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On convolution operator in Orlicz spaces

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ABSTRACT. The Orlicz spaces defined on a locally compact abelian group are considered. The main results consist in presenting sufficient and necessary conditions, expressed in terms of generated functions φ_i, for embeddings $L_{\varphi_1} * L_{\varphi_2} \hookrightarrow L_{\varphi_3}$, $E_{\varphi_1} * E_{\varphi_2} \hookrightarrow E_{\varphi_3}$, $E_{\varphi_1} * E_{\varphi_2} \hookrightarrow L_{\varphi_3}$ and $L_{\varphi_1} * L_{\varphi_2} \hookrightarrow E_{\varphi_3}$, where L_{φ_i} are Orlicz spaces and E_{φ} are their subspaces consisting of all order continuous elements. Some results of the paper are an extension and generalization of those contained in [2], [3], [8] and [10].

INTRODUCTION

The Young theorem ([2], [8]) including sufficient conditions for $L^p * L^q \hookrightarrow L^r (1<p,q,r<\infty)$ has been known for many years. In [8], O'Neil generalizes this result to Orlicz spaces stating sufficient condition for $L_{\varphi_1} * L_{\varphi_2} \hookrightarrow L_{\varphi_3}$. From the other hand there are known sufficient and necessary conditions for the space $L^p (1 \leqslant p<\infty)$ to be a Banach algebra under convolution as multiplication [10]. A generalization of this result for the Orlicz space is included in [3]. Our main topic consists in finding necessary and sufficient conditions for embeddings of $L_{\varphi_1} * L_{\varphi_2}$ into L_{φ_3} and $E_{\varphi_1} * E_{\varphi_2}$ into E_{φ_3}. We investigate also the other embeddings like $E_{\varphi_1} * E_{\varphi_2}$ into L_{φ_3}, $L_{\varphi_1} * E_{\varphi_2}$ into L_{φ_3} and $L_{\varphi_1} * L_{\varphi_2}$ into E_{φ_3}. The Young theorem, the O'Neil's results and the results concerning the Lebesgue and Orlicz spaces as Banach algebras are obtained as corollaries of our results. In particular we get the necessity of the Young theorem, which seems to be not known so far. We also get the answer to the problem given by B. Gramsch in [1].

The first part is devoted to general modular spaces. We give some equivalent conditions in order to a bilinear operator defined on a Cartesian product of modular spaces $X_{\rho_1} \times X_{\rho_2}$ act to another modular space X_{ρ_3}. The results of this part are applied to the second one, where the Orlicz spaces and convolution are investigated as modular spaces and the bilinear operator, respectively. The important role is played by conditions (+) and (++) expresing some connections between Young functions φ_i. There are a few versions of those

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conditions depending on the kind of a group G and the Haar measure μ. In Theorems 8 and 9 there are given sufficient conditions for $L_{\varphi_1}*L_{\varphi_2}\hookrightarrow L_{\varphi_3}$, $E_{\varphi_1}*E_{\varphi_2}\hookrightarrow E_{\varphi_3}$ and $L_{\varphi_1}*L_{\varphi_2}\hookrightarrow E_{\varphi_3}$ by means of the condition (+) and (++). For a discrete group it is posible to prove a converse statement (Theorem 10) without any additional assumptions on the group G, whereas for a nondiscrete group the full converse statement (Theorems 11,14) is obtained under the assumption of the so called condition (*) on the group G. It is not difficult to check that the groups like $(\mathbb{R},+)$, $(\mathbb{K},+)$, $(T,\cdot)$, $(\mathbb{R}\setminus\{0\},\cdot)$, $(\mathbb{K}\setminus\{0\},\cdot)$ satisfy condition (*) (Remark 12). At the end there are a number of corollaries including among others, the Young theorem with necessary and sufficient conditions for a large class of locally compact abelian groups.

1.

Let us now agree on some terminology. Let $\mathbb{R}$, $\mathbb{K}$, $\mathbb{N}$ stand for real, complex and natural numbers respectively. Let X be a complex or real vector space. Recall some notions connected with modular spaces [7]. A functional $\rho: X\to[0,+\infty]$ is called a convex modular if it satisfies the conditions $(1)\rho(0)=0$; $[\forall_\lambda \rho(\lambda x)=0]\Rightarrow x=0$, $(2)\rho(e^{it}x)=\rho(x)$ for all $t\in\mathbb{R}$ ($\rho(-x)=\rho(x)$ in the real case), $(3)\rho(\alpha x+\beta y)\leqslant\alpha\rho(x)+\beta\rho(y)$ if $\alpha,\beta\geqslant 0$ and $\alpha+\beta=1$. For any convex modular define the space $X_\rho=\{x\in X: \lim_{\lambda\to 0}\rho(\lambda x)=0\}=\{x\in X:\rho(\lambda x)<\infty$ for some $\lambda>0\}$ called a modular space and $X_\rho^f=\{x\in X:\rho(\lambda x)<\infty$ for all $\lambda>0\}$ a subspace of X_ρ called the subspace of finite elements. The functional $\|x\|_\rho=\inf\{\varepsilon>0:\rho(x/\varepsilon)\leqslant 1\}$, $x\in X_\rho$, is a norm in X_ρ. The subspace X_ρ^f considered with the same norm is closed in X_ρ.

1.1. Theorem. *Let $\rho_i (i=1,2,3)$ be modulars defined on X and $\gamma: X_{\rho_1}\times X_{\rho_2}\to X$ be a bilinear operator. The following conditions are equivalent*

(*i*) *For every $\varepsilon>0$ there exists $\delta>0$ such that for all $x\in X_{\rho_1}$, $y\in X_{\rho_2}$ if $\rho_1(x)\leqslant\delta$ and $\rho_2(y)\leqslant\delta$ then $\rho_3(\delta\gamma(x,y))\leqslant\varepsilon$.*

(*ii*) *There exists $k_i>0$ $(i=1,2,3)$ such that for $x\in X_{\rho_1}$, $y\in X_{\rho_2}$ if $\rho_1(x)\leqslant k_1$ and $\rho_2(y)\leqslant k_1$ then $\rho_3(k_2\gamma(x,y))\leqslant k_3$.*

(*iii*) *There exists $k>0$ such that if $x\in X_{\rho_1}$, $y\in X_{\rho_2}$ and $\rho_1(x)\leqslant 1$ and $\varepsilon_2(y)\leqslant 1$ then $\rho_3(k\gamma(x,y))\leqslant 1$.*

(*iv*) *There exists $c>0$ such that $\|\gamma(x,y)\|_{\rho_3}\leqslant c\|x\|_{\rho_1}\|y\|_{\rho_2}$ for all $x\in X_{\rho_1}$, $y\in X_{\rho_2}$.*

(*v*) *$\gamma: X_{\rho_1}\times X_{\rho_2}\to X_{\rho_3}$ and γ is continuous.*

If $\gamma: X_{\rho_1}^f\times X_{\rho_2}^f\to X$ then the analogous conditions to the above in which the spaces X_{ρ_i} are replaced by $X_{\rho_i}^f (i=1,2)$ are equivalent.

Proof. It is enough to give a proof only for $\gamma : X_{\rho_1} \times X_{\rho_2} \to X$. The implications $(i) \to (ii)$ and $(iv) \to (v)$ are evident and $(iii) \to (iv)$ results directly from the definition of a norm in a modular space.

$(ii) \to (iii)$ *if* $\rho_1(x) \leqslant 1$ *and* $\rho_2(y) \leqslant 1$ *then*

$\rho_1(min(1,k_1)x) \leqslant k_1$ *and* $\rho_2(min(1,k_1)y) \leqslant k_1$. *So*

$\rho_3(k_2 min^2(1,k_1)\gamma(x,y)) \leqslant max(1,k_3)$. *Then, by convexity of* φ_3,

$\rho_3(k\gamma(x,y)) \leqslant 1$ *under* $k = k_2 min^2(1,k_1)/ \max(1,k_3)$.

$(v) \to (i)$ Suppose γ takes its values in X_{ρ_3} and (i) is not satisfied. So there exist $\varepsilon > 0$ and sequences $(x_n) \subset X_{\rho_1}$, $(y_n) \subset X_{\rho_2}$ such that $\rho_1(x_n) \leqslant 1/n$, $\rho_2(y_n) \leqslant 1/n$ and $\rho_3((1/n)\gamma(x_n, y_n)) \geqslant \varepsilon$. Taking $\bar{x}_n = (1/\sqrt{n})x_n$, $\bar{y}_n = (1/\sqrt{n})y_n$ we have $\rho_1(\lambda \bar{x}_n) \leqslant \rho_1(\sqrt{n}\bar{x}_n) \leqslant 1/n \to 0$ and $\rho_2(\lambda \bar{y}_n) \to 0$, which implies *that* $\|\bar{x}_n\|_{\rho_1} \to 0$ and $\|\bar{y}_n\|_{\rho_2} \to 0$. However, $\rho_3(\gamma(\bar{x}_n, \bar{y}_n)) = \rho_3((1/n)\gamma(x_n, y_n)) \geqslant \varepsilon$. Thus $\|\gamma(\bar{x}_n, \bar{y}_n)\|_{\rho_3}^1 \not\to 0$ and γ is not continuous.

For some kinds of spaces X, modulars ρ_i and operators γ one can show more.

1.2. Theorem. *Let X be a vector lattice. Suppose* $\rho_i(|x_1|) \leqslant \rho_i(|x_2|)$ *if* $|x_1| \leqslant |x_2|$. *Moreover, let* X_{ρ_i} *be complete and a bilinear operator* $\gamma : X_{\rho_1} \times X_{\rho_2} \to X$ *be positive, i.e.* $\gamma(x,y) \geqslant 0$ *if* $x \geqslant 0$ *and* $y \geqslant 0$ *and let* $|\gamma(x,y)| \leqslant \gamma(|x|,|y|)$.

The following conditions are equivalent

1.° $\gamma : X_{\rho_1} \times X_{\rho_2} \to X_{\rho_3}$ *and* γ *is continuous.*

2.° $\gamma : X_{\rho_1} x X_{\rho_2} \to X_{\rho_3}$.

3.° There exists $k > 0$ *such that for* $x \in X_{\rho_1}$, $y \in X_{\rho_2}$ *if* $\rho_1(x) \leqslant 1$ *and* $\rho_2(y) \leqslant 1$ *then* $\rho_3(k\gamma(x,y)) \leqslant 1$.

The above conditions in which X_{ρ_i} *are replaced by* $X^T_{\rho_i}$ *for* $i = 1,2$, *are equivalent, too.*

Proof. By the previous theorem only the implication 2.°$\to$3.° needs a proof. We shall show it in the case of $X^T_{\rho_i}$ $i = 1,2$. For a contrary, let 3.° be not fulfilled. So there exist sequences $(x_n) \subset X^T_{\rho_1}$, $y_n \subset X^T_{\rho_2}$ such that $\rho_1(x_n) \leqslant 1$, $\rho_2(y_n) \leqslant 1$ and $\rho_3((1/2^{2n}n)\gamma(x_n, y_n)) > 1$.

The elements $z = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n}$, $w = \sum_{n=1}^{\infty} \frac{|y_n|}{2^n}$ belong to $X^T_{\rho_1}$, $X^T_{\rho_2}$, respectively, because of convexity of ρ_i and the fact that $X^{\gamma}_{\rho_i}$ are closed subspaces of X_{ρ_i}. However,

$$\rho_3((1/k)\gamma(z,w)) \geqslant \rho_3((1/2^{2k}k)\gamma(|x_k|,|y_k|))$$

$$\geqslant \rho_3((1/2^{2k}k)\gamma(x_k,y_k)) > 1,$$

by assumed properties of γ. Thus $\gamma(z,w) \notin X_{\rho_3}$, which ends the proof.

The results of this section will be applied in the next one to Orlicz spaces and convolution as modular spaces and bilinear operator, respectively.

2.

Let $\varphi : [0,+\infty] \to [0,+\infty]$ be convex, left-continuous not identical to zero and infinity, $\varphi(0)=0$ and $\varphi(+\infty)=+\infty$. In the sequel this function will be called a Young function. We say that a Young function is finite if it is finite on $[0,+\infty)$. A generalized inverse function $\varphi^{-1} : [0,+\infty] \to [0,+\infty]$ is defined as

$$\varphi^{-1}(y) = \inf\{x \geqslant 0 : \varphi(x) > y\}, \text{ where } \inf \emptyset = \infty.$$

Let a,b be reserved for the following numbers

$$a = \sup\{x \geqslant 0 : \varphi(x) = 0\},$$

$$b = \sup\{x \geqslant 0 : \varphi(x) < \infty\}.$$

If a function φ_i is considered instead of φ then a_i, b_i denote the numbers a,b for the function φ_i.

The connections between φ and φ^{-1} are formulated in the following simple lemma.

1. Lemma. *For all* $x \in [0,+\infty]$

$$x \leqslant \varphi^{-1}(\varphi(x)) \text{ and } \varphi(\varphi^{-1}(x)) \leqslant x.$$

Moreover,

$$\varphi(\varphi^{-1}(x)) = x \text{ for } x \in [0,\varphi(b)]$$

$$\varphi(\varphi^{-1}(x)) = \varphi(b) \text{ for } x \in (\varphi(b), \infty].$$

We say that two Young functions φ_1,φ_2 are equivalent for large arguments (small arguments) [all arguments] if

$$\overline{\lim_{u\to\infty}} \frac{\varphi_i(ku)}{\varphi_j(u)}<\infty \; \Big(\overline{\lim_{u\to 0}} \frac{\varphi_i(ku)}{\varphi_j(u)}<\infty\Big) \; \Big[\overline{\lim_{\substack{u\to 0\\ u\to\infty}}} \frac{\varphi_i(ku)}{\varphi_j(u)}<\infty\Big]$$

for some $k>0$ and $i,j=1,2$.

In the sequel the expresions "large arguments", "small arguments", "all arguments" will be always denoted by "l.a.", "s.a." and "a.a.", respectively.

In the rest of the paper, G will be a locally compact abelian group, with Haar measure μ. Let $\mathcal{M}$ the family of all μ-measurable, complex valued functions f defined on G. The Orlicz space L_φ is a modular space generated by the modular $I_\varphi(f)=\int_G\varphi(|f(t)|)d\mu(t)$ defined on $\mathcal{M}$. The subspace of finite elements of the space L_φ is denoted by E_φ. It is well known that, when φ is finite, E_φ consists of those elements of L_φ which are order continuous ([6],[5]). Let us recall that if φ_1 and φ_2 are equivalent for l.a. (s.a.)[a.a.] then $L_{\varphi_1}=L_{\varphi_2}$ when G is nondiscrete and compact (G is nondiscrete and noncompact)[G is discrete]. If φ_1 and φ_2 are finite then equivalence of these functions implies also that $E_{\varphi_1}=E_{\varphi_2}$.

In further considerations the important role will be played by the following two conditions. Let φ_i, $i=1,2,3$, be Young functions.

It is said that φ_i *satisfy condition (+) for l.a.(s.a.)[a.a.] if there exist* $k>0$, $\delta>0$ such that

$$kuv\leqslant\varphi_1(u)\varphi_3^{-1}(\varphi_2(v))+\varphi_2(v)\varphi_3^{-1}(\varphi_1(u))$$

when $\varphi_1(u)\geqslant\delta$ and $\varphi_2(v)\geqslant\delta$ ($\varphi_1(u)\leqslant\delta$ and $\varphi_2(v)\leqslant\delta$)[$u,v\geqslant o$].

It is said that φ_i satisfy condition (++) for l.a. (s.a.)[a.a.] if for every $\alpha>0$ there exist $k>0$, $\delta\geqslant 0$ such that

$$\alpha uv\leqslant\varphi_1(u)\varphi_3^{-1}(k\varphi_2(v))+\varphi_2(v)\varphi_3^{-1}(k\varphi_1(u))$$

when $\varphi_1(u)\geqslant\delta$ and $\varphi_2(v)\geqslant\delta$ ($\varphi_1(u)\leqslant\delta$ and $\varphi_2(v)\leqslant\delta$)[$u,v\geqslant 0$].

The above conditions can be reformulated equivalently. Namely, we have the following proposition. The proof will be omitted because it is analogous to that of Lemma 2.4 in [8].

2. Proposition.

– *Condition* (+) *for l.a.* (*s.a.*) [*a.a.*] *is equivalent to the following one: there exist* $l, \delta > 0$ *such that*

$$\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leqslant lu\varphi_3^{-1}(u)$$

if $u \geqslant \delta (u \leqslant \delta)[u \geqslant 0]$.

– *Condition* (++) *for l.a.* (*s.a.*) [*a.a.*] *is equivalent to the following one: for every* $\alpha > 0$ *there exist* $l, \delta > 0$ *such that*

$$\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leqslant \alpha u\varphi_3^{-}(lu)$$

if $u \geqslant \delta (u \leqslant \delta)[u \geqslant 0]$.

– Condition (+) is invariant under equivalence of the functions φ_i which is shown in the next proposition.

3. Proposition. *If* φ_i *satisfy condition* (+) *for l.a.* (*s.a.*) [*a.a.*] *and* $\bar{\varphi}_i$ *are equivalent to* φ_i *for l.a.* (*s.a.*) [*a.a.*], *then* $\bar{\varphi}_i$ *satisfy condition* (+) *for l.a.* (*s.a.*) [*a.a.*] *again.*

Proof. For instance, we shall show that $\bar{\varphi}_i$ satisfy condition (+) for s.a. Since $\bar{\varphi}_i$ are equivalent to φ_i, there exist δ_i, $l_i > 0$ such that $\varphi_i(l_i u) \leqslant \bar{\varphi}_i(u)$ if $\bar{\varphi}_i(u) \leqslant \delta_i$ ($i = 1,2$) and $\bar{\varphi}_3(l_3 u) \leqslant \varphi_3(u)$ if $\varphi_3(u) \leqslant \delta_3$. Put $l = \min l_i$ and $\delta_o = \min \delta_i$. Without loss of generality suppose $\delta_o \leqslant \delta$, where δ is the constant from condition (+). Then, by condition (+) we have

$$kl^2uv \leqslant \varphi_1(lu)\varphi_3^{-1}(\varphi_2(lv)) + \varphi_2(lv)\varphi_3^{-1}(\varphi_1(lu)) \leqslant \bar{\varphi}_1(u)\varphi_3^{-1}(\bar{\varphi}_2(v)) + \bar{\varphi}_2(v)\varphi_3^{-1}(\bar{\varphi}_1(u))$$

if $\bar{\varphi}_1(u) \leqslant \delta_o$ and $\bar{\varphi}_2(v) \leqslant \delta_o$.

Since $\bar{\varphi}_3(lu) \leqslant \varphi_3(u)$ when $\varphi_3(u) \leqslant \delta_o$, so $\varphi_3^{-1}(u) \leqslant (1/l)\bar{\varphi}_3^{-1}(u)$ if $u \leqslant \delta_o$. Indeed, putting $v = \varphi_3(u)$ we have $lu \leqslant \bar{\varphi}_3^{-1}(\bar{\varphi}_3(lu)) \leqslant \bar{\varphi}_3^{-1}(v)$ for $v \leqslant \delta_o$. But for $\delta_o > v = \varphi_3(u) > 0$, $u = \varphi_3^{-1}(\varphi_3(u)) = \varphi_3^{-1}(v)$ and so $\varphi_3^{-1}(v) \leqslant (1/l)\bar{\varphi}_3^{-1}(v)$. If $v = \varphi_3(u) = 0$ then the inequality is also true because $\bar{\varphi}_3(la_3) \leqslant \varphi_3(a_3) = 0$ implies $la_3 \leqslant \bar{a}_3$, *i.e.* $\varphi_3^{-1}(0) \leqslant (1/l)\bar{\varphi}_3^{-1}(0)$. Thus

$$kl^2uv \leqslant (1/l)\bar{\varphi}_1(u)\bar{\varphi}_3^{-1}(\bar{\varphi}_2(v)) + (1/l)\bar{\varphi}_2(v)\bar{\varphi}_3^{-1}(\bar{\varphi}_1(u))$$

If $\bar{\varphi}_1(u)\leqslant\delta_o$ and $\bar{\varphi}_2(v)\leqslant\delta_o$, which means that $\bar{\varphi}_i$ satisfy ccondition (+) for s.a.

Now we shall discuss the case of a discrete group G in the connection with the Orlicz space and its subspace of finite elements. Traditionally, in this case the notations l_φ and h_φ are used instead of L_φ and E_φ. First let us note the following simple fact.

4. Lemma. *For every Young function* φ *there exists a Young function* $\bar{\varphi}$ *finite and equivalent to* φ *for s.a.*

As a corollary, it is seen that instead of l_φ, where φ can take infinite values, one can always consider the isomorphic space $l_{\bar{\varphi}}$, where $\bar{\varphi}$ is finite. But there are some problems with the subspace of finite elements. If φ is infinite for some real numbers, then $h_\varphi=\{0\}$, whereas h_φ is always different than $\{0\}$ if φ is finite. Thus an equivalent function $\bar{\varphi}$ defines a different subspace of finite elements than the function φ. However, let us note that for any function φ there exists the only subspace of finite elements defined by a function $\bar{\varphi}$ finite and equivalent to φ. This subspace $h_{\bar{\varphi}}$ does not depend on the choice of the function $\bar{\varphi}$, belonging to the class of all Young functions finite and equivalent to φ.

Taking into considerations the above remarks, in the sequel we shall always assume that φ is finite in the case of a discrete group.

The Lemmas 5,6 and 8 are some technical steps to prove Theorems 7 and 9.

5. Lemma. *If* φ_i *are finite and satisfy condition* (+) *for s.a., then there exist functions* $\bar{\varphi}_i$ *finite and equivalent to* φ_i *for s.a. satisfy condition* (+) for a.a., if φ_i *satisfy condicion* (+) *for l.a., then there exist funtions* $\bar{\varphi}_i$ *equivalent to* φ_i *for l.a. and satisfying condition* (+) *for a.a.*

Proof. Let first φ_i satisfy condition (+) for s.a. Put $h(u,v)=\varphi_1(u)\varphi_3^{-1}(\varphi_2(v))+\varphi_2(v)\varphi_3^{-1}(\varphi_1(u))$ and $\bar{h}(u,v)$ if φ_i are replaced by $\bar{\varphi}_i$. Let u_i be such that $\varphi_i(u_i)=\delta$ and put

$$\varphi_i(u)=\begin{cases}\varphi_i(u) \quad , \quad u\in[0,u_i]\\ \varphi'_{i+}(u)u+\varphi_i(u_i)-\varphi'_{i+}(u_i)\,u_i\,, \ u\in(u_i,\infty),\end{cases}$$

where φ'_{i+} is a right-hand derivative of φ_i. We have $\varphi_i(u)\geqslant\bar{\varphi}_i(u)$ and $\varphi_i^{-1}(v)\leqslant\bar{\varphi}_i^{-1}(v)$ for all $u,v\geqslant 0$. We shall show that $\bar{\varphi}_i$ satisfy condition (+) for a.a. For $u\leqslant u_1$, $v\leqslant u_2$ the inequality is immediate. Let $u\geqslant u_1$ and $v\geqslant u_2$. Then we can write $\bar{\varphi}_i(u)=c_iu+d_i$ for $u\geqslant u_i$, where $c_i>0$ and $d_i<0$. Hence we simply obtain

$$\bar{\varphi}_1(u)\bar{\varphi}_3^{-1}(\bar{\varphi}_2(v)) = \frac{c_1c_2}{c_3}uv + \frac{c_1d_2}{c_3}u + \frac{d_1c_2}{c_3}v + M,$$

where M is a constant dependent on c_i, d_i. Since $c_1c_2/c_3 > 0$, there exist $e_1 > 0$ and $u_o > max(u_1, u_2)$ such that

$\bar{\varphi}_1(u)\bar{\varphi}_3^{-1}(\bar{\varphi}_2(v)) \geqslant e_1uv$ whereas $u, v \geqslant u_o$. Moreover,

$$\frac{\bar{\varphi}_1(u)\bar{\varphi}_3^{-1}(\bar{\varphi}_2(v))}{uv} \geqslant \frac{\bar{\varphi}_1(u_1)\bar{\varphi}_3^{-1}(\bar{\varphi}_2(u_2))}{u_o^2} = \frac{\delta u_3}{u_o^2} > 0$$

for $u \in [u_1, u_o]$ and $v \in [u_2, u_o]$. Then for $e = min\,(e_1, \frac{\delta u_3}{u_o^2})$ and $u \geqslant u_1$, $v \geqslant u_2$ we have $\bar{h}(u,v) \geqslant e\,uv$.

Let now $u \geqslant u_1$ and $v \leqslant u_2$. Then

$$\bar{h}(u,v) \geqslant (\varphi_3^{-1}(\varphi_2(v)) + \varphi_2(v))min(c_1u + d_1, \frac{c_1}{c_3}u + \frac{d_1}{c_3} - \frac{d_3}{c_3})$$

However, by the assumption (+) we have

$ku_1v \leqslant \delta\varphi_3^{-1}(\varphi_2(v)) + \varphi_2(v)u_3$ for $v \leqslant u_2$. Hence

$max\,(\varphi_3^{-1}(\varphi_2\,(v)),\ \varphi_2(v)) \geqslant e_2v$, where

$e_2 = min((k/2\delta)u_1,\ ku_1/2u_3) > 0$. Therefore

$$\bar{h}(u,v) \geqslant e_2v\ min(c_1u + d_1,\ \frac{c_1}{c_3}u + \frac{d_1}{c_3} - \frac{d_3}{c_3})$$

for $u \geqslant u_1$ and $v \leqslant u_2$. Since $\bar{\varphi}_1(u_1) = \delta > 0$ and $\bar{\varphi}_3^{-1}(\bar{\varphi}_1(u_1)) = u_3 > 0$ and the functions $\bar{\varphi}_i(u)$ and $\bar{\varphi}_3^{-1}(\bar{\varphi}_1(u))$ are linear for $u \geqslant u_1$, there exists a constant $e_3 > 0$ such that

$$\min(c_1u + d_1,\ \frac{c_1}{c_3}u + \frac{d_1}{c_3} - \frac{d_3}{c_3}) \geqslant e_3u \text{ for } u \geqslant u_1.$$

Hence $\bar{h}(u,v) \geqslant e_2e_3uv$ when $u \geqslant u_1$ and $v \geqslant u_2$. So we proved the first part of the lemma.

Now let φ_i satisfy condition (+) for l.a. It is not difficult to verify that the functions

$$\bar{\varphi}_i(u) = \begin{cases} \varphi_i(u) \text{ if } \varphi_i(u) > \delta \\ (\delta/\varphi_i^{-1}(\delta))u \text{ if } \varphi_i(u) \leqslant \delta \end{cases}$$

satisfy condition (+) for a.a. Moreover, it is evident that they are equivalent to φ_i for l.a. which finishes the proof of the lemma.

6. Lemma. *If φ_i satisfy condition (+) for a.a. and $I_{\varphi_2}(g) \leqslant 1$ and $I_{\varphi_1}((2\lambda/k)f) \leqslant 1$ (or $I_{\varphi_1}(f) \leqslant 1$ and $I_{\varphi_2}((2\lambda/k)g) \leqslant 1$ where k is the constant from (+), then $I_{\varphi_3}(\lambda f * g) < \infty$.*

Proof. Applying (+) we obtain

$$I_{\varphi_3}(\lambda f*g) \leqslant \int_G \varphi_2[\ 1/2 \int_G \varphi_1(\ 2\lambda/k|f(t)|)\varphi_3^{-1}(\varphi_2|g(t^{-1}x)|))d\mu$$

$$+ 1/2 \int_G \varphi_2(|g(t^{-1}x)|)\varphi_3^{-1}(\varphi_1(2\lambda/k|f(t)|))d\mu(t)]d\mu(x)$$

$$\leqslant 1/2 \int_G \varphi_3[\int_G \varphi_1(2\lambda/k|f(t)|)\varphi_3^{-1}(\varphi_2(|g(t^{-1}x)|))d\mu(t)]d\mu(x)$$

$$+ 1/2 \int_G \varphi_3[\int_G \varphi_2(|g(t^{-1}x)|)\varphi_3^{-1}(\varphi_1(2\lambda/k|f(t)|))d\mu(t)]d\mu(x)$$

Since $I_{\varphi_1}(2\lambda/kf) \leqslant 1$ and $I_{\varphi_2}(g) \leqslant 1$, by Jensen's inequality $I_{\varphi_3}(\lambda f*g) \leqslant I_{\varphi_1}(2\lambda/kf) I_{\varphi_2}(g) \leqslant 1 < \infty$.

The next theorems give sufficient cconditions for embeddings of the spaces $L_{\varphi_1} * L_{\varphi_2}$, $E_{\varphi_1} * E_{\varphi_2}$ into L_{φ_3}, E_{φ_3}.

7. Theorem. I. *Let G be nondiscrete and φ_i satisfy condition (+) for l.a. if G is compact and (+) for a.a. if G is noncompact. Then $L_{\varphi_1} * L_{\varphi_2} \hookrightarrow L_{\varphi_3}$. If additionally φ_3 is finite, then $E_{\varphi_1} * E_{\varphi_2} \hookrightarrow E_{\varphi_3}$.*
II. *Let G be discrete and φ_i satisfy condition (+) for s.a. Then $h_{\varphi_1} * h_{\varphi_2} \hookrightarrow h_{\varphi_3}$* and $l_{\varphi_1} * l_{\varphi_2} \hookrightarrow l_{\varphi_3}$.

Proof. By Theorem 1.2 it is enough to prove only inclusions. I. Let first G be noncompact and φ_i satisfy (+) for a.a. The proof of the inclusion $L_{\varphi_1} * L_{\varphi_2} \subset L_{\varphi_3}$ is an immediate consequence of the previous lemma. Really, taking $f \in L_{\varphi_1}$ and $g \in L_{\varphi_2}$ from the unit balls we have $I_{\varphi_1}(f) \leqslant 1$ and $I_{\varphi_2}(g) \leqslant 1$. So we can apply the lemma with $\lambda = k/2$ and thus $I_{\varphi_3}(k/2\, f*g) < \infty$, which means that $f*g \in L_{\varphi_3}$.

To prove the inclusion $E_{\varphi_1} * E_{\varphi_2} \subset E_{\varphi_3}$ take $f \in E_{\varphi_1}$, $g \in E_{\varphi_2}$. Let $\lambda = 3\beta$, where β is arbitrary. Since the Haar measure μ is regular and $I_{\varphi_1}(2\lambda/k\, f) < \infty$ and

$I_{\varphi_2}(2\lambda/k\, g)<\infty$, there exist compact sets G_1, G_2 such that $I_{\varphi_1}(2\lambda/k\, f\chi_{G\setminus G_1})\leqslant 1$ and $I_{\varphi_2}(2\lambda/k\, g\chi_{G\setminus G_2})\leqslant 1$. We can write

(8.1) $I_{\varphi_3}(\beta f*g)\leqslant 1/3 I_{\varphi_3}(\lambda f\chi_{G\setminus G_1}*g)+$

$$1/3 I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G-G_2})+1/3 I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G_2})$$

By the previous lemma, the first two components of the above inequality are finite, so it is enough to show that $I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G_2})<\infty$. Since the support of $f\chi_{G_1}*g\chi_{G_2}$ is contained in G_1G_2,

$I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G_2})\leqslant\int_{G_1G_2}\varphi_3(\lambda\int_{G_1\cap xG_2^{-1}}|f(t)||g(t^{-1}x)|d\mu(t)d\mu(x)$. There exists $u_0\geqslant 0$ such that

(8.2) $I_{\varphi_1}(6\lambda/k f\chi_{\tilde G_1})\leqslant 1$ and $I_{\varphi_2}(g\chi_{\tilde G_2})\leqslant 1$, where

$$\tilde G_1=\{t\in G_1: |f(t)|\geqslant u_0\}\quad \tilde G_2=\{t\in G_2: |g(t)|\geqslant u_0\}.$$

Denote $G_0=G_1\cap xG_2^{-1}$. We have

$$\lambda\int_{G_0-\tilde G_1}|f(t)||g(t^{-1}x)|d\mu(t)\leqslant\lambda u_0\int_{G_0}|g(t^{-1}x)|d\mu(t)$$

$$=\lambda u_0\int_{G_2\cap xG_1^{-1}}|g(u)|d\mu(u)\leqslant\lambda u_0\int_{G_0}|g(u)|d\mu(u)=M_1<\infty,$$

since every function from Orlicz space is locally integrable. Analogously

$$\lambda\int_{G_0\setminus x\tilde G_2^{-1}}|f(t)||g(t^{-1}x)|d\mu(t)\leqslant\lambda u_0\int_{G_1}|f(t)|d\mu(t)=M_2<\infty.$$

Thus,

$$I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G_2})\leqslant\int_{G_1G_2}\varphi_3(M_1+M_2+\lambda\int_{\tilde G_1\cap x\tilde G_2^{-1}}|f(t)||g(t^{-1}x)|d\mu(t))d\mu(x)\leqslant$$

$$1/3(\varphi_3(3M_1)+\varphi_3(3M_2))\mu G_1G_2+1/3\int_{G_1G_2}\varphi_3(1/2\int_{\tilde G_1\cap x\tilde G_2^{-1}}$$

$$k6\lambda/k|f(t)||g(t^{-1}x)|d\mu(t)d\mu(x).$$

Denoting by M_3 the first component of the above sum and applying condition (+) to the second one, we get

$$I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G_2})\leqslant M_3+$$

$$1/2\int_{G_1G_2}\varphi_3(\int_{\tilde G_1\cap x\tilde G_2^{-1}}\varphi_1(6\lambda/k|f(t)|)\varphi_3^{-1}(\varphi_2(|g(t^{-1}x)|))d\mu(t)d\mu(x)$$

$$+1/2\int_{G_1G_2}\varphi_3(\int_{\tilde G_1\cap x\tilde G_2^{-1}}\varphi_2(|g(t^{-1}x)|)\varphi_3^{-1}(\varphi_1(6\lambda/k|f(t)|))d\mu(t)d\mu(x).$$

In virtue of (8.2) and Jensen's inequality,

$$I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G_2})\leqslant M_3+$$

$$1/2\int_{G_1G_2}\left(\int_{G_1\cap xG_2^{-1}}\varphi_1(6\lambda/k|f(t)|)\varphi_2(|g(t^{-1}x)|)d\mu(t)\right)d\mu(x)+$$

$$1/2\int_{G_1G_2}\left(\int_{G_1\cap xG_2^{-1}}\varphi_2(|g(t^{-1}x)|)\varphi_1(6\lambda/k|f(t)|)d\mu(t)d\mu(x)\leqslant\right.$$

$$M_3+I_{\varphi_1}(6\lambda/kf)I_{\varphi_2}(g)<\infty.$$

If G is compact and φ_i satisfy, condition (+) for l.a., then by Lemma 5 there exist functions $\bar{\varphi}_i$ satisfing (+) for a.a. and such that $E_{\varphi_i}=E_{\bar{\varphi}_i}$, $L_{\varphi_i}=L_{\bar{\varphi}_i}$. So without loss of generality one can assume that φ_i satisfy condition (+) for a.a. Thus, the inclusions follow in the same way as above (we can put $G_1=G_2G$).

II. For this case, applying Lemma 5 again, we can also assume condition (+) for a.a. In the inequality (8.1) we can see analogously as in I that the first two components are finite. The third component is also finite since the support of $f\chi_{G_1}*g\chi_{G_2}$ is finite. So it is the end of the proof.

8. Lemma. *If φ_i satisfy condition (++) for s.a. then $l_{\varphi_1}\subset h_{\varphi_3}$ and $l_{\varphi_2}\subset h_{\varphi_3}$.*

Proof. Using the equivalent form of condition (++) expressed in Proposition 2 we have $\varphi_1^{-1}(u)\varphi_2^{-1}(u)\leqslant au\varphi_3^{-1}(ku)$ for $u\leqslant\delta$. But by concavity of φ_2^{-1} there exists $\delta_1>0$ such that $u/\varphi_2^{-1}(u)\leqslant 1$ for $u\leqslant\delta_1$. So $\varphi_1^{-1}(u)\leqslant a\varphi_3^{-1}(ku)$ for sufficiently small u. Putting $v=\varphi_1^{-1}(u)$ we obtain $\varphi_3(1/av)\leqslant k\varphi_1(v)$ for small v, which immediately implies that

$\overline{\lim_{u\to 0}}\dfrac{\varphi_3(\lambda u)}{\varphi_1(u)}<\infty$ for all $\lambda>Q$. But the last condition implies the inclusion

$l_{\varphi_1}\hookrightarrow h_{\varphi_3}$([9]).

9. Theorem.

I. Let G be nondiscrete and φ_i satisfy condition (++) for l.a. if G is compact and (++) for a.a. if G is noncompact. If φ_3 is finite, then $L_{\varphi_1}*L_{\varphi_2}\hookrightarrow E_{\varphi_3}$.

II. Let G be discrete and φ_i satisfy condition (++) for s.a. Then $l_{\varphi_1}*l_{\varphi_2}\hookrightarrow h_{\varphi_3}$.

Proof. I. Suppose G is compact. Take $f \in L_{\varphi_1}$, $g \in L_{\varphi_2}$ such that $I_{\varphi_1}(f) \leq 1$ and $I_{\varphi_2}(g) \leq 1$. Let $\lambda > 0$ be arbitrary and $\delta > 0$ be from condition (++) chosen for $\alpha = 2\lambda$. Put

$$G_1 = \{t \in G : \varphi_1(|f(t)|) \geq \delta\},$$

$$G_2 = \{t \in G : \varphi_2(|g(t)|) \geq \delta\}.$$

The convolution $f*g$ is the sum of functions $f\chi_{G_1}*g\chi_{G_2}$, $f\chi_{G_1}*g\chi_{G\setminus G_2}$ and $f\chi_{G\setminus G_1}*g$. Applying (++) for l.a. and Jensen's inequality, we get analogously as in the proof of Theorem 7, that

$$I_{\varphi_3}(\lambda f\chi_{G_1}*g\chi_{G_2}) \leq \int_G \varphi_3(\alpha/2 \int_G |f(t)|\chi_{G_1}(t)|g(t^{-1}x)|\chi_{xG_2^{-1}}(t)d\mu(t)d\mu(x) \leq$$

$$\int_G \varphi_3[1/2 \int_G \varphi_1(|f(t)|\chi_{G_1}(t)\varphi_3^{-1}(k\varphi_2(|g(t^{-1}x)|\chi_{xG_2^{-1}}(t)))d\mu(t)$$

$$+ 1/2 \int_G \varphi_2(|g(t^{-1}x)|\chi_{xG_2^{-1}}(t))\varphi_3^{-1}(k\varphi_1(|f(t)|\chi_{G_1}(t)))d\mu(t)d\mu(x)$$

$$\leq kI_{\varphi_1}(f)I_{\varphi_2}(g) < \infty.$$

So it is enough to show that e.g. $I_{\varphi_3}(\lambda f\chi_{G\setminus G_1}*g) < \infty$.

By local integrability of g, we have $M = \int_G |g(t)|d\mu(t) < \infty$.

Hence $I_{\varphi_3}(\lambda f\chi_{G\setminus G_1}*g) \leq \varphi_3(\chi_{G_1^{-1}}(\delta)M)\mu G < \infty$.

In the case of noncompact G the proof is similar and even simpler in the sense that $G_1 = G_2 = G$.

II. If G is discrete and f, g, are the same as above, then there exist finite sets $G_1, G_2 \subset G$ such that $I_{\varphi_1}(f\chi_{G\setminus G_1}) \leq \delta$ and $I_{\varphi_2}(gX_{G\setminus G_2}) \leq \delta$, where $\delta > 0$ is the constant from condition (++) chosen for $\alpha = 2\lambda > 0$. We have

$$f*g = (f\chi_{G_1}*g\chi_{G_2}) + (f\chi_{G_1}*g\chi_{G\setminus G_2}) + (f\chi_{G\setminus G_1}*g\chi_{G_2}) + (f\chi_{G\setminus G_1}*g\chi_{G\setminus G_2}).$$

The first component belongs to h_{φ_3} because its support is finite. Applying (++) and Jensen's inequality to the last one similarly as in I, we get $I_{\varphi_3}(\lambda f\chi_{G\setminus G_1}*gX_{G\setminus G_1}) \leq k < \infty$.

To finish the proof note that

$$|(f\chi_{G_1}*g\chi_{G\setminus G_2})(x)| \leq \sum_{t \in G_1} |f(t)||g(t^{-1}x)|$$

for every $x \in G$, where $g(t^{-1}\cdot) \in l_{\varphi_2}$ for all $t \in G$. Thus

$$\sum_{t\in G_1} |f(t)||g(t^{-1}\cdot)| \in l_{\varphi_2} \text{ and so } f\chi_{G_1} * g\chi_{G\setminus G_2} \in l_{\varphi_2}.$$

But by Lemma 8, we have $l_{\varphi_2} \subset h_{\varphi_3}$, so $f\chi_{G_1} * g\, \chi_{G\setminus G_2} \in h_{\varphi_3}$.

Next theorems will be converse to the results obtained in Theorem 7.

10. Theorem. *Let G be discrete. If $h_{\varphi_1} * h_{\varphi_2} \subset l_{\varphi_3}$, then G is finite or condition (+) for s.a. is satisfied.*

Proof. Remember, in the sequence case we have assumed the functions φ_i have been finite. For a contrary, suppose the group G is infinite and condition (+) for s.a. is not satisfied. There exist sequences $(u_n),(v_n)$ such that $\varphi_1(u_n)\to 0$ and $\varphi_2(v_n)\to 0$ and

(10.1) $$1/n\; u_n v_n > \varphi_1(u_n)\varphi_3^{-1}(\varphi_2(v_n)) + \varphi_2(v_n)\varphi_3^{-1}(\varphi_1(u_n)).$$

Without loss of generality assume $\varphi_1(u_n) \geqslant \varphi_2(v_n)$.

We shall consider a few cases.

I. Let $\varphi_1(u_n) \geqslant \varphi_2(v_n) > 0$, Let $\bar{u}_n$ be such that $\varphi_1(\bar{u}_n) = \varphi_2(v_n)$. Since $\varphi_1(u)/u$ is nondecreasing, $1/n\; \bar{u}_n v_n > \varphi_1(\bar{u}_n)\varphi_3^{-1}(\varphi_2(v_n))$, by (10.1). So we can assume about u_n, v_n that $\varphi_1(u_n) = \varphi_2(v_n)$ and

(10.2) $$1/n\; u_n v_n > \varphi_1(u_n)\varphi_3^{-1}(\varphi_2(v_n)).$$

We shall examine two types of the group G.

(a). Let G contain a cyclic subgroup of arbitrary large rank.

There exist natural numbers l_n such that

(10.3) $$(2l_n + 1)\varphi_1(u_n) \leqslant 1 \text{ and } (2l_n + 3)\varphi_1(u_n) > 1.$$

Take a cyclic subgroup S such that $rS > 2l_n + 1$. Let

$$A_n = \{t^i \in S : i = 0, 1, \ldots, l_n, -1, \ldots, -l_n\},$$

and

$$f_n(t) = u_n\chi_{A_n}(t),\; g_n(t) = v_n\chi_{A_n}(t).$$

By (10.3), it is evident that

(10.4) $$1/2 \leqslant I_{\varphi_1}(f_n) \leqslant 1,\ 1/2 \leqslant I_{\varphi_2}(g_n) \leqslant 1.$$

Moreover

$$(f_n * g_n)(x) = u_n v_n \mu(A_n \cap A_n x).$$

If $x \in A_n$ then either $A_n \cap A_n x \supset \{e, t, \ldots, t^{l_n}\}$ or $A_n \cap A_n x \supset \{e, t^{-1}, \ldots, t^{-l_n}\}$. Therefore $\mu(A_n \cap A_n x) \geqslant l_n + 1$ for $x \in A_n$.

Hence

(10.5) $$(f_n * g_n)(x) \geqslant u_n v_n (l_n + 1) \chi_{A_n}(x).$$

But by (10.3), we get the following estimation of l_n

(10.6) $$l_n + 1 \geqslant 1/3 \varphi_1(u_n).$$

Thus in virtue of (10.2)

$$\frac{3}{n}(f_n * g_n)(x) \geqslant \frac{1}{n} u_n v_n \frac{1}{\varphi_1(u_n)} \chi_{A_n}(x) > \varphi_3^{-1}(\varphi_2(v_n)) \chi_{A_n}(x).$$

Hence and by Lemma 1 and the fact that $\varphi_3(b_3) = \infty$ and by (10.4), we have

$$I_{\varphi_3}(3/n\, f_n * g_n) \geqslant \varphi_2(v_n) \mu A_n = I_{\varphi_2}(g_n) \geqslant 1/2.$$

So we found sequences $f_n \in h_{\varphi_1}$, $g_n \in h_{\varphi_2}$ such that $I_{\varphi_1}(f_n) \leqslant 1$, $I_{\varphi_2}(g_n) \leqslant 1$ and $I_{\varphi_3}(3/n\, f_n * g_n) \geqslant 1/2$. Applying Theorem 1.2 we can see that $h_{\varphi_1} * h_{\varphi_2} \not\subset l_{\varphi_3}$.

(b). Let, contrary to (a), the rank of all cyclic subgroups of G be bounded. So there exists a prime number k and infinite number of cyclic subgroups with rank equal to k. Let S_i be an infinite sequence of cyclic subgroups such that $rS_i = k$, $S_i \cap S_j = \{e\}$ for $i = j$. Let $P_n = \overset{n}{\underset{i=1}{\oplus}} S_i$ be a simple sum of $S_1, \ldots, S_n$. The set P_n is a subgroup of G containing k^n different elements. Put

$$l_n = [\log_k \frac{1}{\varphi_1(u_n)}] \text{ and}$$

$$f_n(t) = u_n \chi_{P_{l_n}}(t) \quad \text{and} \quad g_n(t) = v_n \chi_{P_{l_n}}(t).$$

It is clear that

(10.7) $$1/k \leqslant I_{\varphi_1}(f_n) \leqslant 1,\ 1/k \leqslant I_{\varphi_2}(g_n) \leqslant 1.$$

Moreover,

$$(f_n * g_n)(x) \geqslant u_n v_n \mu P_{l_n} \chi_{P_{l_n}}(x) = u_n v_n k^{l_n} \chi_{P_{l_n}}(x).$$

But by the definition of l_n, we have

(10.8) $$k^{l_n} \geqslant 1/k\varphi_1(u_n).$$

Thus in virtue of (10.2), Lemma 1 and (10.7) we obtain

$$I_{\varphi_3}(k/n\, f_n * g_n) \geqslant I_{\varphi_2}(g_n) \geqslant 1/k.$$

So applying Theorem 1.2 we get a contradiction.

II. Suppose $\varphi_1(u_n) > \varphi_2(v_n) = 0$. Hence $a_2 > 0$. We shall consider two cases. Let first $a_3 = 0$. There exists an infinite, countable subgroup G_o of G. There exists an element $(c_n) \in c_o$ such that $(c_n) \notin l_{\varphi_3}(G_o)$. Put

$$u(t) = \begin{cases} c_n \text{ if } t = t_n,\ n = 0,1,2,\dots \\ 0 \text{ if } t \notin G_o \end{cases}$$

$$v(t) = \chi_{\{e\}}(t),$$

where $G_o = \{e, t_1, t_2, \dots,\}$. Since $a_2 > 0$, h_{φ_2} is isomorphic to c_o. So it is clear that $u \in h_{\varphi_2}$, $v \in h_{\varphi_1}$ and $u * v = \notin l_{\varphi_3}$.

Now let $a_3 > 0$. In this case we modify the proof of the part I. Let C_n denote A_n or P_{l_n} and c_n denote l_n or k^{l_n} and c be equal to 3 or k, respectively, when G satisfies *(a)* or *(b)*. Let $f_n(t) = u_n \chi_{C_n}(t)$ and $g_n(t) = v_n \chi_{C_n}(t)$. Then $\min(1/2, 1/k) \leqslant I_{\varphi_1}(f_n) \leqslant 1$, $I_{\varphi_2}(g_n) \leqslant 1$ and

$$(f_n * g_n)(x) \geqslant u_n v_n \frac{1}{c\varphi_1(u_n)} \chi_{C_n}(x),$$

by (10.4),(10.7),(10.5),(10.6) and (10.8). Hence and by (10.1) we get

$$I_{\varphi_3}(2\, c/n\, f_n * g_n) \geqslant \int_G \varphi_3(2\varphi_3^{-1}(\varphi_2(v_n)))\chi_{C_n}(x)d\mu(x)$$

$$= \int_G \varphi_3(2a_3)\chi_{C_n}(x)d\mu(x) = \varphi_3(2a_3)\mu C_n \to \infty, \text{ as } n \to \infty.$$

So by virtue of Theorem 1.2 and the fact that $f_n \in h_{\varphi_1}$ and $g_n \in h_{\varphi_2}$ we get the hypothesis.

III. If $\varphi_1(u_n)=\varphi_2(v_n)=0$ then $a_i>0$ for $i=1,2$. So $h_\varphi(G_o)$ is isomorphic to c_o for $i=1,2$ and any infinite, countable subgroup G_o of G. But it is possible to construct elements $u,v\in c_o$ such that $(u*v)(e)=\infty$.

11. Theorem. *For any noncompact abelian group G condition (+) for s.a. is necessary for the inclusion $L_{\varphi_1}*L_{\varphi_2}\subset L_{\varphi_3}$. If additionally $\varphi_i(i=1,2)$ are finite then condition (+) for s.a. is necessary for the inclusion $E_{\varphi_1}*E_{\varphi_2}\subset L_{\varphi_3}$, too.*

Proof. By the well known results about the structure of an abelian group either the group G contains a compact open subgroup G_o or it contains an element z such that the set $\{z^n : n\in\mathbb{Z}\}$ is an infinite, discrete subgroup of G, where $\mathbb{Z}$ is the set of all integers. Let $\bar{\varphi}_i$ be finite and equivalent to φ_i for s.a. (see Lemma 4).

In the first case G/G_o is infinite and discrete. Suppose $\mu G_o=1$. We shall show that $l_{\varphi_1}(G/G_o)*l_{\varphi_2}(G/G_o)\subset l_{\varphi_3}(G/G_o)$. For $f\in l_{\varphi_1}(G/G_o)$ and $g\in l_{\varphi_2}(G/G_o)$ put $\bar{f}(x)=f(xG_o)$ and $\bar{g}(x)=g(xG_o)$ where xG_o belongs to G/G_o. Then $\bar{f}\in l_{\varphi_1}(G/G_o)$, $\bar{g}\in l_{\varphi_2}(G/G_o)$ and clearly $\bar{f}\in L_{\varphi_1}$ and $\bar{g}\in L_{\varphi_2}$. By the assumption $\bar{f}*\bar{g}\in L_{\varphi_3}$. But $\bar{f}*\bar{g}(x)=f*g(xG_o)$ for all $x\in G$, because $\mu G_o=1$. Therefore $f*g\in l_{\varphi_3}(G/G_o)=l_{\bar{\varphi}_3}(G/G_o)$. Thus by virtue of finiteness of $\bar{\varphi}_i$ and Theorem 10, $\bar{\varphi}_i$ satisfy condition (+) for s.a. But Proposition 3 implies that φ_i satisfy condition (+) for s.a., too.

In the second case denote $G_1=\{z^n : n\in\mathbb{Z}\}$. Analogously as above it is enough to show that $l_{\bar{\varphi}_1}(G_1)*l_{\bar{\varphi}_2}(G_1)\subset l_{\bar{\varphi}_3}(G_1)$. Take arbitrary $a=(a_n)\in l_{\bar{\varphi}_1}(G_1)=l_{\varphi_1}(G_1)$ and $b=(b_n)\in l_{\bar{\varphi}_2}(G_1)=l_{\varphi_2}(G_1)$. Then $a*b=c=(c_n)$ where $c_n=\sum_{k\in\mathbb{Z}} a_k b_{n-k}$.

Let U,V be symmetric neighbourhoods of e such that

$U\cap G_1=\{e\}$ and $V^2\subset U$. Put $f(t)=\sum_{n\in\mathbb{Z}} a_n\chi_{Uz^n}(t)$, $g(t)=\sum_{n\in\mathbb{Z}} b_n\chi_{Uz^n}(t)$. Clearly $f\in L_{\varphi_1}$,

$g\in L_{\varphi_2}$ and so $f*g\in L_{\varphi_3}$. Moreover,

$$|f|*|g|(x)\geqslant\sum_{n\in\mathbb{Z}}|a_k|\,|b_{n-k}|\mu(Uxz^{-n}\cap U).$$

If $x\in Vz^n$ then $xz^{-n}V\subset V^2\subset U$. Hence $\mu(Uxz^{-n}\cap U)\geqslant\mu V$.
Thus $|f|*|g|(x)\chi_{Vz^n}(x)\geqslant|c_n|\mu V$. Therefore there exists $\lambda>0$ such that $\sum_{n\in\mathbb{Z}}\varphi_3(\lambda\mu V|c_n|)\leqslant I_{\varphi_3}(\lambda|f|*|g|)<\infty$, which shows that $c=(c_n)\in l_{\bar{\varphi}_3}(G_1)=l_{\varphi_3}(G_1)$.
Thus the first part of the theorem is proved. The proof of the second one is similar and even simpler because the functions φ_i are finite by the assumption.

Now, le us introduce two conditions for a locally compact group.

We say that a group G satisfies condition (*) if for every sequence $\alpha_i \to \infty$ there exist sequences $(U_i),(V_i)$ of measurable sets and constants $\kappa, k_1, k_2 > 0$ such that

- $k_1 \leqslant \alpha_i \mu U_i \leqslant k_2$,
- $V_i V_i^{-1} \subset U_i$,
- $\mu U_i \leqslant \kappa \mu V_i$,

for every $i \in \mathbb{N}$.

It is said that a sequence (U_i, V_i) is a so called D''-sequence ([2]) if U_i, V_i are measurable sets and there exists $\kappa > 0$ such that

- $U_1 \supset U_2 \supset \ldots, U_i \to e$,
- $V_i V_i^{-1} \subset U_i$,
- $\mu U_i \leqslant \kappa \mu V_i$,

for every $i \in \mathbb{N}$.

Note, in the above two conditions we may always suppose that $V_i \subset U_i$.

12. Remark. The following groups satisfy condition (*) : $(\mathbb{R},+),(\mathbb{K},+),(T,\cdot),(\mathbb{R}\backslash\{0\},\cdot),(\mathbb{K}\backslash\{0\},\cdot)$, where T is a subgroup of $(\mathbb{K}\backslash\{0\},\cdot)$ consisting of all elements belonging to the unit sphere of $\mathbb{K}$.

In [2] there are examples of groups addmiting a D''-sequence. For instance the groups containing an open subgroup of the form $\mathbb{R}^a \times T^b \times F$, where a,b are positive integers and F is a finite group, admit a D''-sequence.

13. Proposition. *If a group G contains an infinite, discrete and cyclic subgroup and G admits a D''-sequence, then the condition (*) is satisfied.*

Proof. Let (α_i), $\alpha_i \geqslant 1$, be an arbitrary sequence tending to infinity and $\{z^n : n \in \mathbb{Z}\}$ be a discrete subgroup of G. If W is neighbourhood of e such that $\{z^n W\}$ is a pairwise disjoint family of sets, then we may assume that $U_i \subset W$, where (U_i, V_i) is a D"-sequence. We find a subsequence (U_{i_j}) and natural numbers k_j such that $1/2 \leqslant \alpha_j(2k_j+1)\mu U_{i_j} \leqslant 1$. Putting

$$P_j = \bigcup_{n=-2k_j}^{2k_j} z^n U_{i_j},\ Q_j = \bigcup_{n=-k_j}^{k_j} z^n V_{i_j},$$ we have

$\mu P_j = (4k_j+1)\mu U_{i_j}$ and $Q_j = (2k_j+1)\mu V_{i_j}$. Hence

$1/4 \leqslant \alpha_j \mu P_j \leqslant 2$ and $\mu P_j \leqslant 2\kappa\mu Q_j$. Moreover,

$$Q_j Q_j^{-1} = \bigcup_{n=-k_j}^{k_j} \bigcup_{m=-k_j}^{k_j} z^{n+m} V_{i_j}^{-1} V_{i_j} = \bigcup_{l=-2k_j}^{2k_j} z^l V_{i_j}^{-1} V_{i_j}$$

$$\subset \bigcup_{l=-2k_j}^{2k_j} z^l U_{i_j} = P_j.$$

Thus the group G satisfies condition (*).

14. Theorem. *Let a group G satisfy condition (*). Then condition (+) for l.a. is necessary for the inclusion $L_{\varphi_1} * L_{\varphi_2} \subset L_{\varphi_3}$.*

If moreover $\varphi_i (i=1,2)$ are finite, then the condition (+) for l.a. is necessary for the inclusion $E_{\varphi_1} * E_{\varphi_2} \subset L_{\varphi_3}$, too.

Proof. Assume $\varphi_i(i=1,2)$ are finite (in another case condition (+) for l.a. is always satisfied). Suppose condition (+) for l.a. is not sastified. Then there exist sequences $(u_i),(v_i)$ such that $\varphi_1(u_i) \to \infty$ and $\varphi_2(v_i) \to \infty$ and

(14.1) $$1/i\ u_i v_i > \varphi_1(u_i)\varphi_3^{-1}(\varphi_2(v_i)) + \varphi_2(v_i)\varphi_3^{-1}(\varphi_1 u_i)).$$

Analogously as in the proof of Theorem 10 one can put $\varphi_1(u_i) = \varphi_2(v_i)$. By the assumed condition (*), one can find a sequence U_i of measurable sets such that

(14.2) $$k_1 \leqslant \varphi_1(u_i)\mu U_i \leqslant k_2,$$

where $k_1, k_2 > 0$. Putting

$$f_i(t) = u_i \chi_{U_i}(t),\ g_i(t) = v_i \chi_{U_i^{-1}}(t),$$

we have

$$f_i * g_i(x) = u_i v_i \mu(xU_i \cap U_i).$$

We can assume that $V_i \subset U_i$, where V_i are sets from condition (*). So, if $x \in V_i^{-1}$ then $xV_i \subset U_i$ and

$$\mu(xU_i \cap U_i) \geqslant \mu(xU_i \cap xV_i) = \mu V_i,$$

for $x \in V_i^{-1}$. Thus

$$f_i * g_i(x) \geqslant u_i v_i \mu V_i \chi_{V_i^{-1}}(x).$$

Then in virtue of (14.1) and (14.2), we get

$$\kappa/k_1 \; 1/i \; (f_i * g_i)(x) \geqslant \kappa/k_1 \; \varphi_1(u_i)\varphi_3^{-1}(\varphi_2(v_i))\mu V_i \chi_{V_i^{-1}}(x) \geqslant \varphi_3^{-1}(\varphi_2(v_i))\chi_{V_i^{-1}}(x).$$

Therefore

$$I_{\varphi_3}(\kappa/k_1 \;\; 1/i \, f_i * g_i) \geqslant 1/\kappa \; \varphi_3[\varphi_3^{-1}(\varphi_2(v_i))]\mu U_i.$$

Now if $b_3 = \infty$ then

$$I_{\varphi_3}(\kappa/k_1 \; 1/i \, f_i * g_i) \geqslant k_1/\kappa \text{ for every } i \in \mathbb{N},$$

if $b_3 < \infty$ then $4\varphi_3^{-1}(\varphi_2(v_i)) > 2b_3$ for sufficiently large i and so

$$I_{\varphi_3}\Big(\frac{4\kappa}{\kappa_1} \; \frac{1}{i} \; f_i * g_i\Big) \geqslant \varphi_3(2b_3)\mu V_i = \infty.$$

Thus we have found sequences $(f_i),(g_i)$ such that $f_i \in E_{\varphi_1}$, $g_i \in E_{\varphi_2}$ and $I_{\varphi_1}(f_i) \leqslant k_2$, $I_{\varphi_2}(g_i) \leqslant k_2$ and $I_{\varphi_3}(\lambda_i f_i * g_i) \geqslant$ const. for some $\lambda_i \to 0$. Then, by Theorem 1. The inclusions $E_{\varphi_1} * E_{\varphi_2} \subset L_{\varphi_3}$ and $L_{\varphi_1} * L_{\varphi_2} \subset L_{\varphi_3}$ are not fulfilled, which ends the proof of the theorem.

The following three corollaries are immediate consequence of Theorems 1.1, 1.2, 2, 7, 10, 11 and 14.

15. Corollary. *Let G be a discrete group. The following conditions are equivalent*

(1) $l_{\varphi_1} * l_{\varphi_2} \hookrightarrow l_{\varphi_3}$

(2) $h_{\varphi_1} * l_{\varphi_2} \hookrightarrow h_{\varphi_3}$

(3) $h_{\varphi_1} * l_{\varphi_2} \hookrightarrow l_{\varphi_3}$

(4) $h_{\varphi_1} * l_{\varphi_2} \hookrightarrow l_{\varphi_3}$

(5) φ *satisfies condition* (+) *for s.a. or G is finite,*

(6) *There exist $l,\delta>0$ such that*

$$\varphi_1^{-1}(u)\varphi_2^{-1}(u)\leqslant l\ u\ \varphi_3^{-1}(u)$$

if $u\leqslant\delta$, or G is finite,

(7) $\|f*g\|_{\varphi_3}\leqslant c\|f\|_{\varphi_1}\|g\|_{\varphi_2}$

for some $c>0$ and all $f\in l_{\varphi_1}$, $g\in l_{\varphi_2}$.

16. Corollary. *Let G be nondiscrete group and $\varphi_i(i=1,2,3)$ be finite. Consider the conditions (1) to (4) and (7) as in Corollary 15, where l_{φ_i}, h_{φ_i} are replaced by L_{φ_i}, E_{φ_i} respectively. Moreover, let*

(5') *φ satisfies condition (+) for l.a. if G is compact or for a.a. if G is noncompact,*

(6') *there exist $l,\delta>0$ such that*

$$\varphi_1^{-1}(u)\varphi_2^{-1}(u)\leqslant lu\varphi_3^{-1}(u)$$

if $u\geqslant\delta$ and G is compact or if $u\geqslant 0$ and G is noncompact.

We have relations: (1)$\leftrightarrow$(7) *and*

$$(6')\leftrightarrow(5')\nearrow(1)\longrightarrow(3)\searrow(4)$$
$$(5')\searrow(2)\nearrow(4)$$

Moreover, if a group G satisfies condition (*) then (4)$\rightarrow$(5'), i.e. all the above conditions are equivalent.

Sufficiency of the next corollary is known as Young theorem (see e.g. [2], [8]).

17. Corollary. *Let $1\leqslant p,q,r<\infty$.*

I. If G is discrete and infinite, then

$$l^p*l^q\hookrightarrow l^r$$

if $1/p+1/q\geqslant 1/r+1$.

II. Let G be nondiscrete compact and $1/p+1/q \leqslant 1/r+1$ or respectively G is noncompat and $1/p+1/q=1/r+1$, then

$$L^p * L^q \hookrightarrow L^r.$$

If additionally G satisfies condition (*), then the converse of the above is also true.

18. Corollary.

I (th.2 in [3]) L_φ is a Banach algebra under convolution as multiplication iff $L_\varphi \hookrightarrow L^1$, i.e. $\lim_{u \to 0} \varphi(u)/u > 0$ or G is compact.

II ([10]) $L^p (1 \leqslant p < \infty)$ is a Banach algebra iff $p=1$ or G is compact.

Proof. I. If we put $\varphi_i = \varphi (i=1,2,3)$, then φ_i satisfy condition (+) for l.a., by convexity of φ. Moreover, if L_φ is a Banach algebra and G is noncompact, then applying Theorem 11 we get condition (+) for s.a. Thus φ_i satisfy (+) for a.a., which means that $\lim_{u \to 0} \varphi(u)/u > 0$. The converse is immediate, by Theorem 7.

The point I of the above Corollary (see also [3]) is the answer to the Gramsch's problem from [1], in the case of convex function Φ.

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