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Positive solutions of an elliptic equation with strongly nonlinear lower order terms

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ABSTRACT. In this paper we study the existence of positive solutions of the equation:

$$\Delta_p u + g(x,u) = 0$$

in the case when the growth of g(x,.) is allowed to be of exponential type.

INTRODUCTION

Let $1 < p < +\infty$ and let Ω be a bounded regular open set in $\mathbb{R}^N$. We look for positive solutions, $u \in W_0^{1,p}(\Omega)$, of the equation:

$$\text{(E)} \qquad \Delta_p u + g(x,u) = 0 \qquad \text{in} \quad \Omega$$

where $F(\nabla u) = |\nabla u|^{p-2}\nabla u$ and $\Delta_p u = \text{div } F(\nabla u)$.
We are specially interested by the case when the growth of g near $u = +\infty$ is not of polynomial type, for example of exponential type.

In the case when Ω is a starshaped domain and $g(x,u) = |u|^{\gamma-2}u$ with $\gamma(N-p) > Np$, it is well known [7,9,10,12,13] that (E) cannot have positive solutions $u \in W_0^{1,p}(\Omega)$.

On the other hand, in the case when $p=2$ and $\Omega = A = \{x \in \mathbb{R}^N : \rho < |x| < R\}$ with $0 < \rho < R < +\infty$, recent papers have shown that (E) has positive solutions:

- either for $g(u) = 0(u^k)$, $k > -1$, near $u = +\infty$ [3];
- or for $R - \rho$ sufficiently small [2].

In this paper, proving that radially symmetric functions in $W_0^{1,p}(A)$ are in $L^\infty(A)$, we can obtain positive solutions of (E) for any $p \in]1, +\infty[$, any $R - \rho > 0$, and any growth of g near $u = +\infty$.

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In the limit case $N = p$, $W_0^{1,p}(\Omega) \not\subset L^\infty(\Omega)$ but [1] $W_0^{1,p}(\Omega) \subset L_M(\Omega)$, where L_M is the Orlicz space associated with the Young function:

$$M(\zeta) = \exp\left(|\zeta|^{p^*} - 1\right), \frac{1}{p} + \frac{1}{p^*} = 1$$

Trudinger [15] has shown that for $p = 2$, any $q \in]0,2[$ and any $c > 0$, there are some $\lambda > 0$ and $u \in W_0^{1,2}(\Omega)$ such that:

$$\Delta u + \lambda u^\sigma \exp(|u|^q) = 0 \text{ and } \int_\Omega \int_0^{u(x)} t^\sigma \exp(t^q) dt\, dx = c$$

In this paper we extend these results to $p \neq 2$ and eliminate this λ.

The particular case $N = p$, $\Omega = B(0,R)$ is interesting because we can prove that, for any growth of g near $u = +\infty$, (E) can have positive radially symmetric solutions if R is sufficiently large; we extend to the case $p \neq 2$ the results of Hempel [4,5] and Nehari [6].

As a conclusion, consider the example:

$$g(x,\zeta) = |\zeta|^\sigma \exp(|\zeta|^q) \text{ with } \sigma > p - 1 \text{ and } q > 0$$

(E) has positive solutions:

- for $p > N$ or $\Omega = A$ (Theorem 1)
- for $p = N$ and $q < p^*$ (Theorem 2)
- or $p = N$, $\Omega = B(0,R)$, $R > R_0$, $\sigma > \max(p-1,1)$ and $q > 1$ (Theorem 3)

1. BOUNDED SOLUTIONS

Let X be a closed subspace of $W_0^{1,p}(\Omega)$. g is asumed to be a Caratheodory function satisfying the following conditions:

(H1) $\forall x \in \Omega$, $\forall \zeta \in \mathbb{R}$, $g(x,\zeta) \geqslant 0$ and $\forall x \in \Omega$, $\forall \zeta > 0$, $g(x,\zeta) > 0$;

(H2) $\forall K > 0$, $\exists M > 0$ such that for any $\zeta \in \mathbb{R}$, $|\zeta| \leqslant K$, and for any $x \in \Omega$, $g(x,\zeta) \leqslant M$;

(H3) There exist some $\sigma_0 > p - 1$ and $\zeta_0 \geqslant 0$ such that:

$$\forall \zeta \geqslant \zeta_0,\ \zeta \to \frac{G(x,\zeta)}{\zeta^{\sigma_0+1}} \text{ is a non decreasing function}$$

$$\text{where } G(x,\zeta) = \int_0^\zeta g(x,s)\, ds.$$

Remark: It is sufficient to suppose that g satisfies (H1) and (H2) on $\mathbb{R}_+$; it can be easily extended to a function satisfying (H1) and (H2) on $\mathbb{R}$.

Theorem 1:

Let g satisfy the conditions (H1), (H2), (H3) and suppose that:

(*i*) $X \subset L^\infty(\Omega)$.

(*ii*) *There exist some* $\zeta_1 > 0$, $\sigma_1 > p-1$ *and* $c > 0$ *such that:*

$$\forall x \in \Omega,\ \forall \zeta \in [0,\zeta_1],\ G(x,\zeta) \leqslant c\, \zeta^{\sigma_1+1}$$

Then there is at least one positive solution $u \in X \cap C^{1,\alpha}(\Omega)$ *of* (*E*).

The condition (*i*) is satisfied for $X = W_0^{1,p}(\Omega)$ and any bounded open set Ω in $\mathbb{R}^N$ in the case when $p > N$; the following proposition gives an other interesting example.

Proposition 1:

Let $0 < \rho < R < +\infty$ *and* Ω *be an annulus in* $\mathbb{R}^N$:

$\Omega = \{ x \in \mathbb{R}^N : \rho < |x| < R \}$. *Let X be the set of radially symmetric functions in* $W_0^{1,p}(\Omega)$.

Then, there exist a positive constant $C(N,p,\rho,R)$ *such that:*

$$\forall u \in X,\ \forall x \in \Omega,\ |u(x)| \leqslant C(N,p,\rho,R) \|\nabla u\|_p$$

Examples:

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive non decreasing continuous function and $g(x,\zeta) = \zeta^\sigma h(\zeta)$ where $\sigma > p-1$; for instance $g(x,\zeta) = \zeta^\sigma \exp(\zeta^q)$, $\sigma > p-1$, $q > 0$; then g satisfies (H1), (H2), (H3) and (*ii*).

In the case when Ω is an annulus and $\sigma > 1$, we obtain positive solutions of

$$\Delta u + u^{\sigma} h(u) = 0 \text{ in } \Omega$$

without any limiting condition as $g(u) = 0(u^k)$ when $u \to +\infty$ (GARAIZAR [3]), neither $R - \rho$ small (BANDLE - PELETIER [2]); besides we obtain analogous results for $p \neq 2$ and $\sigma > p - 1$. On the other hand our conditions are more restrictive than [2], [3] on the growth of g and on the limit of $g(u)$ when $u \to 0$.

Proof of Proposition 1

Let $u(x) = \varphi(|x|)$; we have

$$-\varphi(|x|) = \varphi(R) - \varphi(|x|) = \int_{|x|}^{R} \varphi'(t)\, dt$$

By Hölder's inequality we get:

$$|u(x)| \leqslant \left(\int_{|x|}^{R} |\varphi'(t)|^p t^{N-1} dt\right)^{1/p} \left(\int_{|x|}^{R} \frac{dt}{t^{(N-1)/(p-1)}}\right)^{1/p^*}$$

$$\int_{|x|}^{R} |\varphi'(t)|^p\, t^{N-1}\, dt = \frac{1}{\omega_N} \int_{|x| \leqslant |y| \leqslant R} |\nabla u(y)|^p dy$$

whence the result with:

$$C(N,p,\rho,R) = \frac{1}{\omega_N{}^{1/p}} \left(\int_{\rho}^{R} \frac{dt}{t^{(N-1)/(p-1)}}\right)^{\frac{1}{p^*}} \qquad \square$$

The proof of Theorem 1 needs the following lemmas.

Lemma 1:

For any $u \in X$, let us consider:

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p dx - \int_{\Omega} G(x,u(x))\, dx$$

Suppose that g satisfies (H1), (H2), (H3). Then any sequence $(u_j) \subset X$ such $|J(u_j)| \leqslant K$ and $J'(u_j) \to 0$ in X', is bounded in X.

Proof:

For any $v \in X$, we have:

$$J'(u)(v) = \int_\Omega F(\nabla u).\nabla v - \int_\Omega g(.,u)\, v$$

Ω being a bounded set we set:

$$\|u\|_X = \|\nabla u\|_p = \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}$$

Suppose that a subsequence denoted by u_j be such that $\lim_{j\to+\infty} \|u_j\|_X = +\infty$; we get:

$$-\frac{K}{\|u_j\|_X^p} \leqslant \frac{1}{p} - \frac{\int_\Omega G(.,u_j)}{\|u_j\|_X^p} \leqslant \frac{K}{\|u_j\|_X^p}$$

$$-\frac{\varepsilon}{\|u_j\|_X^{p-1}} \leqslant 1 - \frac{\int_\Omega u_j\, g(.,u_j)}{\|u_j\|_X^p} \leqslant \frac{\varepsilon}{\|u_j\|_X^{p-1}}$$

whence $\lim_{j\to+\infty} \dfrac{\int_\Omega G(.,u_j)}{\int_\Omega u_j\, g(.,u_j)} = \dfrac{1}{p}$

(H3) gives for any $\zeta \geqslant \zeta_0 : \zeta g(.,\zeta) \geqslant (\sigma_0+1)\, G(.,\zeta)$, whence:

$$\int_\Omega G(.,u_j) \leqslant C_1 + \frac{1}{(\sigma_0+1)}\int_\Omega u_j\, g(.,u_j)$$

$$\lim_{j\to+\infty} \frac{\int_\Omega G(.,u_j)}{\int_\Omega u_j\, g(.,u_j)} \leqslant \frac{1}{\sigma_0+1} < \frac{1}{p}$$

A contradiction, whence $\|u_j\|_X$ is bounded. □

Lemma 2:

If the hypothesis of Theorem 1 are satisfied, $J \in C^1(X)$ and satisfies the Palais - Smale condition.

Proof:

An easy conseqence of Lebesgue's theorem shows that for $u_j \to u$, $\lim_{j\to+\infty} \|g(.,u_j) - (g(.,u)\|_{p^*} = 0$, whence $J \in C^1(X)$.

Suppose that $|J(u_j)| \leqslant K$ and $J'(u_j) \to 0$; by lemma 1, $g(.,u_j)$ is bounded, and the injection $X \subset L^p$ being compact, there exists a subsequence denoted by u_j which converges to u in strong L^p.

So, $\lim_{n,m\to+\infty} I_{n,m} = 0$ where

$$I_{n,m} = \int_\Omega [F(\nabla u_n) - F(\nabla u_m)].\nabla(u_n - u_m)$$

$$= (J'(u_n) - J'(u_m))(u_n - u_m) + \int_\Omega [g(.,u_n) - g(.,u_m)](u_n - u_m).$$

On the other hand we have:

$$\|\nabla u_n - \nabla u_m\|_p^p \leqslant c\{I_{n,m}\}^{\frac{\alpha}{2}} \{\|\nabla u_n\|_p^p + \|\nabla u_m\|_p^p\}^{1-\frac{\alpha}{2}}$$

where $\alpha = \min(p,2)$ (for example see [11]).

Whence u_j converges to u in X; the Palais-Smale condition is satisfied. □

Proof of Theorem 1:

We shall apply Pass-Mountain Lemma [8] to the function J defined in Lemma 1. J satisfies Palais-Smale condition and $J(0) = 0$.

Let us show that, for $\|u\|_X = r$ sufficiently small, we have $J(u) \geqslant \alpha > 0$. By (i) there is some $c' > 0$ such that,

$\forall x \in \Omega$, $|u(x)| \leqslant c' \|u\|_X$; for $\|u\|_X \leqslant \frac{\zeta_1}{c'}$ we obtain with (ii):

$$G(x,u(x)) \leqslant c|u(x)|^{\sigma_1+1} \leqslant c(c')^{\sigma_1+1} \|u\|_X^{\sigma_1+1}$$

$$J(u) \geqslant \frac{1}{p} \|u\|_X^p [1 - c''\|u\|_X^{\sigma_1+1-p}]$$

For $\| u \|_x = r \leqslant \min [\frac{\zeta_1}{c'}, \frac{1}{2c''}]$ we get $J(u) \geqslant \frac{r^p}{2p} = \alpha > 0.$

Now, let us consider $u_0 \in X$ such that:

$$\forall x \in \Omega_0,\ u_0(x) \geqslant \alpha_0 > 0 \text{ and meas } (\Omega_0) > 0.$$

For λ sufficiently large, $\lambda\alpha_0 \geqslant \zeta_0$ and by (H3):

$$\int_\Omega G(.,\lambda u_0) \geqslant \int_{\Omega_0} G(.,\lambda u_0) \geqslant \beta\, \lambda^{\sigma_0+1}$$

where $\beta = \frac{1}{\zeta_0^{\sigma_0+1}} \int_{\Omega_0} G(x,\zeta_0) \mid u_0(x) \mid^{\sigma_0+1} dx > 0$

We then obtain

$$\lim_{\lambda\to+\infty} J(\lambda u_0) \leqslant \lim_{\lambda\to+\infty} [\frac{\lambda^p}{p} \| u_0 \|_X^p - \beta\, \lambda^{\sigma_0+1}] = -\infty$$

and there is some $v_0 \in X$, $v_0 \neq 0$, such that $J(v_0) = 0$.

By the Pass-Mountain lemma, there exists some $u_0 \in X$, $u_0 \not\equiv 0$, such that $J'(u_0) = 0$:

$$\forall v \in X, \int_\Omega F(\nabla u_0).\nabla v - \int_\Omega g(.,u_0)\, v = 0.$$

By TOLKSDORF's regularity results $u_0 \in C^{1,\alpha}(\Omega)$ [14], and by VAZQUEZ's maximum principle [16], $u_0 > 0$ in Ω. □

2. SOLUTIONS IN AN ORLICZ SPACE

Let us recall that a Young function M is an even convex function from $\mathbb{R}$ to $\mathbb{R}_+$, such that:

$$\lim_{\zeta\to 0} \frac{M(\zeta)}{\zeta} = 0 \text{ and } \lim_{\zeta\to+\infty} \frac{M(\zeta)}{\zeta} = +\infty.$$

The conjugate M^* of M is defined by:

$$M^*(\zeta) = \operatorname*{Sup}_{s \in \mathbb{R}} [\zeta s - M(s)]$$

The Orlicz space $L_M(\Omega)$ is the set of measurable functions u defined on $\mathbb{R}$ such that there is some $\lambda > 0$ with

$$\int_\Omega M\left(\frac{u}{\lambda}\right) < +\infty.$$

$L_M(\Omega)$ is a Banach space for the following norm:

$$\|u\|_M = \operatorname{Inf}\left\{ \lambda > 0 : \int_\Omega M\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}.$$

Let $E_M(\Omega)$ be the closure of $D(\Omega)$ in $L_M(\Omega)$.

We say that M is superhomogeneous of degree $(\sigma + 1)$ if there exists some $K > 0$ such that [11] :

$$\forall \zeta \in \mathbb{R},\ \forall h \in [0,1],\ M(h\zeta) \leqslant h^{\sigma+1} M(K\zeta).$$

Let Ω be a bounded regular open set in $\mathbb{R}^N$.

In the case when $N = p$, $W_0^{1,p}(\Omega) \not\subset L^\infty(\Omega)$, but $W_0^{1,p}(\Omega) \subset E_{M_1}(\Omega)$ [1] where

$$M_1(\zeta) = \exp|\zeta|^{p^*} - 1, \quad \frac{1}{p} + \frac{1}{p^*} = 1$$

So, we can get the following Theorem.

Theorem 2:

Let g satisfy the conditions (H1), (H2), (H3). Suppose that there exists a Young function of exponential type M such that:

(*i*) *The imbedding $W_0^{1,p} \hookrightarrow E_M(\Omega)$ is compact;*
(*ii*) *M is superhomogeneous of degree $\sigma_1 + 1 > p$;*
(*iii*) *There are some $c_1 > 0$* and $K_1 > 0$ such that:

$$\forall x \in \Omega,\ \forall \zeta \in \mathbb{R},\ \zeta g(x,\zeta) \leqslant c_1 M\left(\frac{\zeta}{K_1}\right);$$

(*iv*) $\forall K > 0$, $\displaystyle\lim_{\zeta \to \infty} \frac{g(x,\zeta)}{M'\left(\frac{\zeta}{K}\right)} = 0$, *uniformly in x.*

Then there is at least one positive solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\Omega)$ *of* (E).

Example:

Let $p = N = 2$; $g(x,\zeta) = \zeta^\sigma \exp(\zeta^q)$ with $\sigma > 1$, $0 < q < 2$, and

$$M(\zeta) = |\zeta|^{q+1-r}(e^{|\zeta|^r} - 1) \text{ with } q < r < 2.$$

$r < 2$ gives (i) [1]; $\zeta \to e^{|\zeta|^r} - 1$ is superhomogeneous of degree r, whence (ii); (iii) is easy and $q < r$ gives (iv).

So, the equation:

$$\Delta u + u^\sigma e^{u^q} = 0$$

has at least one positive solution $u \in W_0^{1,2}(\Omega)$.

In a similar case TRUDINGER [15] proves that for any $m > 0$, there exist $\lambda > 0$ and $u > 0$ such that $\int_\Omega G(.,u) = m$ and

$$\Delta u + \lambda\, g(x,u) = 0.$$

Our method allows us to eliminate this λ.

We obtain the same results for the equation:

$$\Delta_p u + u^\sigma e^{u^q} = 0$$

where $p = N \geqslant 2$, $\sigma > p-1$, $0 < q < \frac{p}{p-1}$

J being defined in lemma 1, the proof of Theorem 2 needs the following lemma.

Lemma 3:

If the hypothesis of Theorem 2 are satisfied, $J \in C^1(W_0^{1,p}(\Omega))$ *and satisfies the Palais Smale condition.*

Proof:

Let (u_j) be a bounded sequence in $W_0^{1,p}(\Omega)$.
By (i) there is some $K > 0$ such that:

$$\forall j, \int_{\Omega} M\left(\frac{u_j}{K}\right) \leqslant 1$$

Let $c > 0$ be such that $M^*(\frac{1}{c})\,\text{meas}(\Omega) < 1$ and:

$$\forall x \in \Omega,\ \forall \zeta \in \mathbb{R},\ |g(x, \zeta)| \leqslant \frac{c}{2} + \frac{1}{2} M'(\frac{\zeta}{K}).$$

We obtain:

$$(1) \int_{\Omega} M^*\left[\frac{g(.,u_j)}{c^2}\right] \leqslant \int_{\Omega} \frac{1}{2} M^*(\frac{1}{c}) + \int_{\Omega} \frac{1}{2} M(\frac{u_j}{K}) \leqslant 1.$$

Let u_j converges to u in $W_0^{1,p}(\Omega)$. For sufficiently small δ and for meas $(A) < \delta$, we have:

$$\int_A M^*\left[\frac{g(.,u_j)}{c^2}\right]$$

$$\leqslant \frac{1}{2} M^*\left(\frac{1}{c}\right) \text{meas}(A) + \frac{1}{4} \int_A M\left(\frac{u_j - u}{K}\right) + \frac{1}{4} \int_A M\left(\frac{u}{K}\right) \leqslant \varepsilon.$$

$M^*\left[\frac{g(.,u_j) - g(.,u)}{c^2}\right]$ is then an equi-summable sequence and

$$\lim_{j \to +\infty} \int_{\Omega} M^*\left[\frac{g(.,u_j) - g(.,u)}{c^2}\right] = 0$$

By (ii) M^* satisfies the "Δ_2-condition" [11], so $\lim \|g(.u_j) - g(.,u)\|_{M^*} = 0$; whence $J \in C^1(W_0^{1,p}(\Omega))$.

Suppose now that $|J(u_j)| \leqslant K_1$ and $J'(u_j) \to 0$. By lemma 1, $\|u_j\|_{W^{1,p}}$ is bounded and, by (i), u_j converges in $E_M(\Omega)$; by relation (1), $g(.,u_j)$ converges for $\sigma(L_{M^*}, E_M)$. So the same proof than for lemma 2 shows that the Palais-Smale condition is satisfied. □

Proof of Theorem 2:

Let us show that for $\|u\|_{W_0^{1,p}} = r$ sufficiently small we have $J(u) \geqslant \alpha > 0$.

By (iii) and (ii), we have

$$\forall x \in \Omega,\ \forall \zeta \in \mathbb{R},\ \forall h \in [0,1],\ G(x,\zeta) \leqslant c_l\, M\left(\frac{\zeta}{K_l}\right) \leqslant h^{\sigma_l+1}\ M\left(\frac{K\zeta}{K_l h}\right)$$

By (i)

$$\forall u \in W_0^{1,p}(\Omega),\ \|u\|_M \leqslant c\, \|u\|_{W_0^{1,p}}.$$

Whence for $\|u\|_{W_0^{1,p}} = r \leqslant \dfrac{K_l}{cK}$ and $h = \dfrac{cKr}{K_l}$:

$$\int_\Omega G(.,u) \leqslant c_l \int_\Omega M\left(\frac{u}{K_l}\right) \leqslant c_l\, h^{\sigma_l+1} \int_\Omega M\left(\frac{u}{cr}\right) \leqslant c_l\, h^{\sigma_l+1} = c'\, \|u\|_{W_0^{1,p}}^{\sigma_l+1}$$

The same proof than for Theorem 1 gives $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, solution of (E). The end of the proof is a consequence of the following lemma. □.

Lemma 4:

If all the hypothesis of Theorem 2 are satisfied, $u \in C^{1,\alpha}(\Omega)$.

Proof:

This proof is very similar to OTANI's one [9] (see also [13]). By (iii) there is some $s > 1$ such that $ug(x,u) \in L^s(\Omega)$.

Consider the following sequences:

$$q_l = 2ps^* = 2ps/(s-1)$$
$$q_{k+1} = 2(p+q_k)$$
$$\theta = s^*\, q_k.$$

Multiplying (E) by $|u|^{q_k} u$, we obtain:

$$\left(\frac{p}{p+q_k}\right)^p \int_\Omega \left|\nabla\left(u^{1+\frac{q_k}{p}}\right)\right|^p = \int_\Omega u\, g(.,u)\, |u|^{q_k}$$

$$\leqslant \|u\, g(.,u)\|_s\, \|\, u^{q_k}\|_{s^*} \leqslant c\, \|u\|_{s^*}^{q_k}.$$

M being of exponential type, $W^{1,p}(\Omega) \hookrightarrow L^{2ps^*}(\Omega)$ and there is some K such that:

$$\|u\|^{p+q_k}_{2s^*(p+q_k)} \leqslant K^p \int_\Omega \left|\nabla\left(u^{1+\frac{q_k}{p}}\right)\right|^p$$

We then obtain:

$$\|u\|^{\theta_{k+1}/2s^*}_{\theta_{k+1}} \leqslant c\left(\frac{K(p+q_k)}{p}\right)^p \|u\|^{\theta_k/s^*}_{\theta_k}$$

This formal proof can be made rigorous by using some regularized equation [13].

Observing that $p + q_k \leqslant 4^{k-1}\, 4ps^*$, we get:

$$\|u\|^{\theta_{k+1}}_{\theta_{k+1}} \leqslant c^{2s^*}\,(4Ks^*)^{2ps^*}\, 4^{2(k-1)ps^*} \|u\|^{2\theta_k}_{\theta_k}$$

Let:

$E_k = \theta_k \operatorname{Log} \|u\|_{\theta_k}$

$a = 4^{2ps^*}$

$b = \operatorname{Log}[c^{2s^*}(2Ks^*)^{2ps^*}]$

$r_k = b + (k-1)\operatorname{Log} a.$

We then obtain:

$$E_{k+1} \leqslant r_k + 2E_k$$

Whence, following OTANI [9], we deduce:

$$\|u\|_\infty \leqslant \overline{\lim_{k\to+\infty}} \exp\left(\frac{E_k}{\theta_k}\right) < +\infty$$

So $u \in L^\infty(\Omega)$ and by TOLKSDORF's results $u \in C^{1,\alpha}(\Omega)$. □

3. A PARTICULAR CASE : Ω IS A BALL

In the particular case when Ω is a ball and $N = p$, we can obtain radially symmetric solutions of (E), for any growth of g near infinity.

For simplicity we suppose that g does not depend on x ; we assume the following conditions:

(H4) $g \in C^1(\mathbb{R})$, $g \geqslant 0$ and $g(0) = 0$;

(H5) g and g' are non decreasing on $\mathbb{R}_+$;

(H6) $\lim_{t \to 0} \frac{g(t)}{t^{p-1}} = 0$

Theorem 3:

Let $N = p \geqslant 2$ and let g satisfy the conditions (H4), (H5), (H6). Then, there exists R_0 such that, for $R \geqslant R_0$, the equation

$$(E) \qquad \Delta_p u + g(u) = 0 \text{ in } \Omega = B(0,R)$$

admits at least one positive radially symmetric solution $u \in W_0^{1,p}(\Omega)$.

Example:

For any $\sigma > \max(1, p-1)$ and any $q > 1$, $g(\zeta) = |\zeta|^\sigma \exp|\zeta|^q$ satisfies (H4), (H5), (H6).

Theorem 3 is a consequence of the following proposition. Let us consider the following system:

$$(S) \begin{cases} v'(x) = |w(x)|^{p^*-2} w(x) \\ w'(x) = -\dfrac{e^{-x}}{p^p} g[v(x)] \end{cases}$$

where $p^* = \dfrac{p}{p-1}$

Submitted to the conditions:

$$(L.C.) \begin{cases} \lim_{x \to +\infty} v(x) = m \\ \lim_{x \to +\infty} w(x) = 0. \end{cases}$$

Proposition 2:

Let $p \geqslant 2$ and let g satisfy the conditions (H4), (H5), (H6). Then, for any $m > 0$, (S) + (L.C.) admits one and only one solution (v,w) ; there exists some $\alpha = \theta(m) \in \mathbb{R}$ such that:

$$v(\alpha) = 0 \text{ and } v > 0 \text{ on }]\alpha, +\infty].$$

Moreover θ *is continuous on* $\mathbb{R}_+$ *and* $\lim_{m\to 0} \theta(m) = -\infty$.

Proof:

Let us consider the following iterations;
$v_0 = 0$ and for $n \in \mathbb{N}$:

$$w_n(x) = \int_x^{+\infty} \frac{e^{-t}}{p^p} g[v_n(t)]dt$$

$$v_{n+1}(x) = m - \int_x^{+\infty} |w_n(t)|^{p^*-2} w_n(t)\, dt.$$

We have:

$$w_1(x) = \frac{g(m)}{p^p} e^{-x} > w_0 = 0$$

$$v_2(x) = m - \frac{g(m)}{p^p(p^*-1)} \exp[-(p^*-1)x] < m = v_1(x)$$

There is some $M(m,p)$ such that:

- $\forall x \geqslant M(m,p),\ v_2(x) \geqslant \frac{m}{2} > v_0(x)$ and

$$w_2(x) = \int_x^{+\infty} \frac{e^{-t}}{p^p} g[v_2(t)]\, dt \geqslant \frac{f(\frac{m}{2})}{p^p} e^{-x}.$$

By induction we can prove that for any $q \in \mathbb{N}$, v_{2q} is a nondecreasing sequence, v_{2q+1} is a noninncreasing sequence and $v_{2q} \leqslant v_{2q+1}$; whence for any n we have either $v_n \leqslant v_{n+1}$, or $v_{n+1} \leqslant v_n$.

Suppose that $n \geqslant 2$ and $v_n \leqslant v_{n+1}$; we have $w_n \leqslant w_{n+1}$ and $v_{n+2} \leqslant v_{n+1}$.

$p^* \leqslant 2$ and $w_n \geqslant w_2$, whence :

$$|w_{n+1}(t)|^{p^*-2} w_{n+1}(t) - |w_n(t)|^{p^*-2} w_n(t) \leqslant (p^*-1)\, |w_2(t)|^{p^*-2}\, [w_{n+1}(t) - w_n(t)].$$

We then obtain:

$$0 \leqslant v_{n+1}(x) - v_{n+2}(x) \leqslant \frac{p^*-1}{p^*-2} \left(\frac{g\left(\frac{m}{2}\right)}{p^p} \right)^{p^*-2} \exp[-(p^*-2)x] \sup_{t \in [x,+\infty]} |w_{n+1}(t) - w_n(t)| .$$

On the other hand, by (H5), we get:

$$0 \leqslant w_{n+1}(x) - w_n(x) \leqslant \frac{g'(m)e^{-x}}{p^p} \sup_{t \in [x,+\infty]} |v_{n+1}(x) - v_n(x)|.$$

Therefore:

$$\sup_{t \in [x,+\infty]} |v_{n+2}(x) - v_{n+1}(x)| \leqslant c(x) \sup_{t \in [x,+\infty]} | v_{n+1}(t) - v_n(t)|.$$

And, for $x \geqslant M_1(m,p) \geqslant M(m,p)$, we have:

$$c(x) = \frac{p^*-1}{p^*-2} \left(\frac{g\left(\frac{m}{2}\right)}{p^p} \right)^{p^*-2} \frac{g'(m)}{p^p} \exp[-(p^*-1)x] < 1$$

By Picard's theorem we obtain a unique solution (v,w) of (S) + (L.C) for $x \geqslant M_1(m,p)$. By classical differential equations theory this solution can be continued for $x < M_1(m,p)$. Since v has increasing gradient, it has a last zero at a point $x = \alpha = \theta(m)$.

Let us set:

$$H(x,m) = \int_x^{+\infty} |w(t,m)|^{p^*-2} w(t,m)\,dt - m.$$

$\frac{\partial H}{\partial x}(\alpha,m) \neq 0$ and by implicit functions theorem θ is continuous.

For $x \in \,]\alpha, +\infty]$, we have $0 < v(x) \leqslant m$, whence:

$$w(x) \leqslant \int_x^{+\infty} \frac{e^{-t}}{p^p} g(m) = \frac{e^{-x}}{p^p} g(m)$$

$$m = \int_{\alpha}^{+\infty} v'(x)\,dx \leqslant \left(\frac{g(m)}{p^{p}}\right)^{p^*-1} \frac{e^{-(p^*-1)\alpha}}{p^*-1}$$

$$\lim_{m\to 0} (p^*-1)\, p^{p} e^{(p^*-1)\theta(m)} \leqslant \lim_{m\to 0} \left(\frac{g(m)}{m^{p-1}}\right)^{p^*-1} = 0$$

whence $\lim_{m\to 0} \theta(m) = -\infty$. □

Proof of Theorem 3:

By proposition 2, there is some α_0 such that for any $\alpha = -p \operatorname{Log} R \leqslant \alpha_0$, (S)+ (L.C.) has one and only one solution such that $v(\alpha, m) = 0$.
The change of variable $x = -p \operatorname{Log} r$, $v(x) = \varphi(r)$ transforms (S) into the equation:

$$\frac{d}{dr} \{ | r\varphi'(r) |^{p-2}\, r\varphi'(r) \} + r^{p-1} g[\varphi(r)] = 0$$

which is the radial form of the equation (E), with boundary condition $\varphi(R) = 0$. □

Remark: The deep study of the case $p = 2$ made by HEMPEL [5] and NEHARI [6] shows that there is no hope to find a solution of (E) for any R, if the growth of g has no bound when $u \to +\infty$.

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