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On some convexity properties of Orlicz spaces of vector valued functions

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ABSTRACT. A "stability" theorem that is a generalization of Th. 6 in [2] for the modulus of convexity of Banach spaces is given. Necessary and sufficient conditions for $\delta_L\Phi(a)>0$, where $a\in(0,2]$, in Orlicz spaces $L^\Phi(\mu,X)$ of vector valued functions are given. The convexity coefficient $\varepsilon_o(L^\Phi(\mu, X))$ is computed for these spaces. The equality $\varepsilon_o(L^\Phi(\mu,X))=\varepsilon_o(X)$ for Orlicz-Bochner spaces generated by uniformly convex Orlicz functions satisfying the Δ_2-condition is showed.

INTRODUCTION

Throughout this paper $(T,\sum,\mu)$ denotes a non-atomic, infinite and complete measure space and X denotes a Banach space. A function $\Phi:X\to[0,+\infty]$ is said to be an *Orlicz function* if it is convex, even, lower semicontinuous, vanishing and continuous at zero, and $\Phi\not\equiv 0$. $F(T,X)$ stands for the space of all equivalence classes of strongly $\sum$-measurable functions from T into X.

Given an Orlicz function Φ, we define the *Orlicz space* $L^\Phi(\mu,X)$ as the set of all functions $f\in F(T,X)$ such that

$$I_\Phi(\lambda f)=\int_T \Phi(\lambda f(t))d\mu<+\infty$$

for some $\lambda>0$ depending on f. This space equipped with the Luxemburg norm

$$\|f\|_\Phi=\inf\{\lambda>0:I_\Phi(\lambda^{-1}f)\leq 1\}$$

is a normed space (see [11-13]), and it is a Banach space if and only if Φ is uniformly large at infinity, i.e.

$\lim\limits_{k\to+\infty}\inf\{\Phi(x):\|x\|=k\}=+\infty$ (see [15]).

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We say an Orlicz function Φ satisfies the Δ_2-condition if there is a constant $K>0$ such that $\Phi(2x)\leq K\Phi(x)$ for all $x\in X$.

The *modulus of convexity* of a normed space $(X,\|\ \|)$ is the function $\delta_X(\cdot)$: $(0,2]\to[0,1]$ defined by

$$\delta_X(\varepsilon)=\inf\{1-\|1/2(x+y)\|:\|x\|\leq 1,\|y\|\leq 1,\|x-y\|\geq \varepsilon\}.$$

The *convexity coeffcient* of a normed space $(X,\|\ \|)$ is defined by

$$\varepsilon_o(X)=\sup\{\varepsilon\in[0,2]:\delta_X(\varepsilon)=0\}(\sup \emptyset \overset{\text{def}}{=} o).$$

(see [2]).

RESULTS

To prove the first theorem we shall need the following

Lemma 1. *If $\delta_X(a)>0$ for a number $a\in(0,2)$, then there is a number $\gamma>1/a$ such that $a\gamma(1-\delta_X(1/\gamma))=1$.*

Proof. By the assumptions and by the continuity of δ_X it follows that there is a number $\alpha>1/a$ such that $\delta_X(1/\alpha)>0$. So, $a\beta(1-\delta_X(1/\alpha))<1$ for a certain $\beta>1/a$. Taking $\alpha_o=\min(\alpha,\beta)$, we have $a\alpha_o(1-\delta_X(1/\alpha_o))<1$.

A function h: $(1/2,+\infty)\to R_+$ defined by $h(\lambda)=a\lambda(1-\delta_X(1/\lambda))$ is continuous and $h(\lambda)\to+\infty$ as $\lambda\to+\infty$. Since $h(\alpha_o)<1$, the Darboux property of h yields $h(\gamma)=a\gamma(1-\delta_X(1/\gamma))=1$ for a certain $\gamma>1/a$ which finishes the proof.

Now, we are able to generalize Th. 6 in [2]. The proof is almost the same but we shall give it for the sake of completness.

Theorem 2. *Let X be a Banach space with $\varepsilon_o(X)$ in the interval $[0,a)$, where $0<a\leq 2$. Let $\gamma>1/a$ be such that $a\gamma(1-\delta_X(1/\gamma))=1$. If Y is a Banach space with Banach-Mazur distance $d(X,Y)<a\gamma$, then $\varepsilon_o(Y)<a$.*

Proof. Without loss of generality we may assume that U is an isomorphism between X and Y such that $\|U^{-1}\|=1$ and $d(X,Y)\leq\|U\|\leq ab\gamma$, where $0<b<1$. Let $y_1, y_2\in S_Y$ (=the unit sphere of Y), $\|y_1-y_2\|\geq\|U\|/\gamma$ and $x_1=U^{-1}y_1$, $x_2=U^{-1}y_2$. Then $\|x_1\|\leq 1, \|x_2\|\leq 1$ and $\|U\|/\gamma\leq\|y_1-y_2\|=\|U(x_1-x_2)\|\leq\|U\|\|x_1-x_2\|$, whence $\|x_1-x_2\|\geq 1/\gamma$. Since $a\gamma>1$, by the equality $a\gamma(1-\delta_X(1/\gamma))=1$, it follows that $\delta_X(1/\gamma)>0$. Therefore

$$\|(y_1+y_2)/2\|=\|U(1/2(x_1+x_2))\|\leq\|U\|\|1/2(x_1+x_2)\|\leq ab\gamma(1-\delta_X(1/\gamma))=b.$$

This means that $\delta_Y(\|U\|/\gamma)\geq 1-b>0$. Thus, $\varepsilon_o(Y)\leq\|U\|/\gamma<a$.

In the fixed point theory the notion of the convexity coefficient is useful (see [2]). We shall give now a basic theorem to compute $\varepsilon_0(L^\Phi(\mu,X))$.

Theorem 3. *Let a be a number in* (0,2]. *The following conditions are equivalent:*

1° $\delta_{L^\Phi}(a)>0$.

2° (*a*) *there is* $\delta\in(0,1)$ *such that for every* $x,y\in X$ *satisfying the equality*

$\Phi((x-y)/a)\geq(1-\delta)\Phi((x+y)/2)$, we have $\Phi((x+y)/2)\leq\frac{1-\delta}{2}\{\Phi(x)+\Phi(y)\}$,

(*b*) Φ *satisfies the* Δ_2*-condition.*

Proof. 2°$\Rightarrow$1°. Assume that $\|f\|_\Phi\leq 1$, $\|g\|_\Phi\leq 1$ and $\|f-g\|_\Phi\geqslant a$. Then $I_\Phi(f)\leq 1$, $I_\Phi(g)\leq 1$ and $I_\Phi((f-g)/a)\geq 1$. Define

$$A=\{t\in T:\Phi((f(t)-g(t))/a)\geq(1-\delta)\Phi((f(t)+g(t))/2)\}.$$

Then

$$I_\Phi(\frac{f-g}{a}\chi_{T\setminus A})\leq\frac{1-\delta}{2}\{I_\Phi(f\chi_{T\setminus A})+(I_\Phi(g\chi_{T\setminus A})\}\leq 1-\delta.$$

Consequently, $I_\Phi(\frac{f-g}{a}\chi_A)\geq\delta$. By the Δ_2-condition, we get

$$I_\Phi(\frac{f-g}{a}\chi_A)\leq K\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\},$$

where K is a constant depending only on a and Φ. Hence,

$$\begin{aligned}1-I_\Phi(\frac{f+g}{2})&\geq(1/2)\{I_\Phi(f)+I_\Phi(g)\}-I_\Phi(\frac{f+g}{2})\\&\geq(1/2)\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}-I_\Phi(\frac{f+g}{2}\chi_A)\\&\geq(1/2)\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}-\frac{1-\delta}{2}\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}\\&=(\delta/2)\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}\geq\frac{\delta^2}{2K}\end{aligned}$$

Equivalently,

$$I_\Phi\left(\frac{f+g}{2}\right) \le 1 - \frac{\delta^2}{2K}.$$

Applying the Δ_2-condition, we get

$$\left\| \frac{f+g}{2} \right\|_\Phi \le 1 - p\left(\frac{\delta^2}{2K}\right),$$

where p is a function from (0,1) into itself (in the real case see [4] and [6]).

This yields $\delta_{L^\Phi}(a) \ge p\left(\frac{\delta^2}{2K}\right) > 0$.

1° ⇒2°. If Φ does not satisfy the Δ_2-condition, then $L^\Phi(\mu,X)$ contains an isometric copy of l^∞ (see [3] and [4]). Therefore, $\delta_{L^\Phi}(a) \le \delta_{l^\infty}(a) = 0$ for every $a \in (0,2]$. Assume now that condition 2°(a) is not satisfied, i.e. for every $\delta \in (0,1)$ there exist $x,y \in X$ such that

$$\Phi(1/a(x-y)) \ge (1-\delta)\Phi(x+y)/2) \text{ and } \Phi((x+y)/2) > 1/2(1-\delta)\{\Phi(x)+\Phi(y)\}.$$

Let $B,C \in \sum$, $B \subset C$, be such that $\mu(C) = \mu(B\backslash C)$ and $(\Phi(x)+\Phi(y))\mu(B) = 2$.

Define

$$f = x\chi_C + y\chi_{B\backslash C},\ g = y\chi_C + x\chi_{B\backslash C}$$

We have $I_\Phi(f) = I_\Phi(g) = 1$, whence $\|f\|_\Phi = \|g\|_\Phi = 1$. Moreover,

$$I_\Phi((f-g)/(1-\delta)^2 a) \ge (1/(1-\delta)^2) \int_B \Phi((x-y)/a)d\mu$$

$$\ge (1/(1\text{-}\delta)^2)\Phi((x-y)/a)\mu(B) \ge \left(1/(1-\delta)\right)\Phi((x+y)/2)\mu(B)$$

$$\ge (1/2)\{\Phi(x)+\Phi(y)\}\mu(B) = 1.$$

Therefore $\|(f-g)/a\|_\Phi \ge (1-\delta)^2$. In an analogous way the inequality $\|(f+g)/2\|_\Phi \ge 1-\delta$ can be proved. Since $\delta \in (0,1)$ was arbitrary, this means that $\delta_{L^\Phi}(a) = 0$. The theorem is proved.

To prove the next theorem, we will need the following

Proposition 4. *Let Φ be an Orlicz function satisfying the Δ_2-condition. Then the following assertions are equivalent:*

(+) *there is* $\delta \in (0,1)$ *such that* $\Phi((x+y)/2 \leq ((1-\delta)/2)\{\Phi(x)+\Phi(y)\}$ *whenever* $x,y \in X$ *satisfy* $\Phi((x-y)/a) \geq (1-\delta)\Phi((x+y)/2)$.

(++) *there is* $\sigma \in (0,1)$ *such that* $\Phi((x+y)/2) \leq ((1-\sigma)/2)\{\Phi(x)+\Phi(y)\}$ *whenever* $x,y \in X$ *satisfy* $\Phi((x-y)/a(1-\sigma)) \geq \Phi((x+y)/2)$.

Proof. (++)$\Rightarrow$(+). Assume that $\Phi((x-y)/a) \geq (1-\sigma)\Phi((x+y)/2)$. Then $\Phi((x-y)/a(1-\sigma)) \geq (1/(1-\sigma))\Phi((x-y)/a) \geq \Phi((x+y)/2)$. In view of condition (++), we have $\Phi((x+y)/2) \leq ((1-\sigma)/2)\{\Phi(x)+\Phi(y)\}$.
Thus, it suffices to put $\delta=\sigma$.

(+)$\Rightarrow$(++). Assume that $\Phi((x-y)/a(1-\sigma_1)) \geq \Phi((x+y)/2)$, where σ_1 is a constant in (0,1) satisfying $\Phi(x/(1-\sigma_1)) \leq (1/(1-\delta))\Phi(x)$ for every $x \in X$ (by the Δ_2-condition such a constant exists) and δ is the constant from condition (+). Then

$$\Phi((x-y)/a)(1/(1-\delta)) \geq \Phi((x-y)/a(1-\sigma_1)) \geq \Phi((x+y)/2).$$

Therefore, by condition (+), we get

$$\Phi((x+y)/2) \leq ((1-\delta)/2)\{\Phi(x)+\Phi(y)\}.$$

It suffices to put $\sigma=\min(\sigma_1, \delta)$.

Theorem 5. *Let* Φ *be a uniformly convex Orlicz function defined on the real line, i.e. for every* $a \in (0,1)$ *there exists* $\delta(a) \in (0,1)$ *such that for every* $u \in R$ *we have* $\Phi((u+au)/2) \leq (1/2)(1-\delta(a))\{\Phi(u)+\Phi(au)\}$, *and let* Φ *satisfy the* Δ_2*-condition and* $(X,\|\ \|)$ *be a Banach space. Then* $\delta_{L^\Phi}(\varepsilon)>0$ *for the Orlicz-Bochner space* $L^\Phi=L^\Phi(\mu,X)$ *if and only if* $\delta_X(\varepsilon)>0$.

Proof. Since X can be isometrically embedded into $L^\Phi(\mu,X)$, the condition $\delta_X(\varepsilon)=0$ implies $\delta_{L^\Phi}(\varepsilon)=0$.

Now, in view of Proposition 4 assume that $\delta_X(\varepsilon)>0$. It suffices to show that there exists a constant $\sigma \in (0,1)$ such that for every $x,y \in X$, we have

(1) $\left\| \frac{x-y}{\varepsilon\sigma} \right\| \geq \left\| \frac{x+y}{2} \right\|$ implies $\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \leq \frac{\sigma}{2}\{\Phi(\|x\|)+\Phi(\|y\|)\}$.

Since $\delta_X(\varepsilon)>0$ by the assumption, there exists $\delta \in (0,1)$ such that

(2) $\left\| \frac{x+y}{2} \right\| \leq \delta$ whenever $x,y \in B_X$ (=the unit ball of X) and

$$\left\| \frac{x-y}{\varepsilon\delta} \right\| \geq \left\| \frac{x+y}{2} \right\|.$$

Assume that $x,y\in X$ and $\left\| \frac{x-y}{\varepsilon\delta} \right\| \geq \left\| \frac{x+y}{2} \right\|$. We can assume without loss of generality that $\|x\|\leq\|y\|$. Then $\frac{x}{\|y\|}, \frac{y}{\|y\|} \in B_X$ and

$\left\| \frac{x-y}{\varepsilon\delta\|y\|} \right\| \geq \left\| \frac{x+y}{2\|y\|} \right\|$. Thus, in virtue of condition (2), we get

$\left\| \frac{x+y}{2} \right\| \leq \delta\|y\|$. Now, we shall consider two cases.

1°. $\sqrt{\delta}\|y\|\leq\|x\|$. Then

$$\left\| \frac{x+y}{2} \right\| \leq \delta\|y\| = \delta\,\frac{\|y\|+\|y\|}{2} \leq \delta\,\frac{\|x\|/\sqrt{\delta}+\|y\|}{2} \leq \delta\,\frac{\|x\|+\|y\|}{2\sqrt{\delta}}$$

$$= \frac{\sqrt{\delta}}{2}(\|x\|+\|y\|).$$

Hence

$$\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \leq \frac{\sqrt{\delta}}{2}\{\Phi(\|x\|)+\Phi(\|y\|)\}.$$

2°. $\|x\|<\sqrt{\delta}\|y\|$. By uniform convexity of Φ, we have

$$\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \leq \Phi\left(\frac{\|x\|+\|y\|}{2}\right) \leq \frac{1-\eta(\delta)}{2}\{\Phi(\|x\|)+\Phi(\|y\|)\}.$$

Therefore, for every $x,y\in X$ such that $\left\| \frac{x-y}{a\delta} \right\| \geq \left\| \frac{x+y}{2} \right\|$, we have

$$\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \leq \frac{\max((1-\eta(\delta)),\sqrt{\delta})}{2}\{\Phi(\|x\|)+\Phi(\|y\|)\}.$$

To prove condition (1) it suffices to put $\sigma = \max((1-\eta(\delta),\sqrt{\delta})$.

Note. The thesis of Theorem 5 means that $\varepsilon_o(L^{\Phi}(\mu,X)) = \varepsilon_o(x)$. It is a generalization of Theorem 9 of Downing and Turett [2]. However, our method of the proof is quite different than the method used there.

An Orlicz function Φ is said to satisfy condition C_a ($a\in(0,2)$) if there exists a number $\sigma\in(0,1)$ such that

$$\Phi((x+y)/2) \leq (\sigma/2)\{\Phi(x)+\Phi(y)\},$$

whenever $x,y \in X$ and $\Phi((x-y)/a\sigma) \geq \Phi((x+y)/2)$.

For any Orlicz function Φ we define the parameter

$$\alpha(\Phi) = \inf\{a \in (0,2): \Phi \text{ satisfies condition } C_a\}.$$

We shall give now an inmediate consequence of Th. 3 and Prop. 4.

Corollary 6. *Let Φ be an Orlicz function. Then $\varepsilon_o(L^\Phi(\mu,X)) = 2$ whenever Φ does not satisfy the Δ_2-condition and $\varepsilon_o(L^\Phi(\mu,X)) = \alpha(\Phi)$ in the opposite case.*

Note. Theorem 3 and Corollary 6 generalize the results of [6] to Orlicz spaces of vector valued functions. Theorem 3 generalizes also some results of [4], [5] and [7]. These results are also connected with the results of [8], [9] and [10].

Corollary 7. *Let Φ be an Orlicz function defined on the real line R and $(X,\| \ \|)$ be a Banach space. Then the Orlicz-Bochner space $L^\Phi(\mu,X)$ is uniformly rotund if and only if both spaces $L^\Phi(\mu,R)$ and X are uniformly rotund.*

Recall that $L^\Phi(\mu,R)$ is uniformly rotund if and only if Φ is uniformly convex and satisifes the Δ_2-condition (see [7]).

Problem. Is the equality $\varepsilon_o(L^\Phi(\mu,X)) = max(\varepsilon_o(L^\Phi(\mu, R), \varepsilon_o(X))$ true for every Orlicz-Bochner space?

Added in proof. The problem has negative answer. We refer to the paper of the author and T. Landes entitled *"Characteristic of convexity of Köthe function spaces"* (preprint).

References

[1] S. CHEN and H. HUDZIK, *On some convexities of Orlicz and Orlicz-Bochner spaces,* Comment. Math. Univ. Carol. 29, No.1(1988), 13-29.

[2] D. J. DOWING and B. TURETT, *Some properties of the characteristic of convexity relating to fixed point theory,* Pacific J. Math. 104, No. 2 (1983), 343-350.

[3] H. HUDZIK, *On some equivalent conditions in Musielak-Orlicz spaces,* Comment. Math. 24 (1984), 57-64.

[4] H. HUDZIK, *Some class of uniformly non-square Orlicz-Bochner spaces,* Comment. Math. Univ. Carol. 26, No.2(1985), 269-274.

[5] H. HUDIZK, *Convexity in Musielak-Orlicz spaces,* Hokkaido Math. J. 14(1985), 85-96.

[6] H. HUDZIK, A. KAMINSKA and J. MUSIELAK, *On the convexity coefficient of Orlicz spaces,* Math. Zeit. 197 (1988), 291-295.

[7] A. KAMINSKA, *On uniform convexity of Orlicz spaces,* Indag. Math. 44 (1982), 27-36.

[8] A. KAMINSKA and B. TURETT, *Uniformly non-$l_n^{(1)}$ Orlicz-Bochner spaces,* Bull. Acad. Polon. Sci. Math. 35, No. 3-4 (1987), 211-218.

[9] M. A. KHAMSI, W.M. KOZLOWSKI and CH. SHUTAO, *Some geometrical properties and fixed point theorems in modular spaces* (to appear).

[10] M. A. KHAMSI, W.M. KOZLOWSKI and S. REICH, *Fixed point theory in modular function spaces* (to appear).

[11] M. A. KRASNOSELSKII and YA. B. RUTICKII, *Convex functions and Orlicz spaces,* Groningen 1961 (translation).

[12] W. A. J. LUXEMBURG, *Banach function spaces,* Thesis, Delft 1955.

[13] J. MUSIELAK, *Orlicz spaces and modular spaces,* Lectures Notes in Math. 1034, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1983).

[14] B. TURETT, *Rotundity of Orlicz spaces,* Indag. Math. 38 (1976), 462-469.

[15] B. TURETT, *Fenchel Orlicz spaces,* Dissertat. Math. 131 (1980), 1-60.

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