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Norm attaining and numerical radius attaining operators

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ABSTRACT. In this note we discuss some results on numerical radius attaining operators paralleling earlier results on norm attaining operators.

For arbitrary Banach spaces X and Y, the set of (bounded, linear) operators from X to Y whose adjoints attain their norms is norm-dense in the space of all operators. This theorem, due to W. Zizler, improves an earlier result by J. Lindenstrauss on the denseness of operators whose second adjoints attain their norms, and is also related to a recent result by C. Stegall where it is assumed that the dual space Y^* has the Radon-Nikodym property to obtain a stronger assertion.

Numerical radius attaining operators behave in quite a similar way. It is also true that the set of operators on an arbitrary Banach space whose adjoints attain their numerical radii is norm-dense in the space of all operators. However no example is known of a Banach space X such that the numerical radius attaining operators on X are not dense. We can prove that such an space X must fail the Radon-Nikodym property.

The content of this paper is merely expository. Complete proofs will be published elsewhere.

1. NORM ATTAINING OPERATORS

Let X and Y be arbitrary Banach spaces over the same scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). By *operator* from X to Y we always mean a bounded linear operator and the space of these operators will be denoted by $L(X,Y)$. Only the norm topology will be considered in $L(X,Y)$. Let us recall that the norm of $T \in L(X,Y)$ is given by

$$\|T\| = \mathrm{Sup}\{\|T(x)\| : x \in X, \|x\| = 1\}$$

and it is said that *T attains its norm* when this supremum is actually a maximum, that is when there is an x_0 in the unit sphere of X such that $\|T(x_0)\| = \|T\|$. We will denote by $P_0(X, Y)$ the set of norm attaining operators from X to Y.

As a particular case of the already classical Bishop-Phelps Theorem we know that $P_0(X, \mathbb{K})$ is norm-dense in $L(X,\mathbb{K}) = X^*$ for all Banach spaces X . J.

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Lindenstrauss [11] began to discuss the general case and showed that this result is no longer true when we put an arbitrary Banach space Y in place of $\mathbb{K}$. More concretely, he even gave an example of a Banach space X such that $P_0(X,X)$ is not dense in $L(X,X)$ [11; Proposition 5]. However the problem becomes easier when we go up to higher duals. Let T^* denote the adjoint of an operator T from X to Y, that is $T^* \in L(Y^*, X^*)$ is given by

$$[T^*(g)](x) = g(T(x))$$

for x in X and g in Y^*. We can consider the set of operators from X to Y whose adjoints attain their norms

$$P_1(X,Y) = \{T \in L(X,Y) : T^* \in P_0(Y^*,X^*)\}$$

and, in a second step, we can consider the set

$$P_2(X,Y) = \{T \in L(X,Y) : T^{**} \in P_0(X^{**}, Y^{**})\}$$

As a simple consequence of the Hahn-Banach Theorem we have

$$P_0(X,Y) \subset P_1(X,Y) \subset P_2(X,Y)$$

and so on.

J. Lindenstrauss [11; Theorem 1] proved that $P_2(X,Y)$ is always dense in $L(X,Y)$. This result was improved by W. Zizler in the following way.

Theorem 1.1. [18; Proposition 4]. *For arbitrary real or complex Banach spaces X and Y, $P_1(X,Y)$ is norm-dense in $L(X,Y)$.*

The proof of this theorem is a successful modification of Lindenstrauss's proof of [11; Theorem 1]. The following example shows that the assertion in Theorem 1.1 is stronger than the one in [11; Theorem 1].

Example 1.2. Let Y be the (real or complex) space c_0 renormed (to be strictly convex) by

$$|||y||| = \|y\| + [\sum_{n=1}^{\infty} \frac{1}{2^n} |y(n)|^2]^{1/2}$$

where $\|.\|$ is the usual norm on c_0 and $y(n)$ denotes the n th term of any sequence y in Y.

Let us define $T \in L(c_0, Y)$ by

$$[T(x)](n) = \alpha_n x(n) \qquad (x \in c_0)$$

where $\{\alpha_n\}$ is any sequence of real numbers such that $0 < \alpha_n < 1$ for all n, and $\{\alpha_n\} \to 1$.

It follows from the proof of [11; Proposition 4] that T does not attain its norm. In fact, a careful examination of the dual norm on Y^* shows that the adjoint of T does not attain its norm, while it is fairly easy to verify that $T \in P_2(c_0, Y)$, so we have that $P_1(c_0, Y) \neq P_2(c_0, Y)$. It follows also from the proof of [11; Proposition 4] that $P_0(c_0, Y)$ is not dense in $L(c_0, Y)$, so the assertion of Theorem 1.1 is the best we have in this case.

To conclude our brief survey on norm attaining operators we mention a result by C. Stegall [17; Corollary 22] which is also related to Theorem 1.1. Under the assumption that Y^* has the Radon-Nikodym property he proves that $P_1(X, Y)$ contains a G-delta dense subset of $L(X, Y)$, the Banach space X being arbitrary. Thus he imposes a severe restriction on Y but obtains an assertion which is stronger than the mere denseness of $P_1(X, Y)$.

For further information about norm attaining operators, a topic which has received a great deal of attention in recent years, we refer the reader to papers by J. Bourgain [5], J.R. Partington [13], W. Schachermayer [14] and the above mentioned papers by C. Stegall [17], W. Zizler [18] and J. Lindenstrauss [11].

2. NUMERICAL RADIUS ATTAINING OPERATORS

Recall that the *numerical range*, $W(T)$, of an operator T from a Hilbert space X into itself is defined by

$$W(T) = \{(T(x) | x) : x \in X, \ \|x\| = 1\}$$

where $(\cdot | \cdot)$ is the inner product. This useful concept has a natural extension to operators from an arbitrary Banach space into itself (we will denote by $L(X)$, instead of $L(X,X)$, the space of these operators). The definition is

$$W(T) = \{f(T(x)) : x \in X, f \in X^*, \ \|x\| = \|f\| = f(x) = 1\}.$$

The *numerical radius* of $T \in L(X)$ is then given by

$$w(T) = \mathrm{Sup}\{|\lambda| : \lambda \in W(T)\}$$

In [3,4] the reader will find a wide discussion of numerical ranges of operators on Banach spaces. We only point out the fact, essentially due to G. Lumer [12], that

$$w(T) = w(T^*)$$

for all T in $L(X)$ [3; Corollary 9.6] (see also [4; Corollary 17.3]).

We are concerned here with *numerical radius attaining* operators. Let us say that an operator $T \in L(X)$ attains its numerical radius if there are $x_0 \in X$, $f_0 \in X^*$ such that

$$\|x_0\| = \|f_0\| = f_0(x_0) = 1 \text{ and } |f_0(T(x_0))| = w(T),$$

that is if the supremum defining $w(T)$ is actually a maximum. Let $R_0(X)$ denote the set of numerical radius attaining operators on a Banach space X. As we did for the norm, we can consider the sets

$$R_1(X) = \{T \in L(X): T^* \in R_0(X^*)\} \text{ and}$$

$$R_2(X) = \{T \in L(X): T^{**} \in R_0(X^{**})\}.$$

From the above mentioned equality between the numerical radii of an operator and its adjoint, through an obvious application of the Hahn-Banach Theorem, we obtain

$$R_0(X) \subset R_1(X) \subset R_2(X)$$

The authors have proved [1] that $R_2(X)$ is dense in $L(X)$ for any Banach space X. This result can be improved in the following way.

Theorem 2.1. *For any real or complex Banach space X, $R_1(X)$ is norm-dense in $L(X)$.*

The proof of this theorem is close to the one of Theorem 1.1, although it is a bit more laborious. Both theorems give us a nice similarity between norm attaining and numerical radius attaining operators. The answer to the following open question, posed by B. Sims [15], could become a striking difference.

Problem 2.2. *Is it true that $R_0(X)$ is dense in $L(X)$ for all Banach spaces X?*

I. Berg and B. Sims [2] gave an affirmative answer to this question for uniformly convex spaces. As an easy consequence, the same result is true for uniformly smooth spaces. This observation is due to C. Cardassi [7], who showed also that the answer is affirmative for several classical Banach spaces, namely c_0, $C(K)$ and $L_1(\mu)$ [8,9,10]. From Theorem 2.1 (or even from the result in [1]) we deduce the following consequence which extends the results in [2,7].

Corollary 2.3. *If X is a reflexive Banach space, then $R_0(X)$ is norm-dense in $L(X)$.*

However we want to discuss here a more general result which requires more sophisticated tools. The crucial one is the following nonlinear optimization theorem by C. Stegall. We refer to [6] for a survey on Radon-Nikodym sets.

Theorem 2.4. (Stegall [16,17]). *Let X be a real Banach space, D a Radon-Nikodym set in X and $\Phi:D\to\mathbb{R}$ a (norm) upper-semicontinuous, bounded above function. Then the set*

$$\{f\in X^*:\Phi+f \text{ strongly exposes } D\}$$

is a dense G-delta subset of X^*.

In order to take advantage of the above theorem in our context, given a Banach space X and $T\in L(X)$, we consider the real function Φ_T defined on the unit sphere of X by

$$\Phi_T(x)=\text{Max}\{|f(T(x))|:\ \|f\|=f(x)=1\}$$

It is plain that

$$w(T)=\text{Sup}\{\Phi_T(x):x\in X,\ \|x\|=1\},$$

so T attains its numerical radius if and only if Φ_T attains its supremum on the unit sphere.

The function Φ_T can be extended to the unit ball of X by defining

$$\Phi_T(x)=\|x\|\Phi_T\left(\frac{x}{\|x\|}\right), \text{ for } 0<\|x\|<1, \text{ and}$$

$\Phi_T(0) = 0$. In this way we have

Lemma 2.5. *Let X be a Banach space and $T\in L(X)$. Then Φ_T is upper-semicontinuous (and bounded above) in the unit ball of X.*

Now, if X is a Banach space with the Radon-Nikodym property, and $T\in L(X)$, Stegall's Theorem gives us an arbitrarily small continuous linear functional f on X such that $\Phi_T + Re\, f$ attains it supremum on the unit ball. After some computations we get the following result.

Theorem 2.6. *Let X be a Banach space satisfying the Radon-Nikodym property. For every $T\in L(X)$ and $\varepsilon>0$ there is a rank-one operator S on X such that*

$\|S\| < \varepsilon$ *and* $T + S$ *attains its numerical radius. In particular,* $R_0(X)$ *is norm-dense in* $L(X)$.

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