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Real interpolation and compactness

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ABSTRACT. The behaviour of compactness under real interpolation is discussed. Classical results due to Krasnosel'skii, Lions-Peetre, Persson and Hayakawa are described, as well as others obtained very recently by Edmunds, Potter, Fernández and the author.

If we have (for example) an integral operator

$$T_K f(x) = \int_\Omega K(x,y) f(y) d\mu(y)$$

usually the function space where the operator is defined is not uniquely established by given conditions. Often investigated is the operator acting between several function spaces. For this reason, it is important to have results which give relationships between properties of a given operator considered in two different spaces. The celebrated Riesz-Thorin theorem is a non-trivial example of such a result. Let us recall its statement.

Let (Ω,μ) be a measure space, with μ a positive measure, and let L_p $(1 \leqq p \leqq \infty)$ denote the space of all (equivalence classes of) μ-measurable functions f on Ω, such that

$$\|f\|_p = \left(\int_\Omega |f(x)|^p d\mu\right)^{1/p}$$

is finite.

Riesz-Thorin Theorem. *Assume that $p_0 \neq p_1$, $q_0 \neq q_1$, and let T be a linear operator which maps L_{p_j} continuously into L_{q_j} $(j = 0,1)$, then T maps L_p continuously into L_q, where p and q are given by*

$$1/p = (1-\theta)/p_0 + \theta/p_1,\ 1/q = (1-\theta)/q_0 + \theta/q_1 \quad (0 < \theta < 1)$$

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This result was proved in 1926 by M. Riesz with the restriction $p \leqq q$. Later on, Thorin in 1938 gave an entirely new proof removing the restriction $p \leqq q$ (see [1] and [13]).

In connection with this theorem, a natural question arises: whether or not the compactness property of a linear operator can also be interpolated.

In 1960 this problem was solved affirmatively by Krasnosel'skii [7]. He proved that under the hypothesis of the Riesz-Thorin theorem, if $q_0 < \infty$ and $T:L_{p_0} \to L_{q_0}$ is a compact mapping, then $T:L_p \to L_q$ is also compact.

Another main result in interpolation theory is the Marcinkiewicz interpolation theorem. Before stating this, let me recall that

$$L_{p,\infty} = \{f : \|f\|_{p,\infty} = \sup_{t>0} \{t^{1/p} f^*(t)\} < \infty\}$$

where f^* is the non-increasing rearrangement of f,

$$f^*(t) = \inf\{\delta : \mu(\{x : |f(x)| > \delta\}) \leqq t\}.$$

Marcinkiewicz Theorem. *Assume that $p_0 \neq p_1$, $q_0 \neq q_1$, and let T be a linear operator which maps L_{p_j} continuously into $L_{q_j,\infty}(j=0,1)$. Let $0 < \theta < 1$ and put $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$. If $p \leqq q$, then T maps L_p continuously into L_q.*

This result appeared in a note by Marcinkiewicz in 1939, without proof. In 1956 Zygmund gave a proof and also applications of the theorem, which cannot be obtained by the Riez-Thorin result. Note that $L_q \hookrightarrow L_{q,\infty}$ (see [1] and [13]).

The study of abstract interpolation theory started in the early 1960's. it was motivated by questions connected with partial differential equations. Two main methods were developed, the complex method (associated with the Riesz-Thorin theorem) and the real method (connected with the Marcinkiewicz theorem). Let me recall the definition of the real interpolation space.

Let (A_0,A_1) be a compatible couple of Banach spaces (meaning that they are continuously embedded in a topological vector space $\mathcal{A}$). Then we can form their sum $A_0 + A_1$ and their intersection $A_0 \cap A_1$. The sum consists of all $x \in \mathcal{A}$ such that we can write $x = x_0 + x_1$ for some $x_0 \in A_0$ and $x_1 \in A_1$.

The intersection and the sum are Banach spaces endowed with the following norms

$$\|x\|_{A_0\cap A_1} = max\{\|x\|_{A_0}, \|x\|_{A_1}\}$$

$$\|x\|_{A_0+A_1} = inf\{\|x_0\|_{A_0} + \|x_1\|_{A_1} : x = x_0 + x_1\}$$

We want to construct a Banach space $\bar{A}_{\theta,q}$ between the intersection and the sum, and which has some properties in common with A_0 and A_1. In this respect, let

$$K(e^m,x) = inf\{\|x_0\|_{A_0} + e^m\|x_1\|_{A_1} : x = x_0 + x_1\},\ m \in \mathbb{Z}$$

(note that for each m this functional is an equivalent norm to that defined in $A_0 + A_1$). Next, for $1 \leqq q \leqq \infty$ and $0 < \theta < 1$, we put

$$\bar{A}_{\theta,q} = \{x \in A_0+A_1 : \|x\|_{\theta,q} = (\sum_{m=-\infty}^{\infty} (e^{-\theta m}K(e^m,x))^q)^{1/q} < \infty\}$$

It turns out that we obtain such a Banach space, which also has the interpolation property:

If T is a linear operator, which maps A_j continuously into $B_j(j = 0,1)$ [(A_0,A_1) and (B_0,B_1) being compatible couples of Banach spaces] then

$$T : \bar{A}_{\theta,q} \to \bar{B}_{\theta,q}$$

is also bounded (see [1] and [13] for details on this method).

As an example, let us mention that the following formulae hold

$$(L_{p_0},L_{p_1})_{\theta,p} = L_p,\ (L_{q_0,\infty},L_{q_1,\infty})_{\theta,q} = L_q$$

(equivalent norms)

where $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$.

Let us focus our attention on the behaviour of compactness under this interpolation method. In other words, let us consider the problem of whether or not Krasnosel'skii's result can be extended to this abstract framework.

The first result in this direction appeared in 1964. It is contained in the famous paper by Lions and Peetre [9] on the real interpolation method.

Lions-Peetre Theorem. *Let (A_0,A_1) and (B_0,B_1) be compatible couples of Banach spaces, and let T be a linear operator such that $T:A_0 \to B_0$ is compact and $T:A_1 \to B_1$ is continuous.*

i) *If* $B_0 = B_1$ *then* $T:\bar{A}_{\theta,q} \to B_0$ *is compact.*

ii) *If* $A_0 = A_1$ *then* $T:A_0 \to \bar{B}_{\theta,q}$ *is compact.*

In fact, they proved a more general result covering all interpolation functors F such that for every Banach couple (A_0,A_1), the space $F(A_0,A_1)$ is a Banach space of class $C(\theta,\bar{A})$ (see [1]). In particular, the result is also true for the complex method.

Note that in this result we always have a degenerate couple. An abstract general result for the case $A_0 \neq A_1$ and $B_0 \neq B_1$ was established in 1964 by A. Persson [12]. In the proof he used the Lions-Peetre theorem and required the following approximation property on the last couple:

There exists a set $\mathcal{P}$ of linear operators $P:B_0 + B_1 \to B_0 + B_1$ and a constant $C > 0$ such that

1) $P(B_j) \subset B_0 \cap B_1$ $(j = 0,1)$.

2) $\|P\|_{\mathcal{L}(B_j,B_j)} \leqq C$ $(j = 0,1)$ for all $P \in \mathcal{P}$.

3) For every $\varepsilon > 0$ and every finite set $\{b_1,\ldots,b_N\} \subset B_0$, there is a $P \in \mathcal{P}$ so that

$$\|Pb_k - b_k\|_{B_0} < \varepsilon \quad (k = 1,\ldots,N).$$

An approximation condition of this kind was also used by Krein-Petunin [8] to prove a compactness theorem between scales of Banach spaces. In fact, as early as Krasnosel'skii's paper the sequence of partial sum operators associated to the Haar basis was used to derive the compactness result. Note that this sequence has the properties (1) – (3).

Again Persson's result is true for the same class of interpolation functors F that we mentioned before.

So far as we are aware the only result for the general case without an approximation hypothesis is that given by Hayakawa [5] in 1969. He states that if T is a linear operator such that the restrictions $T:A_0 \to B_0$ and $T:A_1 \to B_1$ are compact, then $T:\bar{A}_{\theta,q} \to \bar{B}_{\theta,q}$ is compact for all θ and q with $0 < \theta < 1$ and $1 \leqq q < \infty$. Unfortunately his arguments are extremely difficult to follow.

A transparent proof of this result has been given very recently by Edmunds, Potter and the author [2]. The approach developed therein enables us to include the cases $0 < q < 1$ and $q = \infty$ which were not considered by Hayakawa.

Note that if $0 < q < 1$, $\bar{A}_{\theta,q}$ still makes sense and has the interpolation property. The functional $\| \|_{\theta,q}$ is no longer a norm since it does not satisfy the triangle inequality. The funtional $\| \|_{\theta,q}$ is only a quasi-norm.

Let me state our result and describe the main ideas of the proof. It only works for the real method.

Theorem 1. *Let* (A_0,A_1) and (B_0,B_1) *be compatible couples of Banach spaces and let T be a linear operator such that* $T:A_0\to B_0$ *and* $T:A_1\to B_1$ *are compact. Then if* $0 < q \leqq \infty$ *and* $0 < \theta < 1$, $T:\bar{A}_{\theta,q}\to \bar{B}_{\theta,q}$ *is also compact.*

Sketch of the proof. (full details can be found in [2]). Let $\underline{A}_j$ be the closure of $A_0 \cap A_1$ in A_j $(j = 0,1)$ and define $\underline{B}_j$ similarly. First note that the restrictions $T:\underline{A}_0\to \underline{B}_0$ and $T:\underline{A}_1\to \underline{B}_1$ are compact.

Next we are going to embed these last spaces in vector-valued sequences spaces. With this aim, put

$$D_m = [B_0+B_1,K(e^m,\cdot\ )],\ e^{-\theta m}D_m = [B_0+B_1,e^{-\theta m}K(e^m,\cdot)]$$

(here $m \in \mathbb{Z}$ and $0 \leqq \theta \leqq 1$), and consider the following Banach spaces

$$C_0^-(D_m) = \{(x_m) : x_m \in D_m,\ \|(x_m)\|_{C_0^-} = \sup_m K(e^m,x_m) < \infty \text{ and } \lim_{m\to-\infty} K(e^m,x_m) = 0\}$$

$$C_0^+(e^{-m}D_m) = \{(x_m) : x_m \in e^{-m}D_m,\ \|(x_m)\|_{C_0^+} = \sup_m e^{-m}K(e^m,x_m) < \infty \text{ and } \lim_{m\to\infty} ee^{-m}K(e^m,x_m) = 0\}.$$

It is not hard to check that the map

$$j : x \to (\dots,x,x,x,\dots)$$

is a continuous embedding from $\underline{B}_0$ into $C_0^-(D_m)$ and from $\underline{B}_1$ into $C_0^+(e^{-m}D_m)$. Hence, calling

$$\hat{T}x = j(Tx) = (\dots,Tx,Tx,Tx,\dots)$$

we have that

$$\hat{T} : \underline{A}_0\to C_0^-(D_m) \text{ and } \hat{T} : \underline{A}_1\to C_0^+(e^{-m}D_m)$$

are compact.

The next step is to show that the bounded operator

(*) $\hat{T} : (\underline{A}_0,\underline{A}_1)_{\theta,q} \to (C_0^-(D_m), C_0^+(e^{-m}D_m))_{\theta,q}$

is also compact. For this purpose, let us consider the following families of mappings between the sequence spaces. Given $(x_m) \in C_0^-(D_m) + C_0^+(e^{-m}D_m)$ and $n \in \mathbb{N}$, put

$$P_n(x_m) = (\ldots,0,0,x_{-n+1},x_{-n+2},\ldots,x_0,\ldots, x_{n-2},x_{n-1},0,0,\ldots)$$

$$P_+(x_m) = (\ldots,0,0,x_0,x_1,\ldots,x_n,x_{n+1},\ldots)$$

$$P_-(x_m) = (\ldots,x_{-n-1},x_{-n},\ldots,x_{-1},0,0,\ldots)$$

$$R_n(x_m) = (P_-P_n+P_+)(x_m) = (\ldots,0,0,x_{-n+1},x_{-n+2},x_{-n+3},\ldots).$$

We shall see that $\hat{T}$ is the limit of the sequence $(P_n\hat{T})_{n=1}^{\infty}$ and later on that this sequence is formed by compact operators.

We have

$$\begin{aligned}\|P_n\hat{T}-\hat{T}\|_{\theta,q} &= \|P_-(P_n\hat{T}-\hat{T}) + P_+(P_n\hat{T}-\hat{T})\|_{\theta,q} \\ &\leqq C(\|P_-(P_n\hat{T}-\hat{T})\|_{\theta,q} + \|P_+(P_n\hat{T}-\hat{T})\|_{\theta,q})\end{aligned}$$

where C is the constant in the quasi-triangle inequality for $\| \;\|_{\theta,q}$; $C = 1$ if $1 \leqq q \leqq \infty$. Let us fix our attention on $\|P_-(P_n\hat{T}-\hat{T})\|_{\theta,q}$. The interpolation property implies that

$$\begin{aligned}\|P_-(P_n\hat{T}-\hat{T})\|_{\theta,q} &\leqq \|P_-(P_n\hat{T}-\hat{T})\|_0^{1-\theta}\|P_-(P_n\hat{T}-\hat{T})\|_1^{\theta} \\ &\leqq \|\hat{T}\|_1^{\theta}\|P_-(P_n\hat{T}-\hat{T})\|_0^{1-\theta} \\ &= \|\hat{T}\|_1^{\theta}\|R_n\hat{T}-\hat{T}\|_0^{1-\theta}.\end{aligned}$$

From the fact that $\hat{T} : \underline{A}_0 \to C_0^-(D_m)$ is compact and that

$$\lim_{n\to\infty} \|R_n(x_m) - (x_m)\|_{C_0^-} = 0$$

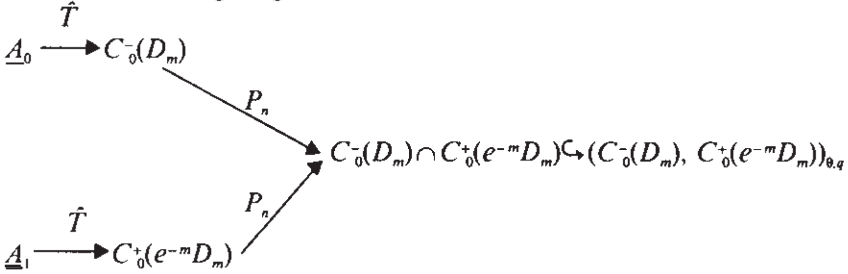
it follows that $\|R_n\hat{T}-\hat{T}\|_0^{1-\theta} \to 0$ as $n\to\infty$, and consequently $\|P_-(P_n\hat{T}-\hat{T})\|_{\theta,q} \to 0$ as $n\to\infty$.

With a similar reasoning, but now using the fact that $\hat{T} : \underline{A}_1 \to C_0^+(e^{-m}D_m)$ is compact, we obtain that

$$\lim_{n\to\infty} \|P_+(P_n\hat{T}-\hat{T})\|_{\theta,q} = 0.$$

Whence $\hat{T}$ is the limit of the sequence $(P_n\hat{T})$ in the operator norm.

Let us see now that the sequence $(P_n\hat{T})$ is formed by compact operators. We have the following diagram



Thus, the Lions-Peetre result implies that

$$P_n\hat{T} : (\underline{A}_0,\underline{A}_1)_{\theta,q} \to (C_0^-(D_m),\ C_0^+(e^{-m}D_m))_{\theta,q}$$

is compact.

This proves (*). To complete the proof, we only need to identify the interpolated spaces.

It is easy to see that

$$(\underline{A}_0,\underline{A}_1)_{\theta,q} = (A_0,A_1)_{\theta,q}.$$

On the other hand, it is shown in [2], Lemma 2.1, that

$$(C_0^-(D_m),\ C_0^+(e^{-m}D_m))_{\theta,q} \hookrightarrow l_q(e^{-\theta m}D_m).$$

This finishes the proof. □

These techniques also allow us to show that the same conclusion holds when the assumption

$$T : A_1 \to B_1 \text{ compactly}$$

is replaced by

$$B_1 \text{ continuously embedded in } B_0$$

(see [2], Theorem 3.2), Note that this last result is a natural extension of the Lions-Peetre theorem (i).

Now the question is whether or not the corresponding extension of part (ii) in the Lions-Peetre theorem also holds. This problem has been solved in the affirmative in a paper by Fernández and the author [3]. To be precise, we have established.

Theorem 2. *Let* (B_0,B_1) *be a compatible couple of Banach spaces and suppose that* A_0,A_1 *are Banach spaces such that* A_1 *is continuously embedded in* A_0. *Let* T *be a linear operator such that* $T: A_0 \to B_0$ *is bounded and* $T: A_1 \to B_1$ *is compact. Then if* $0 < \theta < 1$ *and* $0 < q \leqq \infty$, $T: \bar{A}_{\theta,q} \to \bar{B}_{\theta,q}$ *is compact.*

The proof of this result is based on the description of the real interpolation space through the J-functional

$$J(e^m,x) = max\{\|x\|_{A_0}, e^m\|x\|_{A_1}\},\ x \in A_0 \cap A_1,\ m \in \mathbb{Z}.$$

Let me recall that $x \in \bar{A}_{\theta,q}$ if and only if there exists a sequence $(u_m) \subset A_0 \cap A_1$ with

$$(4) \qquad x = \sum_{m=-\infty}^{\infty} u_m \qquad \text{(convergence in } A_0+A_1)$$

and

$$(5) \qquad |||(u_m)|||_{\theta,q} = \Big(\sum_{m=-\infty}^{\infty} (e^{-\theta m} J(e^m,u_m))^q \Big)^{1/q} < \infty.$$

Moreover

$$\|x\|_{\theta,q} \text{ is equivalent to } \inf\{|||(u_m)|||_{\theta,q}\}$$

where the infimum is extended over all sequences (u_m) satisfying (4) and (5).

Instead of the C_0 spaces modelled on the sum B_0+B_1 that we have used before, we now need vector-valued l_1 spaces modelled on the intersection $A_0 \cap A_1$ (see [3], Theorem 2.1).

The procedures we used in Theorems 1 and 2 also work for the (more general) method of interpolation with a function parameter (see [2], Theorem 3.3). We refer to [11], [4] and [10] for details on this method. Using it one can obtain certain Orlicz spaces as interpolation spaces between L_p-spaces.

Finally, let us mention that in [2] one can find applications of our interpolation results to show that certain integral operators are compact. In particular, we derive a theorem of this kind due to Kantorovich [6].

References

[1] J. BERGH and J. LÖFSTRÖM, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York,1976.

[2] F. COBOS, D.E. EDMUNDS and A.J.B. POTTER, *Real Interpolation and compact linear operators*, J. Funct. Anal. (to appear).

[3] F. COBOS and D.L. FERNÁNDEZ, *On Interpolation of compact operators*, Ark. Mat. (to appear).

[4] J. GUSTAVSSON, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. 42 (1978), 289-305.

[5] K. HAYAKAWA, *Interpolation by the real method preserves compactness of operators*, J. Math. Soc. Japan 21 (1969), 189-199.

[6] L.V. KANTOROVICH and G.P. AKILOV, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[7] M.A. KRASNOSEL'SKII, *On a theorem of M. Riesz*, Soviet Math. Dokl. 1 (1960), 229-231.

[8] S.G. KREIN and JU. I. PETUNIN, *Scales of Banach spaces*, Russian Math. Surveys 21 (1966), 85-159.

[9] J.L. LIONS and J. PEETRE, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. 19 (1964), 5-68.

[10] C. MERUCCI, *Applications of interpolation with a function parameter in Lorentz, Sobolev and Besov spaces, in "Interpolation Spaces and Allied Topics in Analysis"*, Springer Lect. Notes 1070, Berlin 1984, pp. 183-201.

[11] J. PEETRE, *A Theory of Interpolation of Normed spaces*, Lecture notes, Brasilia 1963 [Notas de matematica 39 (1968) 1-86].

[12] A. PERSSON, *Compact linear mappings between interpolation spaces*, Ark. Mat. 5(1964), 215-219.

[13] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

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