \sum p(x_n) \leq \sum \delta_n p(x_n) = \hat{p}(x)$$

and by (b) the family $\hat{F}(X) = \{\hat{p};\, p \in F(X)\}$ also generates the topology of X and each $\hat{p}$ is a norm on X. Note also that a Schauder decomposition which is normal with respect to a family of seminorms $F(X)$ may fail to be normal with respect to an equivalent family $F_1(x)$.

Theorem 1. *If X admits a normal decomposition then X is a quasinormable space.*

Proof. We show that condition (b) in Theorem 0 is satisfied. Obviously, it suffices to check this for a fundamental system of neighbourhoods at the origin 0, and we shall check it for the system N associated to any family $F(X)$ satisfying the normality requirements.

Let $U \in N$ be given. Then $U = \{x \in X;\, p(x) < 1\}$ for some $p \in F(X)$, hence by (*)

$$V = \left\{ y = \sum_n y_n;\, \hat{p}(y) = \sum_n \delta_n\, p(y_n) < 1 \right\}$$

is a neighbourhood of 0, and we claim that the pair (U, V) satisfies (b). Indeed, let $\alpha > 0$ be arbitrary, and choose a positive integer n_0 such that $\sum_{n_0+1}^{\infty} \frac{1}{\delta_n} < \alpha$. If $y = \sum_n y_n \in V$ then $\delta_n p(y_n) < 1$ for all $n \in \mathbb{N}$, and so

$$p\left(\sum_{n_0+1}^{\infty} y_n\right) \leq \sum_{n_0+1}^{\infty} p(y_n) \leq \sum_{n_0+1}^{\infty} \frac{1}{\delta_n} < \alpha$$

which shows that $\sum_{n_0+1}^{\infty} y_n \in \alpha U$.

Let us put

$$M =: \left\{ \sum_0^{n_0} z_k;\, \sum_0^{n_0} \delta_k p(z_k) < 1 \right\}.$$

We claim that M is bounded in X. Indeed, by definition 1, given any k, $0 \leq k \leq n_0$ and given any seminorm $q \in F(X)$, the restriction $q|_{X_k}$ generates the topology that X induces on X_k. In particular $q|_{X_k}$ and $p|_{X_k}$ are equivalent on X_k. Hence there is a constant $C_k > 0$ such that

$$q(u) \leq C_k p(u) \qquad u \in X_k, \qquad 0 \leq k \leq n_0$$

Thus, for $\sum_0^{n_0} z_k \in M$ we have

$$q\left(\sum_0^{n_0} z_k\right) \leq \sum_0^{n_0} q(z_k) \leq \sum_0^{n_0} C_k p(\tilde{z}_k) \leq \sum_0^{n_0} C_k \frac{1}{\delta_k} < \infty$$

which proves that M is bounded in X. Moreover, for $y=\sum_0^{\infty} y_k \in V$ we have $y=\sum_0^{n_0} y_k+\sum_{n_0+1}^{\infty} y_k \in M+\alpha U$ and so $V \subset M+\alpha U$.

Remark. The proof also shows that 0 has a fundamental system $\hat{N}$ of neighbourhoods V such that each projection $\pi_n(V)$, $0 \leq n < \infty$, is bounded in X.

2. QUASINORMABILITY OF THE SPACES OF HOLOMORPHIC MAPPINGS

In this section, E and F are complex Banach spaces and $U \subset E$ is a balanced open subset. $H(U, F)$ denotes the space of all F—valued holomorphic mappings on U, and $P(^nE, F)$, $n \in \mathbb{N}$, is the space of F—valued continuous n—homogeneous polynomials on E. If $f \in H(U, F)$ let $f=\sum_0^{\infty} f_n$ denote the Taylor series of f at 0, where $f_n=: \frac{1}{n!} \hat{d}^n f(0) \in P(^nE, F)$.

We recall that if K is the family of balanced compact subsets of U and c_0^+ is the set of positive null sequences then the ported topology τ_ω on $H(U, F)$ may be defined by the family of seminorms

$$p_{K,\alpha}(f)=: \sum_0^{\infty} \|f_n\|_{K+\alpha_n B} \qquad f=\sum_0^{\infty} f_n \in H(U, F)$$

as K ranges over K and $(\alpha_n)_n$ over c_0^+ where $\|f_n\|_S=\sup_{x \in S} \|f_n(x)\|$ where $S=K+\alpha_n B$ and B is the unit ball of E.

As for the bornological topology τ_δ on $H(U, F)$, no explicit formula is known for any fundamental system of seminors. However, τ_δ is generated by the family of **all** seminorms p on $H(U, F)$ with the following properties

(1) $p(f)=\sum_0^{\infty} p(f_n)$ for $f=\sum_0^{\infty} f_n \in H(U, F)$.

(2) for all $n \in \mathbb{N}$, $p|_{P(^nE, F)}$ is equivalent to the usual norm in $P(^nE, F)$.

By ([2], ch3, §2), for $\tau=\tau_\omega$, τ_δ, the sequence $X_n=: P(^nE, F)$, $n \in \mathbb{N}$, is a Schauder decomposition of $(H(U, F), \tau)$, and the sequence $\delta_0=1$, $\delta_n=n^2$ for $n \geq 1$ satisfies the requirement (*) in the definition of normality. Thus we have proved:

Theorem 2. *If E and F are complex Banach spaces, and U is an open balanced subset of E, then $(H(U, F), \tau)$ is quasinormable for $\tau = \tau_\omega$, τ_δ.*

This gives rise to the following.

Problem 1. Is $(H\, U, F), \tau)$, where U is an arbitrary open set in E, a quasinormable space?

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