

Embedding l_∞^n -cubes in finite-dimensional 1-subsymmetric spaces

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ABSTRACT. In this paper we prove that the l_∞^n -cube can be $(1 + \varepsilon)$ -embedded into any 1-subsymmetric $C(\varepsilon)n$ -dimensional normed space.

Marcus and Pisier in [5] initiated the study of the geometry of finite metric spaces. Bourgain, Milman and Wolfson introduced a new notion of metric type and developed the non-linear theory of Banach spaces (see [2] and [7]). All these themes have been studied more intensively over the last years.

Johnson and Lindenstrauss proved that, given N points in the Euclidean space, they can be $(1 + \varepsilon)$ -embedded into a subspace of dimension $K(\varepsilon) \log N$ (see lemma 1 in [3]). The method they use is based in the isoperimetric inequality of P. Levy. Another proof of the same fact was given by Pisier, using Gaussian processes ([8]). Bourgain, Milman and Wolfson, in the paper before mentioned, studied the l_p^n -cubes and their $(1 + \varepsilon)$ -embeddings in finite metric spaces. More recently, Schechtman obtained estimates for $(1 + \varepsilon)$ -embeddings of finite subsets of L^r into l_p^n -spaces (see [9]).

In this paper we will study $(1 + \varepsilon)$ -embeddings of the l_∞^n -cube in finite-dimensional subsymmetric spaces. The result we prove for the l_p -case $1 \leq p \leq 2$, can be deduced from Johnson and Lindenstrauss's lemma plus a refinement of Dvoretzky's theorem (see for instance [7], Theorem 3.9), but, as far as we know, it is new in other cases. The method we use is in essence of probabilistic nature and the main tool is a well known deviation inequality.

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We begin by recalling some definitions. Given two metric spaces (M, d) and (M', d') , we say that (M, d) $(1 + \varepsilon)$ -embeds into (M', d') if there is a one-to-one map f from M into M' such that $\|f\|_{Lip} \|f^{-1}\|_{Lip} \leq 1 + \varepsilon$, where

$$\|f\|_{Lip} = \sup_{x \neq y} \frac{d'(f(x), f(y))}{d(x, y)}$$

The l_∞^n -cube is the metric space (C_2^n, ρ_∞) where $C_2^n = \{-1, +1\}^n$ and $\rho_\infty(\varepsilon, \varepsilon') = \max_{1 \leq i \leq n} |\varepsilon_i - \varepsilon'_i|$, for any pair of elements $\varepsilon, \varepsilon'$ belonging to C_2^n .

Since $\rho_\infty(\varepsilon, \varepsilon') = 2$, whenever $\varepsilon \neq \varepsilon'$, the problem we are considering may be related with the sphere-packing problem, i.e., how many balls, with radius $\frac{1 - \varepsilon}{2}$, can be packed into the unit ball of a finite dimensional Banach space, in an asymptotic way? (See the paper by Ball [1] for infinite dimensional sphere-packing problem)

In the sequel E_n will denote a finite-dimensional Banach space with a 1-subsymmetric normalized basis $\{e_1, \dots, e_n\}$. We use standard Banach space theory notation as may be found in [4].

The theorem we will prove here is the following

Theorem.—*There exists a numerical constant $C > 0$ such that, for any $\varepsilon > 0$ we can find a subset of N points $\{x_1, \dots, x_N\}$ in E_n verifying*

$$1 - \varepsilon \leq \|x_i - x_j\| \leq 1 + \varepsilon, \quad i \neq j$$

provided that

$$n > \frac{C}{\varepsilon^2} \log N$$

Proof.— Let ε a given positive number verifying $0 < \varepsilon < 1$. Let n be a natural number to be determined after. Consider the function ψ defined by

$$\psi\left(\frac{m}{n}\right) = \frac{\left\| \sum_{i=1}^m e_i \right\|}{\left\| \sum_{i=1}^n e_i \right\|}, \quad \text{if } 0 \leq m \leq n,$$

and by a nondecreasing continuous extension in the other points of the unit interval $[0, 1]$. The function ψ depends on n , but in some particular cases we can choose the same fixed function for all n . This happens, for instance, in the l^p -spaces where we may define $\psi(t) = t^{1/p}$, $0 \leq t \leq 1$.

We note that function ψ verifies $\psi(0) = 0, \psi(1) = 1$ and

$$\psi(2^{-j}) \geq 2^{-j+1}, j = 0, 1, \tag{*}$$

Indeed, if $\frac{m}{n} \leq \frac{1}{2^j} \leq \frac{m+1}{n}$ we have

$$\| \sum_1^n e_i \| \leq \| \sum_1^{2^{j(m+1)}} e_i \| \leq 2 \| \sum_1^{m+1} e_i \| \leq 2^{j+1} \| \sum_1^m e_i \|$$

In general we don't know the behaviour of the derivative of ψ in $[0,1]$, but, by averaging in the interval $[1/4, 1/2]$, given $\delta = \epsilon/128$

$$\int_{1/4}^{1/2} [\psi(t+\delta) - \psi(t-\delta)] dt = \int_{1/4+\delta}^{1/2+\delta} \psi(t) dt - \int_{1/4-\delta}^{1/2-\delta} \psi(t) dt \leq \int_{1/2-\delta}^{1/2+\delta} \psi \leq 2\delta$$

and then, we can pick a number a in the interval $(1/4, 1/2)$ such that $\psi(a+\delta) - \psi(a-\delta) \leq 8\delta$. Hence, for every $x, y \in [a-\delta, a+\delta]$, we have

$$|\psi(x) - \psi(y)| \leq 8\delta = \epsilon/16. \tag{**}$$

Let k be the integer part of $2an$, ($k \leq 2an < k+1$). Then, by (*)

$$\psi\left(\frac{k}{2n}\right) \geq \psi\left(\frac{1}{8}\right) \geq \frac{1}{16} \quad \text{if } n \geq 4 \tag{***}$$

We now define X a random E_n -valued vector by $X(\omega) = \sum_1^k \epsilon_i(\omega)e_i$, where $\{\epsilon_i\}$ is an i.i.d. sequence of symmetric $\{+1, -1\}$ -valued random variables defined in some probability space. If Y is another i.i.d. copy of X , it is clear that the two random variables $\|X - Y\|$ and $2\|\sum_1^k \eta_i e_i\|$ (where $\{\eta_i\}$ is an i.i.d. sequence of random variables uniformly distributed on the set $\{0,1\}$) have the same distribution. Then, if we denote $\lambda(n) = \|\sum_1^n e_i\|$, the 1-subsymmetry of the norm implies that the distribution of the random variables $\psi\left(\frac{1}{n} \sum_1^k \eta_i\right)$ and $\|\frac{1}{2\lambda(n)}(X - Y)\|$ also coincides.

Since $E\left(\frac{1}{n}\sum_{i=1}^k \eta_i\right) = \frac{k}{2n}$ we will compute the probability of deviation of $\left\|\frac{1}{2\lambda(n)}(X - Y)\right\|$ from $\psi\left(\frac{k}{2n}\right)$.

$$\begin{aligned} A &= P\left\{\omega; \left|\left\|\frac{1}{2\lambda(n)}(X - Y)\right\| - \psi\left(\frac{k}{2n}\right)\right| > \varepsilon \psi\left(\frac{k}{2n}\right)\right\} \leq_{(n \geq n)} \\ &\leq P\left\{\omega; \left|\psi\left(\frac{1}{n}\sum_{i=1}^k \eta_i\right) - \psi\left(\frac{k}{2n}\right)\right| > \varepsilon \frac{1}{16}\right\} \text{ by (***)}. \end{aligned}$$

Note that $a - \frac{1}{2n} \leq \frac{k}{2n}$, and so $\frac{k}{2n} \in [a - \frac{\delta}{2}, a + \frac{\delta}{2}]$ if

$n > \frac{128}{\varepsilon}$. Thus, $\left|\frac{1}{n}\sum_{i=1}^k \eta_i - \frac{k}{2n}\right| \leq \frac{\delta}{2}$ implies

$$\left|\psi\left(\frac{1}{n}\sum_{i=1}^k \eta_i\right) - \psi\left(\frac{k}{2n}\right)\right| < \varepsilon \frac{1}{16} \text{ by (**).}$$

Since

$$\frac{1}{n}\sum_{i=1}^k \eta_i - \frac{k}{2n} = \frac{1}{2n}\sum_{i=1}^k \varepsilon_i$$

we have

$$\begin{aligned} A &\leq P\left\{\omega; \left|\psi\left(\frac{1}{n}\sum_{i=1}^k \eta_i\right) - \psi\left(\frac{k}{2n}\right)\right| > \varepsilon \frac{1}{16}\right\} \leq \\ &\leq P\left\{\omega; \left|\frac{1}{n}\sum_{i=1}^k \varepsilon_i(\omega)\right| > \varepsilon \frac{1}{128}\right\} \leq 2 \exp\left(-\frac{\varepsilon^2 n^2}{Ck}\right) \leq \\ &\leq 2 \exp\left(-\frac{\varepsilon^2 n}{C}\right) \end{aligned}$$

where C is a numerical constant. In this last step we have used the well known probabilistic deviation inequality,

$$P\left\{\omega; \sum_{i=1}^m \varepsilon_i(\omega) > \lambda\sqrt{m}\right\} \leq \exp\left(-\frac{\lambda^2}{2}\right) \quad \lambda > 0, m \in \mathbb{N}$$

(see, for instance, [6] Theorem III.15).

Consider now a natural number N such that $n > \frac{2C}{\varepsilon^2} \log N$. If $\{X_i\}_i^N$ is an

i.i.d. sequence of copies of X , then

$$P\{\omega; \|\frac{1}{2\lambda(n)} (X_i - X_j)\| - \psi(\frac{k}{2n}) \leq \epsilon \psi(\frac{k}{2n}), \text{ for all } i \neq j\} \geq \\ \geq 1 - \binom{N}{2} 2 \exp(-\frac{\epsilon^2 n}{2C}) > 0$$

Hence, there exists ω in the probability space, such that the corresponding points

$$x_i = \frac{X_i(\omega)}{2\lambda(n)\psi(\frac{k}{2n})} \quad 1 \leq i \leq N$$

satisfy the conclusion of the theorem.

Corollary.- *The l_∞^n -cube is $(1 + \epsilon)$ -embedded in any finite-dimensional 1-subsymmetric space E , provided that $\dim E > \frac{C}{\epsilon^2} n$. (C is an absolute constant)*

Remarks.-

i) Since

$$\|\sum_1^k e_i\| \leq 2 \|\sum_1^{\lfloor k/2 \rfloor} e_i\| + 1$$

it is easy to prove that $\|x_i\| \leq \frac{3}{2} \quad 1 \leq i \leq N$.

ii) The asymptotic estimate $n > K \log N$ is essentially best possible. Indeed, in a ball of radius r of E_n , the number N of balls of radius $r/2$ we can pack into (with disjoint interior) is given by

$$r^n \text{vol}(B_1) \geq N (\frac{r}{2})^n \text{vol}(B_1)$$

($\text{vol}(B_1)$ is the n -dimensional volume of the unit ball)

iii) When $E = l_p$, $1 \leq p < \infty$, we can improve slightly the numerical constant. Indeed, by taking $a = 1/2$ and using the mean value theorem we obtain the following:

$$\text{a) If } \varepsilon < \frac{2}{p2^{1/p}} \text{ then } n > \frac{C}{\varepsilon^2 p^2} \log N$$

$$\text{b) If } \varepsilon > \frac{1}{p2^{1/p}} \text{ then } n > C \log N$$

(C is a numerical constant). These expressions say that $p = \infty$ is the best possible situation, because, an isometric embedding ($\varepsilon=0$) is possible in this case.

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