Analytic Functions on $c_0$

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ABSTRACT. Let $F$ be a space of continuous complex valued functions on a subset of $c_0$ which contains the standard unit vector basis $\{e_n\}$. Let $R:F \rightarrow C^n$ be the restriction map, given by $R(f) = (f(e_1), \ldots, f(e_n), \ldots)$. We characterize the ranges $R(F)$ for various "nice" spaces $F$. For example, if $F = P(c_0)$, then $R(F) = l_1$, and if $F = A^*(B(c_0))$, then $R(F) = l_\infty$.

Let $c_0$ be the Banach space of complex valued null sequences $x = (x_n)$, with the normal sup-norm and usual basis vectors $e_n = (0, \ldots, 0, 1, 0, \ldots)$, and let $F$ be a space of continuous complex valued functions on some subset of $c_0$ which contains the standard basis of $c_0$. Let $R:F \rightarrow C^n$ be the mapping which assigns to each function $f \in F$ the sequence $(f(e_1), \ldots, f(e_n), \ldots)$. Our attention in this article will be focussed on characterizing the range of $R$ for various spaces $F$ of interest. For example, if $F = C(c_0)$, the space of all continuous complex valued functions on $c_0$, then a trivial application of the Tietze extension theorem shows that $R(F) = C^n$. On the other hand, $c_0$ is weakly normal (Corson [6], see also Ferrera [9]). Since $\{0\} \cup \{e_n : n \in \mathbb{N}\}$ is weakly compact, we see that $R(F) = c_0$, the space of convergent sequences, if we take $F$ to be the subspace of $C(c_0)$ consisting of weakly continuous functions. Recently Jaramillo [11] has examined the relationship between reflexivity of the space $F$ and the range of $R$, for certain spaces of real valued infinitely differentiable functions and polynomials on a Banach space $E$ with unconditional basis $\{e_n : n \in \mathbb{N}\}$.

We concentrate here on analogous spaces of complex valued functions on $c_0$. After a review of relevant notation and definitions, we show in Section 1 that $R(F) = l_1$, when $F = P(c_0)$, $n \in \mathbb{N}$. As a consequence, we prove that if $F = \{f \in H_c(B(c_0)) : f(0) = 0\}$, then $R(F) = l_1$. Taking $n = 2$ in the above result, we see that every $2-$homogeneous polynomial $P$ on $c_0$ satisfies $\sum_{n=1}^{\infty} |P(e_n)| < \infty$. This result is reminiscent of classical work of Littlewood [13], who proved that every continuous bilinear form $A$ on $c_0 \times c_0$ satisfies $(A(c_0, c_0))_{n=1}^{\infty} \in l_\infty$. Littlewood’s work was extended by Davie [7], who showed...
that every continuous $n$-linear form $A: c_0 \times \cdots \times c_0 \to C$ satisfies $(A(e_1, \ldots, e_n)) = l_{\infty}$. In Section 2, we prove that $R(A(B(c_0))) = l_{\infty}$ and as a corollary of the proof of this result we show $R(A(B(c_0))) = l_1$.

Our notation for analytic functions is standard and follows, for example, Dineen [8] and Mujica [14]. For a Banach space $E$, $B(E)$ denotes the open $R$-ball centered at $0$ in $E$ with $B_1(E)$ abbreviated to $B(E)$. $L(E)$ denotes the Banach space of continuous $n$-linear forms $A: E \times \cdots \times E \to C$, equipped with the norm $\|A\| = \sup\{|A(x_1, \ldots, x_n)| : x_j \in E, \|x_j\| \leq 1, j = 1, \ldots, n\}$. $P(E)$ denotes the Banach space of continuous $n$-homogeneous polynomials on $E$. Each such polynomial $P$ is associated with a unique symmetric continuous $n$-linear form $A$, by $P(x) = A(x_1, \ldots, x_n)$, and $\|P\|$ is defined to be $\sup_{x \in B(E)} |P(x)|$. A function $f$ from an open subset $U$ of $E$ to $C$ is said to be holomorphic if $f$ has a complex Fréchet derivative at each point of $U$. Equivalently, $f$ is holomorphic if for all points $a \in U$, the Taylor series $f(x) = \sum_{j=0}^{\infty} P_j(x - a)$, converges uniformly for all $x$ in some neighborhood of $a$, where each $P_j \in P(E)$.

$H(B(E))$ is the space of all holomorphic functions on $B(E)$ which are bounded on $B(E)$ for every $0 < R$. A useful characterization of $H(B(E))$ is that it consists of all holomorphic functions $f$ on $B(E)$ such that $\limsup_{n \to \infty} \|P_n\|^n \leq 1/R$, where $\{P_n\}_{n \in \mathbb{N}}$ represents the Taylor polynomials of $f$ at the origin. The spaces $A^\infty(B(E))$ and $A_0(B(E))$ have been studied by Cole and Gamelin [4,5], Globevnik [10] and others [1]. $A^\infty(B(E)) = \{f: B(E) \to C \, | \, f$ is holomorphic on $B(E)$ and continuous and bounded on $B(E)\}$. Unless $E$ is finite dimensional, this space is always strictly larger than $A_0(B(E)) = \{f: B(E) \to C \, | \, f$ is holomorphic and uniformly continuous on $B(E)\}$. Both of these spaces are natural infinite dimensional analogues of the disc algebra.

SECTION 1

We show here that for all $P \in P(c_0)$ and all $n \in \mathbb{N}$, $\sum_{j=0}^{\infty} |P(e_1)| \leq \|P\|$. This has already been done by K. John [12], in the case $n = 2$. In [13], Littlewood showed that for every $A \in L(c_0)$, $(A(e_1, e_2))_{n \in \mathbb{N}} \in l_{10}$, and that $4/3$ is best possible; thus, Littlewood's 4/3 result notwithstanding, John's result is that every $A \in L(c_0)$ has a trace. Our proof will make use of a generalization of the classical Rademacher functions, which seems to be well-known to probabilists (see, for example, Chatterji [3]).

Definition 1.1. Fix $n \in \mathbb{N}$, $n \geq 2$, and let $\alpha_1 = 1$, $\alpha_2, \ldots, \alpha_n$ denote the $n$th roots of unity. Let $s_{\alpha_1} : [0,1] \to C$ be the step function taking the value $\alpha_1$ on $(j-1/n, j/n)$, for $j = 1, \ldots, n$. Assuming that $s_{\alpha_1}$ has been defined, define $s_\alpha$ in the following natural way. Fix any of the $n$th sub-intervals $I_{j}$ of $[0,1]$ used in the definition of $s_{\alpha_1}$. Divide $I$ into $n$ equal intervals $I_{1}, \ldots, I_{n}$ and set $s_\alpha(t) = \alpha_j$ if $t \in I_{j}$.
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(The endpoints of the intervals are irrelevant for this construction and we may, for example, define $s_0$ to be 1 on each endpoint.)

Of course, when $n=2$, Definition 1.1 gives us the classical Rademacher functions. The following lemma lists the basic properties of the functions $s_n$. Its proof is similar to the usual, induction proof for the Rademacher functions, and is omitted.

**Lemma 1.2.** For each $n=2,3,...$, the associated functions $s_n$ satisfy the following properties:

(a) $|s_n(t)| = 1$, for all $k \in \mathbb{N}$ and all $t \in [0,1]$.

(b) For any choice of $k_1,...,k_n$,

$$\int_0^1 s_{k_1}(t)...s_{k_n}(t)dt = \begin{cases} 1 & \text{if } k_1=...=k_n \\ 0 & \text{otherwise} \end{cases}$$

We are grateful to Andrew Tonge for suggesting an improvement in the proof of the following result.

**Theorem 1.3.** Let $P \in P(c_0)$. Then $\|(P(e))\|_{\ell_1} \leq \|P\|_\ell$

**Proof.** Let $A \in L(c_0)$ be the symmetric $n$-linear form associated to $P$. Fix any $m \in \mathbb{N}$. For each $i=1,...,m$, let $\lambda_i = |A(e,..., e)| / |A(e,..., e)|$, if $A(e,..., e) \neq 0$, and 1 otherwise. Furthermore, let $\beta_i$ denote any $n^{th}$ root of $\lambda_i$. Thus, $\lambda_i A(e,..., e) = |P(e)|$ for each $i = 1,...,m$. Adding and applying Lemma 1.2 for the integer $n$, we get $\sum_{i=1}^m |P(e_i)| = \sum_{i=1}^m \lambda_i A(e,..., e_i)$

$$= \sum_{i=1}^m \int_0^1 \lambda_i s_1(t) s_2(t)...s_n(t) A(e,..., e_i) dt$$

$$= \int_0^1 A(\sum_{i=1}^m \lambda_i s_1(t) e,..., \sum_{i=1}^m \lambda_i s_n(t) e_i) dt$$

$$= \int_0^1 \left[ A(\sum_{i=1}^m \beta_i s_1(t) e,..., \sum_{i=1}^m \beta_i s_n(t) e_i) \right] dt.$$

Since $\|\sum_{i=1}^m \beta_i s_1(t) e_i\| \leq 1$ for all $t$, the last expression is clearly less than or equal to $\|P\|$. Since $m$ was arbitrary, the proof is complete.

Rephrasing the above result in terms of the mapping $R$ mentioned in the introduction, Theorem 1.3 implies that for any $n$, $R(P(c_0)) \subseteq l_1$. In fact, $R$ is onto $l_1$, since any $Y=(\lambda_1,...,\lambda_n) \in l_1$ equals $R(P)$, where $P \in P(c_0)$ is given by $P(x) = \sum_{i=1}^n \lambda_i x_i$. 


We conclude this section by proving that, up to a normalizing factor, $R(H_j(B_R(c_0))) = l_1$, for every $R > 1$. Since $H_j(B_R(c_0))$ "approaches" $A^w(B(c_0))$ as $R \downarrow 1$, it is tempting to guess that Corollary 1.4 below is also true for the latter space. We will see in the next section that this is completely false.

**Corollary 1.4.** Let $R > 1$ and let $f \in H_j(B_R(c_0))$, with $f(0) = 0$. Then $(f(e_n))_{n=1}^\infty \in l_1$.

**Proof.** By the characterization given earlier of $H_j(B_R(c_0))$, we see that if $S$ is such that $1 < S < R$, then $\|P_m\|^{1/m} < 1/S$, for all large $m$. Therefore,

$$
\sum_{n=1}^\infty \|f(e_n)\| = \sum_{n=1}^\infty \sum_{m=1}^\infty \|P_m(e_n)\| 
\leq \sum_{n=1}^\infty \sum_{m=1}^\infty \|P_m\| \|P_m(e_n)\| 
\leq \sum_{n=1}^\infty \|P_m\| < \infty. \tag{1}
$$

**SECTION 2**

The following fundamental lemma shows in effect that any sequence of 0's and 1's can be interpolated by a norm one function in $A^w(B(c_0))$.

**Lemma 2.1.** (i). Let $S \subset N$ be an arbitrary set. There exists a function $F \in A^w(B(c_0))$ with the following properties:

$$
\|F\| = \sup_{x \in B(c_0)} |F(x)| = 1, 
F(e_n) = \begin{cases} 
1 & \text{if } n \in S \\
0 & \text{if } n \notin S 
\end{cases}
$$

(ii). If $S$ is finite, then a function $F \in A_j(B(c_0))$ can be found which satisfies the above conditions.

**Proof.** Let $\alpha_1^{1/\infty}$ so quickly that the following three conditions are satisfied:

(i). The function $\Phi(x) = \prod_{x \in S} (1 - x)^{\alpha}$ converges for all $x \in B(c_0)$.

(ii). Re $\Phi(x) \geq 0$, for all $x \in B(c_0)$.

(iii). $\Phi(x) = 0$ for some $x \in B(c_0)$ if and only if Re $\Phi(x) = 0.$

Note that $\Phi \in A^w(B(c_0))$ and, if $S$ is finite then in fact $\Phi \in A_j(B(c_0))$. Also,

$$
\Phi(e_n) = \begin{cases} 
0 & \text{for } n \in S \\
1 & \text{for } n \notin S 
\end{cases}
$$
Now, let \( G(x) = e^{-4x} \). From the above, it is clear that \( G \in A^\infty(B(c_0)) \) for arbitrary \( S \) and that \( G \in A_d(B(c_0)) \) for finite \( S \). In addition, \( |G(x)| \leq 1 \) for all \( x \) and

\[
G(e) = \begin{cases} 
1 & \text{for } n \in S \\
1/e & \text{for } n \notin S 
\end{cases}
\]

Finally, let \( T : \Delta \to \Delta \) be the Mobius transformation \( T(z) = \frac{z + i}{1 - z} \) (where \( \Delta \) is the complex unit disc.) It is clear that \( F \circ T \circ G \) satisfies all the conditions of the lemma.

We come now to the analogue of Corollary 1.4, for the polydisc algebras \( A^\infty(B(c_0)) \) and \( A_d(B(c_0)) \). Note that here the situation is completely different from the situation in Section 1.

**Theorem 2.2.** (i). \( R(A^\infty(B(c_0))) = l_\infty \). In fact, given \( (a_\alpha) \in l_\infty \) there is \( F \in A^\infty(B(c_0)) \) such that \( F(e) = a_\alpha \) for all \( n \in N \) and such that \( \|F\| \leq 4\|((a_\alpha))\|_{\infty} \).

(ii). \( R(A_d(B(c_0))) = c \). In fact, given \( (a_\alpha) \in c \), there is \( F \in A_d(B(c_0)) \) such that \( F(e) = a_\alpha \) for all \( n \in N \) and such that \( \|F\| \leq 8\|((a_\alpha))\|_{\infty} \).

**Proof.** (i). Without loss of generality, \( \|(a_\alpha)\| \leq 1 \). Let us first suppose that \( a_\alpha \geq 0 \) for all \( n \). Write \( a_\alpha = \sum_{n=1}^{\infty} 2^{n}a_{n,n} \), where each \( a_{n,n} = 0 \) or 1. Let \( S_{n} = n \in N: a_{n,n} = 1 \), and let \( F_j \) be the associated function obtained using Lemma 2.1. It is easy to see that \( F = \sum_{j=1}^{\infty} 2^{-j} F_j \) is the required function in this case, and that \( \|F\| \leq \|((a_\alpha))\| \). The case of general \( a_\alpha \)'s is treated by writing \( a_\alpha = p_\alpha - q_\alpha + iu_\alpha - iv_\alpha \).

(ii). Suppose first that \( (a_\alpha) \in c \) with \( \|(a_\alpha)\| \leq 1 \), and write each \( a_\alpha = l + \beta_\alpha \), where \( l = \text{lim}_{n \to \infty} a_{n,n} \). As above, if each \( \beta_\alpha \) is expressed in binary series form, then each of the associated sets \( S_{n} \) is finite. As a result, each \( F_j \) is in \( A_d(B(c_0)) \) by Lemma 2.1 (ii), so that \( F \in A_d(B(c_0)) \). The required function is \( G = F + l \).

Finally, note that for any \( F \in A_d(B(c_0)) \), \( F(x) \) can be approximated uniformly for \( x \in B(c_0) \) by \( F(x) = F(rx) \) for \( r \) sufficiently close to 1. Next, \( F(rx) \) can be uniformly approximated on the unit ball of \( c_0 \) by a finite Taylor series, say \( \sum_{k=0}^{\infty} P_k(x) \) (where \( P_k \) is a constant). Next, it is well known (see, for example, [15]) that any \( k \)-homogeneous polynomial \( P_k \) on \( c_0 \) can be uniformly approximated on \( B(c_0) \) by an \( k \)-homogeneous polynomial \( Q_k \), which is a finite sum of products of \( k \) continuous linear functionals on \( c_0 \). Summarizing, we see that the original function \( F \) can be uniformly approximated on \( B(c_0) \) by \( \sum_{k=0}^{\infty} Q_k \). Now, since, \( (e) \to 0 \) weakly if follows that for each \( k = 1, \ldots, M, Q_k(\epsilon) \to 0 \) as \( n \to \infty \). Hence \( R(F) \in c \), and the proof is complete.

It would be interesting to determine the best possible estimates in Theorem 2.2. In [2], we note that in this situation, the best estimate must be strictly
larger than 1. To see this, suppose that there is $F \in \mathcal{A}^n(B(c_0))$ such that $\|F\| = 1$ and such that $F(e_j) = 1$, $F(e_{j+1}) = -1$, and $F(e_{j+2}) = 0$ for all $j \geq 3$. Then the function $f_j(z) \equiv F(1, z, 0, \ldots)$ would be in the disc algebra $A(\Delta)$, and $f_j$ would attain its maximum at 0. Hence, $f_j$ would be a constant and, in particular, $1 = f_j(1) = F(1, 1, 0, \ldots)$. Similarly, the function $f_j(z) \equiv F(0, 1, 0, \ldots)$ would be constant, and so $-1 = f_j(1) = F(1, 1, 0, \ldots)$, a contradiction. In [2], the authors find necessary and sufficient conditions on the sequence $(x_j) \in c_0$ in order for the mapping $F \in \mathcal{A}^n(B(c_0)) \rightarrow (F(x_j)) \in l_\infty$, be surjective and satisfy the following condition: For each $(a_n) \in l_\infty$ there is $F \in \mathcal{A}^n(B(c_0))$ such that $F(x_n) = a_n$ for each $n \in \mathbb{N}$ and $\|F\| = \sup_n |a_n|$.

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