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Some normability conditions on Fréchet spaces

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ABSTRACT. We define two new normability condition on Fréchet spaces and announce some related results.

The purpose of this note is to announce some results related to two normability conditions on Fréchet spaces, which will be given subsequently. Results stated here and their proofs are contained in [5] and [6].

We use the standard terminology and notation of the theory of locally convex spaces as in [3]. By $(\| \|_k)$ we always denote an increasing sequence of seminorms defining the topology of a Fréchet space E and $U_k=\{x\in E: \|x\|_k\leq 1\}$. $A=(a_{j,k})$ stands for an infinite matrix of real numbers which always satisfies

$$0\leqq a_{j,k}\leqq a_{j,k+1} \text{ and } \sup_k a_{j,k}>0$$

for all j and k. For $1\leqq p<+\infty$ we set

$$\lambda^p(A)=\{\xi=(\xi_j): \|\xi\|_k^p=\sum|\xi_j|^p a_{j,k}^p<+\infty \text{ for all } k\}.$$

Equipped with seminorms $\| \|_k$, $k=1,2,\ldots$, $\lambda^p(A)$ is a Fréchet space. For $\lambda^1(A)$ we simply write $\lambda(A)$.

We call a Fréchet space $E[\tau]$ *locally normable* [5] if there is a continuous norm $\| \|$ on E such that on every bounded subset τ coincides with the topology defined by this norm $\| \|$. This means that if (x_n) is a bounded sequence in E and $\lim\|x_n\|=0$ then we have $\lim x_n=0$.

For a locally normable Fréchet space E we may assume without loss of generality that $\| \|_1$ is the norm specified in the definition. So for every bounded subset B and k there is a $\varepsilon>0$ with

$$(\varepsilon U_1)\cap B\subset U_k.$$

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Upon polarization we obtain

$$U_k^0 \subset E'[U_1^0] + B^0$$

where $E'[U_1^0]$ is the subspace of the dual E' spanned by U_1^0. Hence in this case the gauge of U_1^{00} defines a continuous norm on the bidual E''. This observation yields our first result.

(1) *The bidual of a locally normable Fréchet space admits a continuous norm.*

In the case of Köthe spaces we can characterize local normability in terms of the defining matrix.

(2) *The following are equivalent:*
(i) $\lambda^p(A)$ is locally normable for some $1 \leqq p < +\infty$
(ii) $\lambda^p(A)$ is locally normable for all $1 \leqq p < +\infty$
(iii) A satisfies the following condition:
(*) there is a k_0 such that for any choice of $C_n > 0$ and for any k there is a $C > 0$ so that

$$a_{j;k} \leqq \max\{C\, a_{j;k_0}, \quad \sup_n \frac{a_{j;n}}{C_n}\}$$

holds for all j.

The case $p = 1$ is of special importance.

(3) *$\lambda(A)$ is locally normable if and only if the bidual $\lambda(A)''$ admits a continuous norm.*

Dierolf and Moscatelli [1] gave an example of a Fréchet space whose topology is defined by a sequence of norms but whose bidual admits no continuous norm. Using (3) and the condition (*), it is easy to construct examples of Köthe spaces which exhibit the same phenomenon. In fact the well-known Grothendieck-Köthe example ([3];§31,7) of a non-distinguished Fréchet space is a Köthe space $\lambda(A)$ which admits a continuous norm, but since A violates (*), the bidual $\lambda(A)''$ does not admit a continuous norm. For this matrix A, $\lambda^2(A)$ is a reflexive Fréchet space with a continuous norm, but it is not locally normable by (2). Hence the converse of (1) is false in general. Let us remark that one can also construct a quasinormable Köthe space which admits a continuous norm but whose bidual again has no continuous norm [5].

Let $\phi:(0,\infty) \to (0,\infty)$ be a strictly increasing function. We say E satisfies (DN_ϕ) if we have a k_0 with the property that for each k there is a p and $C > 0$ so that

$$\|x\|_k \leqq C\phi(r)\, \|x\|_{k_0} + \frac{1}{r}\, \|x\|_p$$

holds for every $x \in E$ and $r > 0$ ([8], [10]). We call a Fréchet space *asymptotically normable* [6] if it satisfies (DN_ϕ) for some function ϕ. For the role played by asymptotically normable Fréchet spaces in the vanishing of the functor Ext we refer to [10] and [12].

If a Fréchet space E is a asympotically normable then for $k \geqq k_0$ on the ball U_p, $\| \ \|_{k_0}$ and $\| \ \|_k$ define equivalent topologies, where k_0, k and p are as in the definition given above. In particular, if (x_n) is Cauchy with respect to $\| \ \|_p$ and $\lim \|x_n\|_{k_0} = 0$ then $\lim \|x_n\|_k = 0$. From this we obtain

(4) *An asymptotically normable Fréchet is locally normable and also countably normable.*

It is known that a countably normable Fréchet-Schwartz space is asymptotically normable [10]. It is also easy to show that a Fréchet-Montel space which admits a continuous norm is locally normable. However, even a nuclear Fréchet space which admits a continuous norm need not be asymptotically normable, a fact which was used in constructing nuclear Fréchet spaces without the bounded approximation property [2], [9].

One can characterize asymptotical normability of a Köthe space $\lambda(A)$ in terms of the defining matrix and also construct a Köthe-Montel space which admits a continuous norm but which is not asympotically normable [6]. Hence a locally normable Fréchet space need not be asymptotically normable.

We note that for $\phi(r) = r$, the condition (DN_ϕ), denoted by (DN), was introduced already in [7] to characterize the subspaces of the space (s) of rapidly decreasing sequences. Subsequently in [11] it was proved that every Fréchet space of type (DN) is isomorphic to a subspace of $l_\infty(I)\hat{\otimes}_\pi(s)$ for some index set I. As a generalization of this result we have

(5) *For every increasing function $\phi: (0,\infty) \to (0,\infty)$ there is a nuclear Köthe space $\lambda(A)$ with a continuous norm such that every Fréchet space which satisfies (DN_ϕ) is isomorphic to a subspace of $l_\infty(I)\hat{\otimes}_\pi\lambda(A)$ for some index set I.*

This means that the class of asymptotically normable Fréchet spaces is precisely the class of all Fréchet spaces which contains all Banach spaces as well as all nuclear Köthe spaces admitting continuous norms and which is closed under the operations of taking complete projective tensor products and passing to subspaces. The parallelism between this and the characterization of the quasinormable Fréchet spaces given by Meise and Vogt [4] is worth noting.

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