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Analytic Functions on c_0

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ABSTRACT. Let F be a space of continuous complex valued functions on a subset of c_0 which contains the standard unit vector basis $\{e_n\}$. Let $R:F\to C^N$ be the restriction map, given by $R(f)=(f(e_1),...,f(e_n),...)$. We characterize the ranges $R(F)$ for various "nice" spaces F. For example, if $F=P(^nc_0)$, then $R(F)=l_1$, and if $F=A^\infty(B(c_0))$, then $R(F)=l_\infty$,

Let c_0 be the Banach space of complex null sequences $\vec{x}=(x_n)$, with the normal *sup*-norm and usual basis vectors $\vec{e}_n=(0,..., 0,1,0,...)$, and let F be a space of continuous complex-valued functions on some subset of c_0 which contains the standard basis of c_0. Let $R:F\to C^N$ be the mapping which assigns to each function $f\in F$ the sequence $(f(e_1),...,f(e_n),...)$. Our attention in this article will be focussed on characterizing the range of R for various spaces F of interest. For example, if $F=C(c_0)$, the space of all continuous complex valued functions on c_0, then a trivial application of the Tietze extension theorem shows that $R(F)=C^N$. On the other hand, c_0 is weakly normal (Corson [6], see also Ferrera [9]). Since $\{0\}\cup\{e_n:n\in N\}$ is weakly compact, we see that $R(F)=c$, the space of convergent sequences, if we take F to be the subspace of $C(c_0)$ consisting of weakly continuous functions. Recently Jaramillo [11] has examined the relationship between reflexivity of the space F and the range of R, for certain spaces of *real* valued infinitely differentiable functions and polynomials on a Banach space E with unconditional basis $\{e_n ; n\in N\}$.

We concentrate here on analogous spaces of *complex* valued functions on c_0. After a review of relevant notation and definitions, we show in Section 1 that $R(F)=l_1$ when $F=P(^nc_0)$, $n\in N$. As a consequence, we prove that if $F=\{f\in H_b(B_R(c_0)): f(0)=0\}$, then $R(F)=l_1$. Taking $n=2$ in the above result, we see that every 2-homogeneous polynomial P on c_0 satisfies $\sum_{j=1}^\infty|P(e_j)|<\infty$. This result is reminiscent of classical work of Littlewood [13], who proved that every continuous bilinear form A on $c_0\times c_0$ satisfies $(A(e_j,e_k))_{j,k=1}^\infty\in l_{4/3}$. Littlewood's work was extended by Davie [7],who showed

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that every continuous n-linear form $A:c_0\times\ldots\times c_0\to C$ satisfies $(A(e_{\alpha_1},\ldots,e_{\alpha_n}))\in l_{2n/n+1}$. In Section 2, we prove that $R(A^\infty(B(c_0)))=l_\infty$ and as a corollary of the proof of this result we show $R(A_U(B(c_0)))=l_1$.

Our notation for analytic functions is standard and follows, for example, Dineen [8] and Mujica [14]. For a Banach space E, $B_R(E)$ denotes the open R-ball centered at 0 in E with $B_1(E)$ abbreviated to $B(E)$. $L(^nE)$ denotes the Banach space of continuous n-linear forms $A:E\times\ldots\times E\to C$, equipped with the norm $\|A\|=sup\{|A(x_1,\ldots,x_n)|:x_j\in E, \|x_j\|\le 1, j=1,\ldots,n\}$. $P(^nE)$ denotes the Banach space of continuous n-homogeneous polynomials on E. Each such polynomial P is associated with a unique symmetric continuous n-linear form A, by $P(x)=A(x,\ldots,x)$, and $\|P\|$ is defined to be $sup_{\|x\|\le 1}|P(x)|$. A function f from an open subset U of E to C is said to be *holomorphic* if f has a complex Fréchet derivative at each point of U. Equivalently, f is holomorphic if for all points $a\in U$, the Taylor series $f(x)=\sum_{n=0}^{\infty}P_n(x-a)$, converges uniformly for all x in some neighborhood of a, where each $P_n\in P(^nE)$.

$H_b(B_R(E))$ is the space of all holomorphic functions on $B_R(E)$ which are bounded on $B_r(E)$ for every $r<R$. A useful characterization of $H_b(B_R(E))$ is that it consists of all holomorphic functions f on $B_R(E)$ such that $\limsup_{n\to\infty}\|P_n\|^{1/n}\le 1/R$, where $\{P_n:n\in N\}$ represents the Taylor polynomials of f at the origin. The spaces $A^\infty(B(E))$ and $A_U(B(E))$ have been studied by Cole and Gamelin [4,5], Globevnik [10] and others [1]. $A^\infty(B(E))=\{f:B(E)\to C:f$ is holomorphic on B(E) and continuous and bounded on $\overline{B(E)}\}$. Unless E is finite dimensional, this space is always strictly larger than $A_U(B(E))=\{f:B(E)\to C:f$ is holomorphic and uniformly continuous on $B(E)\}$. Both of these spaces are natural infinite dimensional analogues of the disc algebra.

SECTION 1

We show here that for all $P\in P(^nc_0)$ and all $n\in N$, $\sum_{j=1}^{\infty}|P(e_j)|\le\|P\|$. This has already been done by K. John [12], in the case $n=2$. In [13], Littlewood showed that for every $A\in L(^nc_0)$, $(A(e_j,e_k))_{j,k=1}^{\infty}\in l_{4/3}$, and that 4/3 is best possible; thus, Littlewood's 4/3 result notwithstanding, John's result is that every $A\in L(^nc_0)$ has a trace. Our proof will make use of a generalization of the classical Rademacher functions, which seems to be well-known to probabilists (see, for example, Chatterji [3]).

Definition 1.1. *Fix $n\in N$, $n\ge 2$, and let $\alpha_1=1,\alpha_2,\ldots,\alpha_n$ denote the n^{th} roots of unity. Let s_1: [0,1] $\to C$ be the step function taking the value α_j on (j-1/n, j/n), for $j=1,\ldots,n$. Assuming that s_{k-1} has been defined, define s_k in the following natural way. Fix any of the n^{k-1} sub-intervals I of* [0,1] *used in the definition of s_{k-1}. Divide I into n equal intervals $I_1,\ldots,I_n$, and set $s_k(t)=\alpha_j$ if $t\in I_j$,*

(The endpoints of the intervals are irrelevant for this construction and we may, for example, define s_k to be 1 on each endpoint.)

Of course, when $n=2$, Definition 1.1 gives us the classical Rademacher functions. The following lemma lists the basic properties of the functions s_k. Its proof is similar to the usual, induction proof for the Rademacher functions, and is omitted.

Lemma 1.2. *For each $n=2,3,\ldots$, the associated functions s_k satisfy the following properties:*

(*a*). $|s_k(t)| = 1$, *for all $k \in N$ and all $t \in [0,1]$.*
(b). *For any choice of $k_1,\ldots,k_n$,*

$$\int_0^1 s_{k_1}(t)\ldots s_{k_n}(t)dt = \begin{cases} 1 & \text{if } k_1 = \ldots = k_n \\ 0 & \text{otherwise} \end{cases}$$

We are grateful to Andrew Tonge for suggesting an improvement in the proof of the following result.

Theorem 1.3. *Let $P \in P(^n c_0)$. Then $\|(P(e_j))\|_{l_1} \leq \|P\|$.*

Proof. Let $A \in L(^n c_0)$ be the symmetric n-linear form associated to P. Fix any $m \in N$. For each $i=1,\ldots,m$, let $\lambda_i = |A(e_i,\ldots, e_i)| / A(e_i,\ldots,e_i)$, if $A(e_i,\ldots, e_i) \neq 0$, and 1 otherwise. Furthermore, let β_i denote any n^{th} root of λ_i. Thus, $\lambda_i A(e_i,\ldots,e_i) = |P(e_i)|$ for each $i= 1,\ldots,m$. Adding and applying Lemma 1.2 for the integer n, we get $\sum_{i=1}^m |P(e_i)| = \sum_{i=1}^m \lambda_i A(e_i,\ldots,e_i)$

$$= \sum_{i,j_2,\ldots,j_n=1}^m \int_0^1 \lambda_i s_i(t) s_{j_2}(t)\ldots s_{j_n}(t) A(e_i, e_{j_2},\ldots,e_{j_n})dt$$

$$= \int_0^1 A\left(\sum_{i=1}^m \lambda_i s_i(t)e_i,\ldots, \sum_{j_n=1}^m s_{j_n}(t)e_{j_n}\right)dt$$

$$= \int_0^1 A\left(\sum_{j_1=1}^m \beta_{j_1} s_{j_1}(t)e_{j_1},\ldots, \sum_{j_n=1}^m \beta_{j_n} s_{j_n}(t)e_{j_n}\right)dt.$$

Since $\|\sum_{j=1}^m \beta_j s_j(t)e_j\| \leq 1$ for all t, the last expression is clearly less than or equal to $\|P\|$. Since m was arbitrary, the proof is complete.■

Rephrasing the above result in terms of the mapping R mentioned in the introduction, Theorem 1.3 implies that for any n, $R(P(^n c_0)) \subset l_1$. In fact, R is onto l_1, since any $\vec{\lambda} = (\lambda_1,\ldots,\lambda_j,\ldots) \in l_1$ equals $R(P)$, where $P \in P(^n c_0)$ is given by $P(x) = \sum_{j=1}^{\infty} \lambda_j x_j^n$.

We conclude this section by proving that, up to a normalizing factor, $R(H_b(B_R(c_0)))=l_1$, for every $R>1$. Since $H_b(B_R(c_0))$ "approaches" $A^\infty(B(c_0))$ as $R\downarrow 1$,it is tempting to guess that Corollary 1.4 below is also true for the latter space. We will see in the next section that this is completely false.

Corollary 1.4. *Let $R>1$ and let $f\in H_b(B_R(c_0))$, with $f(0)=0$. Then $(f(e_n))_{n=1}^\infty\in l_1$.*

Proof. By the characterization given earlier of $H_b(B_R(c_0))$, we see that if S is such that $1<S<R$, then $\|P_m\|^{1/m}<1/S$, for all large m. Therefore,

$$\sum_{n=1}^\infty|f(e_n)| =\sum_{n=1}^\infty\left| \sum_{m=1}^\infty P_m(e_n)\right|$$
$$\leq\sum_{m=1}^\infty\sum_{n=1}^\infty| P_m(e_n)| \leq\sum_{m=1}^\infty \|P_m\|<\infty.\blacksquare$$

SECTION 2

The following fundamental lemma shows in effect that any sequence of 0's and l's can be interpolated by a norm one function in $A^\infty(B(c_0))$.

Lemma 2.1. (*i*). *Let $S\subset N$ be an arbitrary set. There exists a function $F\in A^\infty(B(c_0))$ with the following properties:*

$$\|F\|=\sup_{x\in B(c_0)}|F(x)| =1,$$

$$F(e_n)=\begin{cases}1 & if\ n\in S\\ 0 & if\ n\notin S\end{cases}$$

(ii). If S is finite, then a function $F\in A_u(B(c_0))$ can be found which satisfies the above conditions.

Proof. Let $\alpha_j\uparrow\infty$ so quickly that the following three conditions are satisfied:

(i). The function $\Phi(x)\equiv\Pi_{j\in S}(1-x_j)^{1/\alpha_j}$ converges for all $x\in \overline{B(c_0)}$,

(ii). Re $\Phi(x)\geq 0$, for all $x\in B(c_0)$,

(iii). $\Phi(x)=0$ for some $x\in\overline{B(c_0)}$ if and only if Re $\Phi(x)=0$.

Note that $\Phi\in A^\infty(B(c_0))$ and, if S is finite then in fact $\Phi\in A_u(B(c_0))$. Also,

$$\Phi(e_n)=\begin{cases}0 & \text{for } n\in S\\ 1 & \text{for } n\notin S\end{cases}$$

Now, let $G(x) \equiv e^{-\Phi(x)}$. From the above, it is clear that $G \in A^\infty(B(c_0))$ for arbitrary S and that $G \in A_U(B(c_0))$ for finite S. In addition, $|G(x)| \leq 1$ for all x and

$$G(e_n) = \begin{cases} 1 \text{ for } n \in S \\ 1/e \text{ for } n \notin S \end{cases}$$

Finally, let $T: \bar{\Delta} \to \bar{\Delta}$ be the Mobius transformation $T(z) = \frac{z - 1/e}{1 - z/e}$ (where Δ is the complex unit disc.) It is clear that $F \equiv T \circ G$ satisfies all the conditions of the lemma.■

We come now to the analogue of Corollary 1.4, for the polydisc algebras $A^\infty(B(c_0))$ and $A_U(B(c_0))$. Note that here the situation is completely different from the situation in Section 1.

Theorem 2.2. *(i). $R(A^\infty(B(c_0))) = l_\infty$. In fact, given $(\alpha_n) \in l_\infty$, there is $F \in A^\infty(B(c_0))$ such that $F(e_n) = \alpha_n$ for all $n \in N$ and such that $\|F\| \leq 4\|(\alpha_n)\|_{l_\infty}$.*

(ii). $R(A_U(B(c_0))) = c$. In fact, given $(\alpha_n) \in c$, there is $F \in A_U(B(c_0))$ such that $F(e_n) = \alpha_n$ for all $n \in N$ and such that $\|F\| \leq 8\|(\alpha_n)\|_{l_\infty}$.

Proof. (i). Without loss of generality, $\|(\alpha_n)\| \leq 1$. Let us first suppose that $\alpha_n \geq 0$ for all n. Write $\alpha_n = \sum_{j=1}^\infty 2^{-j} \alpha_{n_j}$, where each $\alpha_{n_j} = 0$ or 1. Let $S_j = \{ n \in N: \alpha_{n_j} = 1 \}$, and let F_j be the associated function obtained using Lemma 2.1. It is easy to see that $F \equiv \sum_{j=1}^\infty 2^{-j} F_j$ is the required function in this case, and that $\|F\| \leq \|(\alpha_n)\|$. The case of general α_n's is treated by writing $\alpha_n = p_n - q_n + iu_n - iv_n$.

(ii). Suppose first that $(\alpha_n) \in c$ with $\|(\alpha_n)\| \leq 1$, and write each $\alpha_n = l + \beta_n$ where $l = lim_{n\to\infty} \alpha_n$. As above, if each β_n is expressed in binary series form, then each of the associated sets S_j is finite. As a result, each F_j is in $A_U(B(c_0))$ by Lemma 2.1 (ii), so that $F \in A_U(B(c_0))$. The required function is $G \equiv F + l$.

Finally, note that for any $F \in A_U(B(c_0))$, $F(x)$ can be approximated uniformly for $x \in B(c_0)$ by $F_r(x) = F(rx)$ for r sufficiently close to 1. Next, $F(rx)$ can be uniformly approximated on the unit ball of c_0 by a finite Taylor series, say $\sum_{k=0}^M P_k(x)$ (where P_0 is a constant). Next, it is well known (see, for example, [15]) that any k-homogeneous polynomial P_k on c_0 can be uniformly approximated on $B(c_0)$ by an k-homogeneous polynomial Q_k which is a finite sum of products of k continuous linear functionals on c_0. Summarizing, we see that the original function F can be uniformly approximated on $B(c_0)$ by $\sum_{k=0}^M Q_k$. Now, since, $(e_n) \to 0$ weakly if follows that for each $k = 1, \ldots, M$, $Q_k(e_n) \to 0$ as $n \to \infty$. Hence $R(F) \in c$, and the proof is complete. ■

It would be interesting to determine the best possible estimates in Theorem 2.2. In [2], we note that in this situation, the best estimate must be strictly larger than 1. To see this, suppose that there is $F \in A^\infty(B(c_0))$ such that $\|F\| = 1$ and such that $F(e_1) = 1$, $F(e_2) = -1$, and $F(e_j) = 0$ for all $j \geq 3$. Then the function $f_1(z) \equiv F(1,z,0,...)$ would be in the disc algebra $A(\Delta)$, and f_1 would attain its maximum at 0. Hence, f_1 would be a constant and, in particular, $1 = f_1(1) = F(1,1,0,...)$. Similarly, the function $f_2(z) \equiv F(z,1,0,...)$ would be constant, and so $-1 = f_2(1) = F(1,1,0,...)$, a contradiction. In [2], the authors find necessary and sufficient conditions on the sequence $(x_n) \subset c_0$ in order that the mapping $F \in A^\infty(B(c_0)) \to (F(x_n)) \in l_\infty$ be surjective and satisfy the following condition: For each $(\alpha_n) \in l_\infty$, there is $F \in A^\infty(B(c_0))$ such that $F(x_n) = \alpha_n$ for each $n \in N$ and $\|F\| = sup_n|\alpha_n|$.

References

[1] R. M. ARON, B. COLE, and T. GAMELIN. *The spectra of algebras of analytic functions associated with a Banach space,* to appear.

[2] R. M. ARON and J. GLOBENIK, *Interpolation by analytic functions on c_0,* Math. Proc. Camb. Phil . Soc. 104(1988)295-302.

[3] S. D. CHATTERJI, *Continuous functions representable as sums of independent random variables,* Z. Wahrsch., 13 (1969) 338-341.

[4] B. COLE and T. GAMELIN, *Representing measures and Hardy spaces for the infinite polydisc algebra,* Proc. London Math. Soc (3) 53 (1986) 1, 112-142.

[5] B. COLE and T. GAMELIN, *Weak-star continuous homomorphisms and a decomposition of orthogonal measures,* Ann. Inst. Fourier 35 (1985) 149-189.

[6] H. H. CORSON, *The weak topology of a Banach space,* Trans. Amer. Math. Soc. 101 (1961), 1-15.

[7] A. M. DAVIE, *Quotient algebras of uniform algebras,* J. London Math. Soc. 7 (1973) 31-40.

[8] S. DINEEN, *Complex analysis in locally convex spaces,* Mathematics Studies 83, North Holland, Amsterdam, 1981.

[9] J. FERRERA, *Spaces of weakly continuous functions,* Pacific J. Math. 102(1982) 285-291.

[10] J. GLOBEVNIK, *Boundaries for polydisc algebras in infinite dimensions,* Math. Proc. Camb. Phil. Soc. 85 (1979) 291-303.

[11] J. A. JARAMILLO, *Algebras de funciones continuas y diferenciables,* PhD dissertation, Univ. Complutense de Madrid, 1988.

[12] K. JOHN, *On tensor product characterization of nuclear spaces,* Math. Ann. 257(1981) 341-353.

[13] J. LITTLEWOOD, *On bounded bilinear forms in an infinite number of variables,* Quart. Journ. of Math. (2) 2 (1930), 164-174.

[14] J. MUJICA, *Complex analysis in Banach spaces,* Math. Studies 120, North Holland, Amsterdam, 1986.

[15] A. PELCZYNSKI, *A theorem of Dunford-Pettis type for polynomial operators,* Bull. Acad. Pol. Sc. XI, 6 (1963) 379-386.

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