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Quasipositivity and new knot invariants

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ABSTRACT. This is a survey (including new results) of relations —some emergent, others established— among three notions which the 1980s saw introduced into knot theory: *quasipositivity* of a link; the *enhanced Milnor number* of a fibered link; and the new *link polynomials.* The Seifert form fails to determine these invariants; perhaps there exists an «enhanced Seifert form» which does.

CONTENTS

0. PRELIMINARIES

Manifolds, in the absence of indications to the contrary, are understood to be oriented, compact, and smooth; maps are smooth.

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0.1. Knots, links, surfaces. A *link* is a 1-submanifold of S^3, non-empty and without boundary; a *knot* is a connected link. A *link type* is an ambient isotopy class of links. A *surface* is a 2-manifold which is connected relative to its non-empty boundary; a surface F embedded in S^3 is a *Seifert surface (for ∂F)*. A knot O for which some Seifert surface is a 2-disk is called an *unknot*. The *mirror image* of $X \subset S^3$, denoted by Mir X, is the image of X by an orientation-reversing diffeomorphism of S^3.

Let F be a Seifert surface. A push-off map $F \to S^3 \backslash F: x \to x^+$ (unique up to ambient isotopy) is defined by a field of positive normal vectors on F; if b and c are 1-cycles on F then link(b, c^+) depends only on the classes of b and c in $H_1(F;\mathbb{Z})$; if 1-cycles $c_1, \ldots, c_d$ give a basis for $H_1(F;\mathbb{Z})$, then $[\mathrm{link}(c_i, c_j^+)] =: L_F$ is a *Seifert matrix* for F (or ∂F), and represents the *Seifert form* of F.

If K is a knot and n is an integer, let $A(K, n)$ denote an annulus of *type* K *with* n *twists*--that is, $A(K, n)$ is a Seifert surface containing K, the class of K generates $H_1(A(K, n);\mathbb{Z})$, and $L_{A(K,n)}$ is the 1-by-1 matrix $[n]$. (Note that the linking number of the two components of $A(K, n)$ is $-n$.) More generally, if L is a link and f is an *integer framing* of L (that is, f assigns an integer to each component of L), then $A(L, f)$ denotes the corresponding union of annuli.

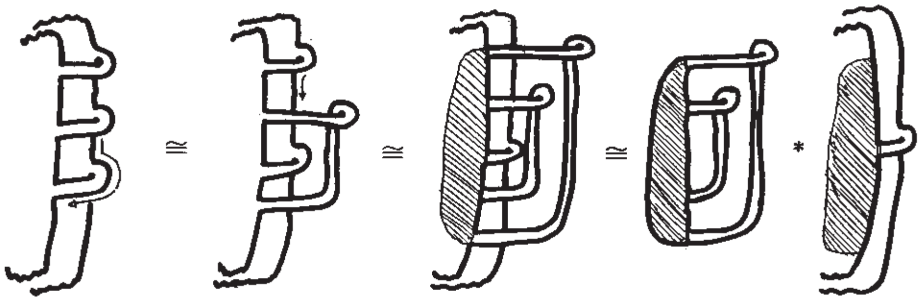
The *positive Hopf annulus* is $A(O, -1)$ (its oriented boundary is a pair of fibers of a positive Hopf fibration of S^3); the *negative Hopf annulus* is $A(O, 1) = \mathrm{Mir}\, A(O, -1)$.

A link L is *fibered* if there is a fibration of $S^3 \backslash L$ over S^1 such that the closure of each fiber is a Seifert surface for L (for example, O is fibered); a *fiber surface* is any such Seifert surface. The fibration of a fibered link is unique up to isotopy and a fibered link determines its fiber surface up to isotopy.

Let K be a knot with tubular neighborhood $N(K)$. A *cable of type (m, n) on K*, where $m > 0$ and n are integers, is a link $K\{m, n\}$ which lies on $\partial N(K)$, is homologous in $N(K)$ to mK, has linking number n with K, and has GCD(m, n) components. If K is fibered then $K\{m, n\}$ is fibered if and only if $n \neq 0$ or $m = 1$. In particular, $O\{2, 2\} = \partial A(O, -1)$ and $O\{2, -2\} = \partial A(O, 1)$ are fibered. The positive and negative Hopf annuli are the only annuli which are fiber surfaces.

If the Seifert surface F is the union of subsurfaces F_1 and F_2, whose intersection is a 2k-gonal 2-disk with alternate edges on ∂F_1 and ∂F_2, and if there is a 3-disk D^3 in S^3 with $F_1 = D^3 \cap F$, $F_2 = (S^3 \backslash \mathrm{Int}\ D^3) \cap F$, then F is called a *Murasugi sum* of F_1 and F_2, denoted $F = F_1 * F_2$; if also $k = 2$ and $F_2 = A(K, n)$ and the 4-gon $F_1 \cap A(K, n)$ meets both components of $\partial A(K, n)$, then F is a *plumbing of $A(K, n)$ to F_1*.

A Murasugi sum of two fiber surfaces is a fiber surface. If K is fibered, $m>1$, and $n\neq 0$, then ([N & R3]) the fiber surface for $K\{m,n\}$ is the Murasugi sum of a fiber surface for $K\{m,n/|n|\}$ and a fiber surface for $O\{m,n\}$, cf. Figure 0.2.



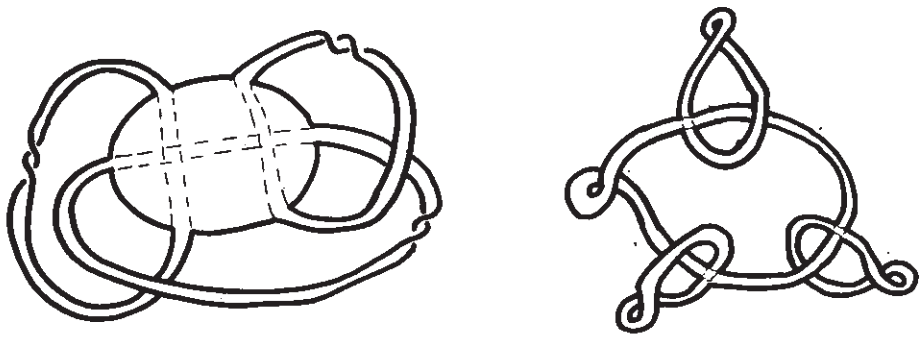
$K\{2,3\}=\partial F$ $O\{2,3\} * K\{2,1\}$

0.2. FIGURE

A *Hopf-plumbed* surface is either a disk or a plumbing of $A(O,\pm 1)$ to a Hopf-plumbed surface; a Hopf-plumbed surface F is *flat* if

$$F=(...((D^2 * A_1) * A_2...) * A_m,\ A_i=A(O,s(i)),\ s(i)=\pm 1,$$
$$\text{where } ((...((D^2 * A_1) * A_2...) * A_{i-1}) \cap A_i \subset D^2 \text{ for } i=2,...,m.$$

Figure 0.3 illustrates some Hopf-plumbed surfaces.



flat not flat

0.3. FIGURE

0.4. Braids and bands. For elements x, y of any group, we will write ${}^{x}y:=xyx^{-1}$, $[x,y]:={}^{x}yy^{-1}$, $c(x,y):=xyxy^{-1}x^{-1}y^{-1}$. The *usual presentation of the n–string braid group* B_n has generators $\sigma_i (1\leq i\leq n-1)$, and relators $[\sigma_i, \sigma_j]$ $(1\leq i<j-1\leq n-1)$ and $c(\sigma_i, \sigma_{i+1})$ $(1\leq i\leq n-2)$.

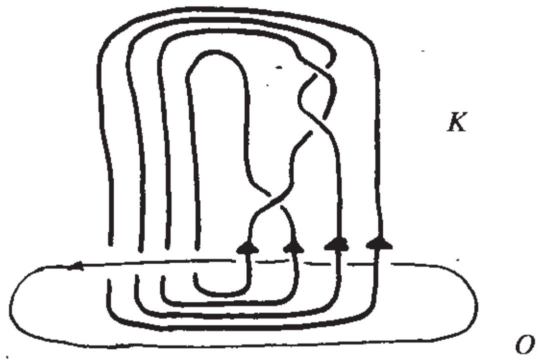
It is convenient to generalize this ([Rul 3,15]). Let T be a tree with vertex set $\{1,...,n\}$ (the edges of T are unordered pairs of vertices); T is *espaliered* if, whenever $1\leq i<j<k<m\leq n$, then $\{i,k\}$ and $\{j,m\}$ are not both edges of T. To an espaliered tree T corresponds a *T-standard* group presentation, as follows: there is a (*T-standard*) generator $\sigma_{\mathbf{e}}$ for each edge $\mathbf{e}$ and a relator for each pair $\{\mathbf{e},\mathbf{f}\}$ of distinct edges; for $\mathbf{e}$ and $\mathbf{f}$ disjoint, the corresponding relator is $[\sigma_{\mathbf{e}}, \sigma_{\mathbf{f}}]$; for $\mathbf{e}$ and $\mathbf{f}$ with one common vertex, the corresponding relator is $c(\sigma_{\mathbf{e}}, \sigma_{\mathbf{f}})$.

Let $I=\{\{1,2\},...,\{i,i+1\},...,\{n-1,n\}\}$. If we abbreviate $\sigma_{\{i,i+1\}}$ to σ_i, then the I-standard presentation is exactly the usual presentation of B_n; more generally, for any espaliered tree T, the group of the T-standard presentation is isomorphic to B_n, and becomes identical with it if, for $1\leq i<j\leq n$, we identify $\sigma_{\{i,j\}}$ with $\sigma_{i,j}:=(\sigma_i\sigma_{i+1}...\sigma_{j-2})\sigma_{j-1}(\sigma_i\sigma_{i+1}...\sigma_{j-2})^{-1}\in B_n$. The *exponent sum* $e(\beta)$ of $\beta\in B_n$ with respect to the T-standard generators is independent of T; in fact $e: B_n\to\mathbb{Z}$ is the abelianization homomorphism.

The $\binom{n}{2}$ elements $\sigma_{i,j}$ of B_n are called *positive embedded bands* in B_n; their inverses are *negative embedded bands*. A *positive* (resp., *negative*) *band* is any conjugate $w^{\pm 1}_{\sigma_{i,j}}$ of a positive (resp., negative) embedded band; all bands of a given sign are, in fact, mutually conjugate. A *band representation* of $\beta\in B_n$ is a word $\mathbf{b}=(b(1),...,b(k))$ where each $b(s)$ is a band in B_n and $\beta=\beta(\mathbf{b}):=b(1)...b(k)$; $\mathbf{b}$ is *embedded* if each $b(s)$ is embedded. A braid is *quasipositive* if it is a product of positive bands.

0.5. Closed braids. With respect to a given fibration $\pi: S^3\backslash O\to S^1$ for the unknot, a link L (disjoint from O) is a *closed braid (on n strings)* if π/L is an orientation-preserving covering map (of degree n). Figure 0.6 (where the *axis* O is drawn in, and π is left to the imagination) is a reminder of the familiar way to construct a closed braid $\beta^{\wedge}$, called the *closure* of β, from a braid $\beta\in B_n$; it also establishes orientation conventions. Conjugate elements of B_n determine closed braids of the same link type, and, conversely, a closed braid determines a conjugacy class in B_n. A well-known theorem of Alexander says that (with O and π fixed) every link type contains closed braids.

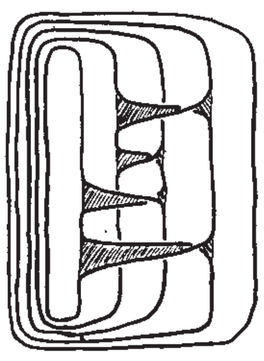
0.7. Braided surfaces. Figure 0.8 shows how to construct a Seifert surface $S(\mathbf{b})$, equipped with a handle decomposition into n 0-handles and k 1-handles, from an embedded band representation $\mathbf{b}=(b(1),...,b(k))$ in B_n; the boundary of $S(\mathbf{b})$ is $\beta^{\wedge}(\mathbf{b}):=\beta(\mathbf{b})^{\wedge}$. Such an $S(\mathbf{b})$ is called a *braided*



$K = (\sigma_1 \sigma_2^{-2})^\wedge$, $n = 4$

0.6. FIGURE

surface (as always, with respect to a given fibration $\pi: S^3 \backslash O \to S^1$ for the unknot). Every ambient isotopy class of Seifert surfaces in S^3 contains braided surfaces $S(\mathbf{b})$; it is not true, however, that given a Seifert surface S for $\beta^\wedge$, there is necessarily an embedded band representation $\mathbf{b}$ of β such that $S(\mathbf{b})$ is ambient isotopic to S by an isotopy fixing $\beta^\wedge$.



$S(\mathbf{b})$

0.8. FIGURE

(Note: in [Ru4, 15] it is shown how to construct a «Seifert *ribbon*» $S(\mathbf{b})$ --that is, according to taste, either a ribbon-immersed surface in S^3, or a ribbon-embedded surface in D^4, in either case bounded by $\beta^\wedge(\mathbf{b})$ --from a not-necessarily-embedded band representation $\mathbf{b}$. Except for Remark 4.6, we will ignore this more general situation.)

0.9. Generalized homogeneous braids. A *T-braidword* is an embedded band representation $\mathbf{b}$ such that, for every s, either $b(s)$ or $b(s)^{-1}$ is a T-standard generator. If every T-standard generator appears either as some $b(s)$ or as some $b(s)^{-1}$, then $\mathbf{b}$ is *strict;* if no generator of the T-standard presentation appears both as some $b(s)$ and as some $b(t)^{-1}$, then $\mathbf{b}$ is *homogeneous.* A *T-braidword surface* $S(\mathbf{b})$ is a fiber surface if and only if $\mathbf{b}$ is strict and homogeneous. (The «if» statement is proved by [St] and [B&W], though these authors treat explicitly only the case $T=I$; cf. also [Rul 4]. Here is a sketch proof of the «only if» statement: (1) if $\mathbf{b}$ is not strict, then $S(\mathbf{b})$ is easily seen to be disconnected; (2) if $\mathbf{b}$ is not homogeneous, then $S(\mathbf{b})$ is almost as easily seen to be compressible; but (3) a fiber surface is connected and incompressible.) According to [Rul 3], the class of strict homogeneous T-braidword surfaces (for all possible n and T) is coextensive with the class of flat Hopf-plumbed surfaces. (This is somewhat sharper than the combination of the two well-known facts that (1) a strict homogeneous T-braidword surface $S(\mathbf{b})$ is an iterated Murasugi sum of surfaces $S(\mathbf{b}_1), \ldots, S(\mathbf{b}_{n-1})$ where each $\mathbf{b}_i$ is a strict homogeneous I-braidword in B_2, and (2) each strict homogeneous I-braidword surface in B_2 is a flat Hopf-plumbed surface.)

1. REVIEW OF QUASIPOSITIVITY

1.1. Definition. *A Seifert surface F is* quasipositive *if it is ambient isotopic to a braided surface* $S(\mathbf{b})$ *where each* $b(s)$ *in the embedded band representation* $\mathbf{b}$ is positive. A link L is quasipositive *if L is ambient isotopic to the closure of a quasipositive braid, and* strongly quasipositive *if there is a quasipositive Seifert surface for L.* ❙

1.2. It is known that no invariant of the Seifert form (e.g., Alexander polynomial, equivariant signatures) can detect the presence or absence of (even strong) quasipositivity.

Theorem [Ru 5]. *Let* F *be a Seifert surface,* $\mathrm{L_F}$ *its Seifert matrix (with respect to some homology basis of 1-cycles). Then there is an embedding* $\mathrm{i}:\mathrm{F}\to\mathrm{S}^3$ *such that* $\mathrm{i(F)}$ *is quasipositive and the Seifert matrix* $\mathrm{L_{i(F)}}$ *(with respect to the corresponding homology basis of 1-cycles) equals* $\mathrm{L_F}$. ❙

1.3. A subsurface G of a surface F is *full* if every simple closed curve on G which bounds a disk on F already bounds a disk on G.

Theorem [Rul]. *A full subsurface of a quasipositive surface is quasipositive.* ❙

1.4. For integers m, n, with $m>0$, $n\neq 0$, let $\mathbf{d}\{m,n\}$ be the I-braidword of length $(m-1)|n|$ in B_m with $d\{m,n\}(i+(m-1)j)=\sigma_i$ $(i=1,\ldots,m-1,$

$j=0,\dots,n-1$) if $n>0$, $d(i+(m-1)j)=\sigma_i^{-1}$ ($i=1,\dots,m-1$, $j=0,\dots,-n-1$) if $n<0$. Then $\mathbf{d}\{m n\}$ is strict and homogeneous, so $S(\mathbf{d}\{m,n\})$ is a fiber surface (in fact its boundary is $O\{m,n\}$, the torus link of type (m,n)).

Theorem [Ru1]. *A Seifert surface F is quasipositive if and only if, for some* $n>0$, F *is ambient isotopic to a full subsurface of* $S(\mathbf{d}(n,n))$. ∎

This should be compared with a theorem of Herbert Lyon [*L*], which shows that, for *any* Seifert surface *F*, there is an $n>0$ such that *F* is ambient isotopic to a subsurface of the boundary— connected sum of $S(\mathbf{d}(n,n))$ with its mirror image $S(\mathbf{d}(n,-n))$.

1.5. Theorem [Ru6]. *For any* K, *there exists* $q\in\mathbb{Z}$ *such that* $A(K,n)$ *is quasipositive if* $n\leqq q$. *(More generally, for any link* L, *there exists a framing* f *of* L *such that* $A(L,f')$ *is quasipositive if* f′ *is less than or equal to* f *componentwise.)* ∎

1.6. Theorem [Ru13]. *A plumbing* $F=F_1 * A(K,n)$ *is quasipositive if (and, by 1.3, only if) both* F_1 *and* $A(K,n)$ *are.* ∎

1.7. Conjecture. *An arbitrary Murasugi sum of quasipositive surfaces is quasipositive.*

APPENDIX to Section 1: Knot theory of complex plane curves.

For a more detailed survey of the knot theory of complex plane curves, up to 1982, the reader is referred to [Ru14] (where, regrettably, Suzuki's 1974 paper [Su] went unnoticed). Some post-1982 references are included below, as appropriate, but I make no claims for completeness.

Let $\Gamma\subset\mathbb{C}^2$ be a complex-algebraic curve (reduced but not necessarily non-singular or irreducible), $(0,0)\in\Gamma$. For $r>0$, set $D^4(r):=\{(z,w)\in\mathbb{C}^2: |z|^2+|w|^2\leq r^2\}$, $S^3(r):=\partial D^4(r)$.

1A.1. Problem. *Describe the topological type of the pair* $(D^4(r), D^4(r)\cap\Gamma)$.

In other words, study complex curves in complex 2—space *via* their topological placement in the large--i.e., not necessarily either «in the small» (infinitesimally) or globally, but in a «middle range» (which at its limits encompasses both extremes).

By «passing to the boundary of the situation» we may pose a more specific problem. The (dense open) subset $R(\Gamma)$ of regular points of Γ is of course a smooth 2-submanifold of $\mathbf{C}^2$; for all but finitely many radii $r \in]0, \infty[$, $S^3(r)$ intersects Γ only at points of $R(\Gamma)$, and there transversely, so that the (naturally oriented) intersection $L(\Gamma, r) := S^3(r) \cap \Gamma$ is a link (it is not empty, by the maximum modulus principle for Γ).

1A.2. Problem. *Describe the link type of* $L(\Gamma, r)$.

In the extreme cases 1A.2 is solved, or nearly solved; sometimes its solution implies a solution for 1A.1.

1A.3. Example. For fixed Γ, and all sufficiently small $r > 0$, the link type of $L(\Gamma, r)$ is constant, known as the *link of the singularity of* Γ *at* (0, 0); let $L(\Gamma, 0)$ denote a representative of this link type. (For instance, if (0, 0) $\in R(\Gamma)$ then $L(\Gamma, 0) = O$.) Links of singularities are completely classified, and their topology is very well understood (cf. [E&N], [M&W], and references cited therein). Here are some facts: $L(\Gamma, 0)$ is an *iterated torus link,* obtained from an unknot O by successive cabling operations; $L(\Gamma, 0)$ is fibered, [*Mi*]; if $L(\Gamma, 0)$ is a slice knot (i.e., the boundary of some smoothly embedded 2-disk in the 4-disk) then it is trivial [*Lê*]). Furthermore [*Mi*], for small *r*, the pair $(D^4(r), D^4(r) \cap \Gamma)$ is homeomorphic to the cone on $(S^3, L(\Gamma, 0))$, so in this case Problem 1 also is solved. ∎

1A.4. Example. For fixed Γ, and all sufficiently large *r*, the link type of $L(\Gamma, r)$ is constant, known as the *link-at-infinity of* Γ; let $L(\Gamma, \infty)$ denote a representative of this link type. Links-at-infinity have been much less studied than links of singularities; they are partially classified, and a good understanding of their topology is beginning to emerge (cf. [Su], [Ru7], [N&R3], and especially the beautiful paper [*Ne*]). Here are some facts: $L(\Gamma, \infty)$ is an iterated torus link; $L(\Gamma, \infty)$ need not be fibered, but is often «approximated» by a fibered link (or «fibered multilink»), [*Ne*]; if $L(\Gamma, \infty)$ is a slice knot then it is trivial, [Ru7]. Furthermore, according to [Ne], $L(\Gamma, \infty)$ often (but not always) determines $(D^4(r), D^4(r) \cap \Gamma)$ for large *r* (and non-singular Γ)--again, a solution to 1A.1 in an extreme case. ∎

In contrast with links of singularities and links-at-infinity, general links $L(\Gamma, r)$ seem hard to get one's hands on (although some progress has been made by Fiedler [*F*1-2]). They need not be iterated torus links, they need not be fibered (even approximately), and they can be slice but highly nontrivial, [Ru2].

1A.5. Caution. In [Ru11] it is shown that, given an arbitrary pair (D^4, S), where S is an oriented surface (without closed components) smoothly and properly embedded in D^4, there is a smooth embedding $i: D^4 \hookrightarrow \mathbf{C}^2$ and a complex-algebraic curve Γ such that $i(S)$ is a connected component of $\Gamma \cap i(D^4)$. Thus 1A.1 and 1A.2 may become uninteresting if modified to omit such geometrical hypotheses as the roundness of $D^4(r)$ (the point is that one has no control over the geometry of the embedding; $i(D^4)$ may be very twisted--e.g., not convex or even pseudoconvex). ∎

To make further progress, we change the terms of the problems. Given Γ, after an arbitrarily small unitary change of coordinates in $\mathbb{C}^2$ we may assume that its reduced defining polynomial $f \in \mathbb{C}[z, w]$ is monic in w when written in Weierstrass form, i.e., $f(z, w) = w^n + f_1(z) w^{n-1} + \ldots + f_n(z)$ for some $n > 0$ and $f_i(z) \in \mathbb{C}[z]$. There is a finite set Z such that, if $z \in \mathbb{C} \setminus Z$, then $\{w : f(z, w) = 0\}$ contains n distinct points. Let $R \subset \mathbb{C}$ be a smooth 2-disk such that $\partial R \subset \mathbb{C} \setminus Z$. There exists $M > 0$ such that, if $z \in \partial R$ and $f(z, w) = 0$, then $|w| < M$; a maximum modulus argument (using the monicity of f in w) shows that then $|w| < M$ whenever $z \in R$ and $f(z, w) = 0$. The product $D := R \times \{w \in \mathbb{C}: |w| \leq M\}$ is a piecewise-smooth 4-disk; ∂D is a 3-sphere-with-corners equipped with a natural genus-1 Heegard splitting into smooth solid tori $\partial_1 D := \partial R \times \{w \in \mathbb{C}: |w| \leq M\}$ and $\partial_2 D$, and we have just seen that $\Gamma \cap \partial D = \Gamma \cap \partial_1 D$. In fact (with the notion of closed braid modified in an obvious way) *the link* $\Gamma \cap \partial D$ *is a closed braid* in ∂D.

1A.6. Theorem [Ru2]. *Such a closed braid is quasipositive. Conversely (even if* D *is restricted to be* $\{(z, w) : |z| \leq 1, |w| \leq 1\}$*), up to isotopy through closed braids every quasipositive closed braid can be realized as* $\Gamma \cap \partial D$ *for some (non-singular) complex algebraic curve* Γ*. Furthermore (after identifying* ∂D *with* S^3 *by rounding its corners), every quasipositive closed braid* $\Gamma \cap \partial D$ *can be realized as* $L(\Gamma', r)$ *for some* Γ' *and* $r > 0$. ∎

This was the original motivation for studying quasipositive links.

1A.7. The link of a singularity is quasipositive; that is, though $L(\Gamma, 0)$ is defined *a priori* as the intersection of Γ with a small round sphere, a link of the same type can be realized (after at worst a linear change of coordinates) as the closed braid intersection of a curve and a bidisk boundary, so 1A.6 applies. (This is a standard trick, cf. [*Lê*2], which basically boils down to the existence of tangent lines for the branches of a singularity.) In fact, it can be seen that the fiber surface of $L(\Gamma, 0)$ is quasipositive, so *the link of a singularity is strongly quasipositive.*

A link-at-infinity is quasipositive (again, the proof is easy), but *need not be strongly quasipositive* (cf. the last paragraph of §4).

As in [Ru5], it is still not known (to me) whether or not every link $L(\Gamma, r)$ is quasipositive.

2. REVIEW OF THE ENHANCED MILNOR NUMBER

Although the theory of the enhanced Milnor number can be extended in various ways (to fibered links and multilinks in other 3-manifolds, [Ru8], [N&R1]; to fibered links in higherdimensional spheres, [N&R2]; perhaps, using work of Gabai, to arbitrary non-split links in S^3), this review will be limited to fibered links in S^3, approached *via* isolated critical points.

We identify the three real vectorspaces $\mathbb{R}^4$, $\mathbb{C}^2$, and $\mathbb{H}$ (the real quaternions), in the usual way. Then the group $S^3 = S^3(1) \subset \mathbb{H}$ of unit quaternions contains the (quaternionic) square roots of -1 as its great 2-sphere S^2 of pure unit quaternions.

If M is a 2-by-4 real matrix of rank 2, let $(\mathbf{u}(M), \mathbf{v}(M))$ be the orthonormal frame obtained by applying the Gram-Schmidt process to the rows of M: so $\mathbf{u}(M)$ and $\mathbf{v}(M)$ belong to S^3 and are mutually orthogonal. Then $\mathbf{p}(M) := \mathbf{v}(M)\mathbf{u}^{-1}(M) \in S^2$.

(The referee has kindly contributed this geometrical interpretation of $\mathbf{p}(M)$: «represent S^3 by stereographic projection as $\mathbb{R}^3 + \infty$», where the space of pure quaternions $\mathbb{R}^3$ is also the tangent space to S^3 at 1; «then $\mathbf{p}(M)$ is the helix turn of angle $\pi/2$, pushing forward $\pi/2$ and sending $\mathbf{u}(M)$ to $\mathbf{v}(M)$.»)

Let $f: (\mathbb{R}^4, \mathbf{0}) \longrightarrow (\mathbb{R}^2, \mathbf{0})$ be continuous, and smooth in a punctured neighborhood of $\mathbf{0} \in \mathbb{R}^4$. Then *$f$ has an isolated critical point* (at $\mathbf{0}$) if the 2-by-4 matrix $Df(\mathbf{X})$ (i.e., the total differential of f at $\mathbf{X}$) has rank 2 for all $\mathbf{X} \neq \mathbf{0}$ of sufficiently small norm. In this case, all the maps $(\mathbf{u} \circ Df, \mathbf{p} \circ Df) | S^3(\epsilon) : S^3(\epsilon) \longrightarrow S^3 \times S^2$, for sufficiently small $\epsilon > 0$, determine the same element of $\pi_3(S^3 \times S^2) = \pi_3(S^3) \oplus \pi_3(S^2)$.

Of course $\pi_3(S^3) \oplus \pi_3(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. We choose the isomorphism so that (id,*) corresponds to $(1,0)$ and $(*, H)$ to $(0,1)$, where $H: S^3 \to S^2 : (z, w) \to (|z|^2 - |w|^2, 2zw)$ is a negative Hopf fibration, and we let $(\mu(f; \mathbf{0}), \lambda(f; \mathbf{0}))$ denote the homotopy class in question. Direct calculation now shows that, if $f(z, w) = z^2 + w^2$ (complex coordinates), then $(\mu(f; \mathbf{0}), \lambda(f; \mathbf{0})) = (1, 0)$, whereas if $f(z, w) = z^2 + \bar{w}^2$ then $(\mu(f; \mathbf{0}), \lambda(f; \mathbf{0})) = (1, 1)$. More generally, the following is readily established [Ru8], [N&R3].

2.1. Theorem. *Let* f *have an isolated critical point at* $\mathbf{0}$*. Then* $\lambda(\mathrm{f}; \mathbf{0}) = 0$ *if* f *is complex-analytic near* $\mathbf{0}$*. Let* $\mathrm{Q}(\mathrm{z}, \mathrm{w}) = (\mathrm{z}, \bar{\mathrm{w}})$*. Then* $\mu(\mathrm{f} \circ \mathrm{Q}; \mathbf{0}) = \mu(\mathrm{f}; \mathbf{0})$ *and* $\lambda(\mathrm{f} \circ \mathrm{Q}; \mathbf{0}) = \mu(\mathrm{f}; \mathbf{0}) - \lambda(\mathrm{f}; \mathbf{0})$. ∎

Now we are ready to introduce (μ, λ) for fibered links.

Let $f: (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ have an isolated critical point at $\mathbf{0}$. Following Kauffman & Neumann, we define the isolated critical point of f at $\mathbf{0}$ to be *tame* if for all sufficiently small $\epsilon > 0$, (1) the set $f^{-1}(\mathbf{0})$ (which is a smooth 2-manifold in a punctured neighborhood of $\mathbf{0}$) intersects $S^3(\epsilon)$ transversely, and

(2) for all sufficiently small $\delta = \delta(\epsilon) > 0$, $D(f;\delta,\epsilon) := D^4(\epsilon) \cap f^{-1}(D^2(\delta))$ is a 4-disk-with-corners; and we observe that, if (1) and (2) hold, then $f^{-1}(\mathbf{0}) \cap \partial D(f;\delta,\epsilon)$ is a fibered link in the (piecewise-smooth) 3-sphere $\partial D(f;\delta,\epsilon)$, with ambient isotopy type $L(f;0)$ depending only on (the germ of) f. Furthermore ([L], [K&N]), every type of fibered link occurs as $L(f;\mathbf{0})$ for some tame f, and if $L(f;\mathbf{0}) = L(g;\mathbf{0})$ then ([K&N]) f and g are equivalent in a sense strong enough to make 2.2 work.

2.2. Definition. *If L is a fibered link, then $(\mu(L), \lambda(L)) = (\mu(f;\mathbf{0}), \lambda(f;\mathbf{0}))$ for any f with $L = L(f;\mathbf{0})$.* ∎

2.3. Theorem. *If F is a fiber surface, then $\mu(\partial F)$ is the first Betti number of F.* ∎

Theorem 2.3 is due to Milnor [*Mi*] in the complex-analytic case; a proof in the general case can be given along exactly the same lines [N&R3]. It is now standard to call the first Betti number of the fiber surface of a fibered link the *Milnor number* of the link. We will call (μ, λ) the *enhanced Milnor number*, and λ the *enhancement*.

The next result, proved quite otherwise in [Ru8], is immediate from 2.1.

2.4. Theorem. $\lambda(K) + \lambda(\mathit{Mir}\ K) = \mu(K)$. ∎

2.5. Example. The *positive Hopf link* $\partial A(O,-1)$ is $L(z^2+w^2;(0,0))$, the simplest non-trivial link of a complex plane curve singularity. Its Milnor number is 1. By 2.1, $\lambda(\partial A(O,-1)) = 0$; by 2.4 (or direct computation), $\lambda(\partial A(O,1)) = 1$.

2.6. The development of the enhanced Milnor number through isolated critical points ties it suggestively to the geometry of two complex variables. (Another way to think of the enhancement is as the obstruction to extending the almost-complex structure «left multiplication by $\mathbf{p} \circ Df$» over $\mathbf{0}$, up to homotopy.) It is also useful, however, to have methods of calculation which take place purely «in the 3-sphere». One such (whose proof, though, does involve an excursion into the 4-disk) is the following.

Theorem [N&R3]. *The enhanced Milnor number is additive over Murasugi sums.* ∎

(Actually, it is additive over a more general composition of fibered links, *unfolding*, which was introduced in [N&R3].)

2.7. Corollary. *If* F *is a Hopf-plumbed surface, then* $\lambda(\partial F)$ *is the number of negative Hopf annuli in any plumbing presentation of* F. *(Of course* $\mu(\partial\mathrm{F})$ *is the total number of Hopf annuli.)*

Proof: Immediate from 2.5 and 2.6. ∎

In particular, one has the formula for $\lambda(\partial S(\mathbf{b}))$, **b** a strict homogeneous T-braidword, which was derived by entirely different methods in [Ru9]: $\lambda(\partial S(b)) = \#\{s, 1 \leq s \leq k : b(s)$ is the inverse of a T-standard generator $\} - \#\{\{i,j\} \in T$: for some s with $1 \leq s \leq k$, $b(s) = \sigma_{\{i,j\}}^{-1}\}$.

2.8. A second method of calculation «in the 3-sphere», this one from [N&R1], applies to cables on fibered knots, and has an important corollary.

Theorem. *Let* K *be a fibered knot,* $\mathrm{m}>0$ *and* $n \neq 0$ *integers. Then* $\lambda(\mathrm{K}\{\mathrm{m},\mathrm{n}\}) = \lambda(\mathrm{K})$ *if* $\mathrm{n}>0$, $\lambda(\mathrm{K}\{\mathrm{m},\mathrm{n}\}) = \lambda(\mathrm{K}) + (\mathrm{m}-1)\ (\mu(\mathrm{K})-\mathrm{n}-1)$ *if* $\mathrm{n}<0$. ∎

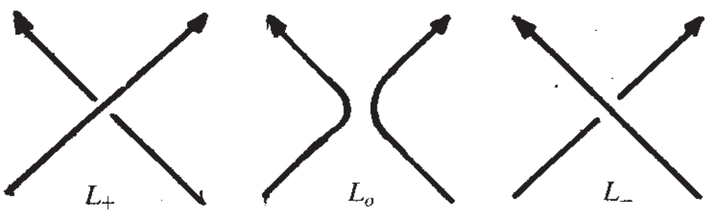
Corollary. *The enhancement is not determined by the Seifert form. (Of course the Milnor number of* F *is determined by the Seifert form of* F*--in fact, by the Alexander polynomial of* ∂F.)

In fact, if K is any fibered knot other than the unknot, and m is any integer greater than 1, then the fiber surfaces of $K\{m, 1\}$ and $K\{m, -1\}$ have identical Seifert matrices (with respect to an obvious diffeomorphism of the surfaces), but different enhancements.

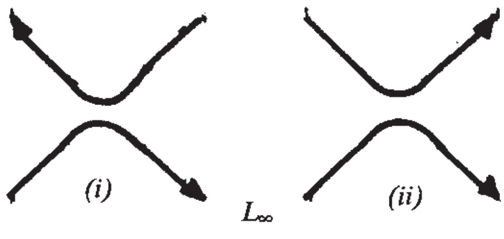
2.9. Another result of [N&R1] is that the enhancement can take on any value in $\mathbb{Z}$. In light of 2.7, this provides graphic evidence of how far the class of Hopf-plumbed links is from exhausting the class of all fibered links.

3. REVIEW OF THE NEW LINK POLYNOMIALS

This exposition follows [Ru 3], which is closely based (except for the framed polynomial) on [*Li*].



3.1. FIGURE



3.2. FIGURE

Let L_+, L_0, and L_- be three links with diagrams identical except as indicated in 3.1. If the visible crossing (of L_+ and L_-) involves just one component, then segments of two components of L_0 are visible; let p be the linking number of the right-hand component of L_0 with the rest of L_0, and let L_∞ be the link indicated in 3.2 *(i)*. If the visible crossing involves two components, let q be the linking number of the bottom-right to top-left component of L_+ with the rest of L_+, and let L_∞ be the link indicated in 3.2 *(ii)*.

3.3. Theorem ([FYHLMO], [P&T]). *There is one and only one way to assign each link* L *an element* $\mathrm{P_L}$ *of* $[v^{\pm 1}, z^{\pm 1}]$ *so as to satisfy:*

($P1$) $P_O = 1$;
($P2$) $P_{L_+} = vzP_{L_0} + v^2 P_{L_-}$ *for all instances of* 3.1. ∎

In 3.3, the choice of variables v and z follows [Mo]. Though I would wish it otherwise, Morton's evocative name «twisted Alexander polynomial» for P_L has not caught on; I will follow [Li] and call P_L the *oriented polynomial* of L.

3.4. Theorem ([K]). *There is one and only one way to assign each link* L *an element* $\mathrm{F_L}$ *of* $[a^{\pm 1}, x^{\pm 1}]$ *so as to satisfy:*

($F1$) $F_O = 1$;
($F2$) $a\,F_{L_+} + a^{-1}F_{L_-} = x(F_{L_0} + a^{-4p}F_{L_\infty})$ for all instances of Case 1 (resp., $aF_{L_+} + a^{-1}F_{L_-} = x(F_{L_0} + a^{-4q+2}F_{L_\infty})$ for all instances of Case 2) of 3.1 and 3.2.

Again following [Li], I will call F_L the *semi-oriented polynomial* of L.

Each of these 2-variable Laurent polynomials can of course be specialized to a 1-variable Laurent polynomial in infinitely many ways. In particular, $P_L(1, t^{-1/2} - t^{1/2}) = \Delta_L(t)$ is the classical Alexander polynomial of L; $P_L(t, t^{1/2} - t^{-1/2}) = V_L(t) = F_L(t^{-3/4}, -(t^{-1/4} + t^{1/4})$ is the *Jones polynomial* of L (see [J]; the second equality is due to Lickorish, [Li]); and $F_L(1, x) = Q_L(x)$ is the *absolute polynomial* of L ([BLM], [H]). Examples show that, of all these polynomials, only the Alexander polynomial can be calculated from a Seifert matrix for L.

In [Ru 3], I introduced some technical modifications of P_L, which turn out to be useful in the theory of quasipositivity.

3.5. Notations. Let $[L,f]$ denote $P_{\partial A(L,f)}$, where f is a framing of L (cf. 0.1). Let u denote $[O,0]=(v^{-1}-v)z^{-1}$. (Of course $\partial A(O,0)$ is the unlink of two components.)

Although u is not invertible in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, it can be handy to invert it formally, and with discretion to interpret u^{-1} as P_ϕ, where ϕ is «the empty link» (which is «the unlink of 0 components»).

3.6. Proposition. (1) *Let* $-\mathrm{L}$ *denote* L *with its orientation reversed; then* $\mathrm{P}_{-\mathrm{L}}=\mathrm{P}_\mathrm{L}$. (2) *Let* Mir L *denote the mirror image of* L; *then* $\mathrm{P}_{\mathrm{Mir\,L}}(v,z)=\mathrm{P}_\mathrm{L}(-\mathrm{v}^{-1},\mathrm{z})$. (3) *For* $\mathrm{n}\geqq 0$, *let* O_n *denote an unlink of* n *components; then* $\mathrm{P}_{\mathrm{O}_\mathrm{n}}=\mathrm{u}^{\mathrm{n}-1}$. (4) *Let* $\mathrm{L}_1 \natural \mathrm{L}_2$ *denote the split sum of* L_1 *and* L_2; *then* $\mathrm{P}_{\mathrm{L}_1 \natural \mathrm{L}_2}=\mathrm{u}\,\mathrm{P}_{\mathrm{L}_1}\,\mathrm{P}_{\mathrm{L}_2}$. (5) *Let* f *and* f′ *be framings of* L *which differ only on the component* K, *with* $\mathrm{f}'(\mathrm{K})=\mathrm{f}(\mathrm{K})+\mathrm{c}$; *then* $[\mathrm{L},\mathrm{f}']=(1-\mathrm{v}^{-2\mathrm{c}})[\mathrm{L}-\mathrm{K},\mathrm{f}]+\mathrm{v}^{-2\mathrm{c}}[\mathrm{L},\mathrm{f}]$. ∎

Proof: These are all well-known consequences of $(P1)$ and $(P2)$ (to see (5), consider 3.7, where $c=-1$).

$\partial A(K,f'(K)) \subset \partial A(L,f')$ $\quad O \subset O \natural \partial A(L-K,f)$ $\quad \partial A(K,f(K)) \subset \partial A(L,f)$

3.7. FIGURE

3.8. Definition. *Let L be a link with n components, f a framing of L. The framed polynomial $\{L,f\}\in\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ is u times the sum, over all sublinks K of L (including ϕ), of $(-1)^{n-k}[K,f]$ (where K has k components, $0\leqq k\leqq n$); Bd $A(\phi,f)$ is of course ϕ.*

3.9. Proposition [Ru 3]. (1) $\{\mathrm{L},\mathrm{f}\}$ *is independent of the orientation of* L. (2) $\{\mathrm{Mir\ L},\mathrm{f}\}(\mathrm{v},\mathrm{z})=\{\mathrm{L},\mathrm{f}\}(-\mathrm{v}^{-1},\mathrm{z})$. (3) $\{\mathrm{O},0\}=\mathrm{u}^2-1$.
(4) $\{\mathrm{L}_1 \natural \mathrm{L}_2,\mathrm{f}\}=\{\mathrm{L}_1,\mathrm{f}\}\{\mathrm{L}_2,\mathrm{f}\}$. (5) $\mathrm{v}^{2\mathrm{f}(\mathrm{L})}\{\mathrm{L},\mathrm{f}\}$ *is independent of* f, *where* f(L) *denotes the total framing, that is, the sum of the integers which* f *assigns to the components of* L. (6) $[\mathrm{L},\mathrm{f}]$ *is* u^{-1} *times the sum of* $\{\mathrm{K},\mathrm{f}\}$ *over all sublinks* K *of* L *(in paticular, this sum is divisible by* u *in* $\mathbb{Z}[\mathrm{v}^{\pm 1},\mathrm{z}^{\pm 1}]$). ∎

3.10. Remark. Up to the normalizing factor u, $\{L,f\}$ is a «Möbius transform» of $\{L,f]$. Yamada has made a general study of Möbius transforms of link polynomials; 3.9 (6) is essentially Proposition 1 of [*Y*].

3.11. Notation. Let 0 be the framing which assigns 0 to each component of L; then $\{L\}$ will denote $\{L, 0\}$.

In this notation, 3.9 (5) becomes the attractively simple formula $\{L,f\} = v^{-2f(L)}\{L\}$.

3.12. Congruence theorem [Ru 3]. $(1+(v^{-2}+v^2)z^{-2})F_L(v^{-2},z^2)$ *is congruent* modulo 2 *to* $v^{4t(L)}\{L\}(v,z)$. ▌

(Here $t(L)$ denotes the *total linking* of L, that is, the sum of the linking numbers of all pairs of components of L.)

3.13. Remark. As remarked by the referee of [Ru 3], the Congruence Theorem is «a generalization, to all values of v and z, but only modulo 2,» of Prop. 10 of [*Y*], which relates--by equality, not congruence--a certain specialization of the semioriented polynomial of L and a Möbius transform of the Jones polynomial of $\partial A(L, 0)$.

4. QUASIPOSITIVITY AND THE NEW LINK POLYNOMIALS

As already mentioned (1.2), the Alexander polynomial of a quasipositive link is utterly undistinguished among all Alexander polynomials. Of course, the Alexander polynomial is also insensitive to handedness. Intuitively, quasipositivity seems to be deeply related to handedness. This intuition might give some reason to hope that the oriented and semi-oriented polynomials (and their common specialization, the Jones polynomial), which are sensitive to handedness, should also be sensitive to quasipositivity. We will see in this section that, in fact, such a hope is to some extent justified.

4.1. Notation. For any coefficient ring R and indeterminates x, y, if $S(x,y) = \sum_{i=m}^{M} S_i(y)x^i \in R[x^{\pm 1}, y^{\pm 1}] = R[y^{\pm 1}][x^{\pm 1}]$ and $S_m, S_M \in R[y^{\pm 1}]$ are non-zero, then $\operatorname{ord}_x S := m$, $\deg_x S := M$. Trivially, for any quotient ring R/I, if $S^*(x,y) \in (R/I)[x^{\pm 1}, y^{\pm 1}]$ denotes the reduction of S modulo I, then $\operatorname{ord}_x S^* \geqq \operatorname{ord}_x S$, $\deg_s S^* \geqq \deg_x S$.

4.2. Theorem. ([Mo], [F&W]). *For all* $\beta \in B_n$, $\operatorname{ord}_v P_{\beta^\cdot} \geqq e(\beta) - n + 1$. ▌

4.3. Corollary. *Let* **b** *be a quasipositive embedded band representation in* B_n. *Then* $\operatorname{ord}_v P_{\beta^\cdot}(\mathbf{b}) \geq 1 - d$, *where* d *is the number of components of* $S(\mathbf{b})$

which are 2-disks. Also, if $S(\mathbf{b})$ *is a fiber surface, then* $\operatorname{ord}_v P_{\beta^\wedge(\mathbf{b})} \geq \mu(\beta^\wedge(\mathbf{b}))$.

Proof: Because $\mathbf{b}$ is quasipositive, its length is $e(\beta)$; thus $n-e(\beta)$ is the Euler characteristic of $S(\mathbf{b})$, so by 4.1, $\operatorname{ord}_v P_{\beta^\wedge(\mathbf{b})} \geq 1 - \dim H_0(S(\mathbf{b}); \mathbb{R}) + \dim H_1(S(\mathbf{b}); \mathbb{R})$. The contribution of each non-disk component of $S(\mathbf{b})$ to $-\dim H_0(s(\mathbf{b}); \mathbb{R}) + \dim H_1(S(\mathbf{b}); \mathbb{R})$ is non-negative, whereas the contribution of each of the d disks is -1; the first conclusion follows. The second is similar by 2.3. ∎

4.4. Corollary [Ru 3]. *Let* f *be a framing of the link* L. *If* A(L, f) *is quasipositive, then* $\operatorname{ord}_v\{\mathrm{L}, \mathrm{f}\} \geq 0$.

Proof: Immediate from 4.3, the definition of the framed polynomial, and standard properties of ord_v. ∎

4.5. Corollary. *If* K *is a strongly quasipositive knot other than the unknot, then* $\operatorname{ord}_v\{\mathrm{K}, 0\} \geq 0$.

Proof: Let S be a quasipositive Seifert surface for K. Then a regular neighborhood of K on S is $A(K, 0)$ (the Seifert self-linking of K is 0 because K bounds on S). If S is not a disk, then $A(K, 0)$ is full on S and therefore quasipositive by 1.3, so $\operatorname{ord}_v\{K, 0\} \geq 0$ by 4.4. ∎

4.6. Remark. In fact, 4.3. remains true (with the same proof) in the context of not-necessarily-embedded band representations and their associated Seifert ribbons (cf. the end of 0.8). This shows, for instance, that if a knot K and its mirror image Mir K are both quasipositive then they are slice (actually ribbon); thus, *any non-slice knot which is its own mirror image* (e.g., the figure-8 knot) *is not quasipositive.* This was the first proof that non-quasipositive knots exist.

More can be said. According to Morton, if a knot K and Mir K are both quasipositive, then $P_K(v, z) = 1$ $(= P_{\mathrm{Mir}\,K}(v, z))$. It is not known if any link other than O has $P_K(v, z) = 1$.

Conjecture. *If a link* L *is such that* L *and* Mir L *are both quasipositive, then* L *is an unlink (i.e., it has a Seifert surface which is the union of disjoint 2-disks).* ∎

4.7. Corollary (converse to 1.5). *For any knot* K, *there exists* $\mathrm{q} \in \mathbb{Z}$ *such that* A(K, n) *is not quasipositive if* $\mathrm{n} > \mathrm{q}$. *(More generally, for any link* L, *there exists* $\mathrm{q} \in \mathbb{Z}$ *such that* A(L, f) *is not quasipositive if* $\mathrm{f(L)} > \mathrm{q}$.)

Proof: By 4.4, this is the case for $q = (1/2) \operatorname{ord}_v\{K, 0\}$. ∎

4.8. In light of 1.5 and 4.7, we may define the *modulus of quasipositivity* $q(K)$ of the knot K to be the greatest integer q such that $A(K, q)$ is quasipositive. (Essentially this definition appears in [Ru5], where, however, the possibility of an infinite modulus of quasipositivity was left open.) The proof of 4.7 shows that $q(K) \leq (1/2)\,\mathrm{ord}_v\{K, 0\}$. By 3.12, this implies the weaker (but more easily calculated) bound $q(K) \leq -1 - \deg_a F_K^*(a, x)$, where F_K^* denotes the reduction of F_K modulo 2.

4.9. Examples. (1) Since $A(O, -1) = S(\mathbf{b})$ where $\mathbf{b} = (\sigma_1, \sigma_1)$ in B_2, the modulus of quasipositivity of the unknot is ≥ -1. By 4.8, $q(O) \leq (1/2)\,\mathrm{ord}_v \{K, 0\} = (1/2)\,\mathrm{ord}_v(1-u) = -1/2$. So $q(O) = -1$. (2) Since $A(O\{2, -3\}, -6) = S(\mathbf{b})$ where $\mathbf{b}$ is the quasipositive embedded band representation $(\sigma_{2,4}, \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,4}, \sigma_{1,3})$ in B_5, $q(O\{2, -3\}) \geq -6$. By 4.8 and a consultation of the table of semioriented polynomials in [K], $q(O\{2, -3\}) \leq -6$; so $q(O\{2, -3\}) = -6$. (The corresponding calculation using the framed polynomial, without reducing the coefficients, can be done by hand--barely; the forbidding prospect of similar calculations, for knots with more crossings than the mere three of $O\{2, -3\}$, was the original motivation for the investigation which led to the Congruence Theorem.)

4.10. Remark. There is some evidence that $q(K)$ is the maximum Maslov index of a knot of type K which is Legendrian with respect to the standard contact structure on S^3, cf. [Ar]; this is the case for O, [Be].

4.11. Corollary. *Let* O *be an unknot lying on a quasipositive surface* F. *Let* $n = \mathrm{link}(O, O+)$ *be the Seifert self-linking of* O *on* F. *Then* $n \geq 0$, *and* $n = 0$ *if and only if* O *bounds a disk on* F.

Proof: A regular neighborhood N of O on F is an annulus $A(O, -n)$. If N is not a full subsurface of F, then O bounds a disk on F, and $n = 0$. If N is full, then (by 1.3) N is quasipositive, so $-n \leq -1$ (by 4.9). ❙

4.12. The next result can extracted from [Be], where it is proved with different machinery (although, tantalizingly, the quantity $e(\beta) - n + 1$ of [Mo] and [F&W] is prominent in [Be] also).

Corollary. *A quasipositive surface is incompressible.*

Proof: The boundary of a compressing disk would be an unknot of self-linking 0 which bounds no disk on F. ❙

4.13. Corollary. *A fibered link is strongly quasipositive (if and) only if its fiber surface is quasipositive.*

Proof (of «only if»): A fiber surface is the unique incompressible Seifert surface for its own boundary. ❙

In particular, a fibered link is not strongly quasipositive if it has $A(O, 1)$ as a Murasugi summand, or if it can be subjected to a Stallings-Harer $+1$—twist or a (non-trivial) Stallings-Harer 0-twist (cf. [St.], [Ha], and--for the present notation--[N&R1]).

Must a quasipositive fibered link be strongly quasipositive? I do not know. However, a *non*-fibered quasipositive link need *not* be strongly quasipositive. For instance the closure K of $\sigma_1\sigma_2^2\sigma_1\sigma_2^{-2}\in B_3$ has 3 unknotted components O_1, O_2, O_3, where (say) link $(O_2, O_1+O_3)=0$, so that on any Seifert surface F for K, the Seifert self-linking of O_2 is 0, yet O_2 cannot bound a disk on F since link$(O_2, O_1)\neq 0$. (It is interesting to note that K is the link-at-infinity of $\Gamma=\{(z,w)\in\mathbb{C}^2: z(zw+1)=0\}$, cf. 1A.7; this is most easily seen in the boundary of a large bi-disk.)

4.14. Remark. In a recent preprint [*F*3], Fiedler derives various interesting results on the Jones polynomial $V_L(t)$, and states a conjecture which can be phrased as follows: *if* **b** *is a band representation in* B_n *with* p *positive and* q *negative bands, then* ord$_t$ $V_{\beta^\cdot(\mathbf{b})}\leq(p+q+1-n)/2$ *and* $-(p+q+1-n)/2\leq\deg_t V_{\beta^\cdot(\mathbf{b})}$. This would imply, for instance, that if K is a strongly quasipositive knot, then ord$_t$ V_K and $-\deg_t V_K$ are bounded above by the genus of K. It also implies (as Fiedler points out) the affirmative answer to the «question of Milnor» on the unknotting number of the link of a singularity.

5. QUASIPOSITIVITY AND THE ENHANCEMENT OF THE MILNOR NUMBER

5.1. Theorem [Rul3]. *A Hopf-plumbed fibered link is strongly quasipositive if and only if its enhancement is* 0.

Proof: By 1.6 and 2.7, a Hopf-plumbed fiber surface is quasipositive if and only if its enhancement is 0; the theorem follows from 4.13. ∎

5.2. A *torus knot* is a cable $O\{m, n\}$, GCD$(m, n)=1$, on an unknot; an *iterated torus knot* is $O\{m_1, n_1; m_2, n_2; \ldots; m_k, n_k\} := O\{m_1, n_1\}\{m_2, n_2\}\ldots\{m_k, n_k\}$ with $m_i>0, n_i\neq 0$, GCD$(m_i, n_i)=1$ $(i=1,\ldots,k)$. Without loss of generality we may assume that $m_i>1, |n_1|>1$ (else the same knot could be realized with strictly smaller k). An iterated torus knot is fibered, but need not be be Hopf-plumbed (e.g., $O\{2,3;2,1\}$, [N&R3]).

Theorem. *An iterated torus knot is strongly quasipositive if and only if its enhancement is* 0.

Proof: For $j=1,\ldots k$, let F_j be the fiber surface of $O\{m_1, n_1; m_2, n_2; \ldots; m_j, n_j\}$; there are m_j (linked, disjoint) copies of F_{j-1} embedded in F_j as full subsurfaces. Using induction (starting from 1.4), it is straightforward to write down an embedded band representation $\mathbf{O}\{m_1, n_1; m_2, n_2; ...; m_k, n_k\}$ in $B_{m_1...m_k}$ (not necessarily a braidword if $k > 1$), which is quasipositive if and only if all n_i are positive, with $S(\mathbf{O}\{m_1, n_1; m_2, n_2; ...; m_k, n_k\}) = F_k$. Since (by 2.8) it is also the case that the enhancement of $O\{m_1, n_1; m_2, n_2; ...; m_k, n_k\}$ is 0 if and only if all n_i are positive, it remains to show that, if some n_i is negative, then F_k is not quasipositive.

This is easy if for some j we have $n_j < -1$: then F_j has $A(O, 1)$ as a Murasugi summand (0.1), so F_j is not quasipositive (4.8), so F_k is not quasipositive (1.3). A more finicky proof (which also works in the preceding case) is needed if $n_i \geq -1$ for $i = 1, ..., k$ and $n_j = -1$ for some j greater than 1: inspection shows that F_j contains an annulus $A(O\{2, 3\}, f)$ with $f \geq -5$, which is not quasipositive (by calculation and 4.2). ❙

5.3. The following seems credible, though the evidence for it is essentially limited to 5.1-5.2.

Conjecture. *The enhancement of a quasipositive (resp., strongly quasi-positive) fibered link is non-positive (resp., zero).* ❙

APPENDIX to Section 5: Complex plane curves and the enhancement

The link $L(\Gamma, 0)$ of a singularity is both quasipositive and fibered (in fact it is Hopf-plumbed), and has enhancement 0 (e.g., by 2.1). A link-at-infinity $L(\Gamma, \infty)$ is quasipositive, but need not be fibered; and, when $L(\Gamma, \infty)$ is fibered, it is not known whether its enhancement is necessarily 0, although this is the case when $L(\Gamma, \infty)$ is *regular* in the sense of [Ne]. If $L(\Gamma, \infty)$ is connected, then it is regular, and thus fibered with enhancement 0, as asserted in [N&R3]; note that the proof there is incomplete, [N&R3, corrigendum].

As remarked in 1A.7, it is not known whether or not a general link $L(\Gamma, r)$ (the transverse intersection of a complex plane curve with a round sphere which need be neither very small nor very large) is quasipositive. Certainly $L(\Gamma, r)$ need not be fibered. Nonetheless, in analogy with 5.3, we may ask whether, when $L(\Gamma, r)$ is fibered, its enhancement must be non-positive.

6. THE ENHANCEMENT AND THE NEW LINK POLYNOMIALS

6.1. Fantasy. Imagine that J. W. Alexander, whose «Note on Riemann Spaces» [Al] essentially introduced open-book structures, had developed a bit more of the geometrical theory of fibered links and knots before he discovered (quite combinatorially) the polynomial invariant which now bears his name [A2]. In such an alternative universe, how might the search for the geometrical underpinnings of the Alexander polynomial have proceeded? I suggest a scenario like the following. Immediately, it would have been noticed that the degree of the Alexander polynomial of a fibered link equals the first Betti number of its fiber surface; and, soon thereafter, that (up to normalization) the leading coefficient is 1. This would have suggested, correctly, that the polynomial of a fibered link is an invariant of the fiber surface--namely, the characteristic polynomial of its algebraic monodromy. The interpretation for general links, *via* Seifert surfaces and infinite cyclic coverings, would have followed very naturally.

6.2. At present, the enhancement and the new link polynomials of a fibered link are known to be related only in certain cases, and there only by an inequality. I still cherish a hope that the analogy «Milnor number : Alexander polynomial :: enhanced Milnor number : oriented polynomial» will be fruitful, and not just in the fibered case.

6.3. The following estimate is derived in [Ru13].

Theorem [Ru13]. *If* L *is either a generalized strict homogeneous braid or a fibered arborescent link, then*

$$(*) \quad \operatorname{ord}_v P_L \geq -4\lambda(L) + \mu(L)$$

(and equality can occur for all possible values of λ*).* ∎

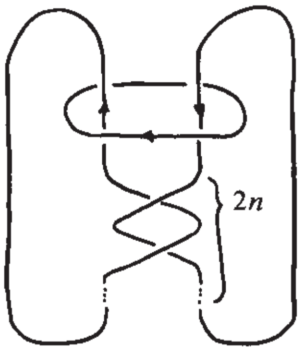
The proof uses 5.1 and 4.1, and inductions on the Milnor number (slightly different for the two cases) within the class of fibered links being considered.

Since generalized strict homogeneous braids and fibered arborescent links are Hopf-plumbed, the following seems reasonable.

Conjecture. *The inequality* (*) *holds for all Hopf-plumbed fibered links.*

The obstacle to generalizing the proof of the Theorem to cover the Conjecture is the inductive step.

6.4. The hope expressed in 6.2 would be distinctly encouraged if the Conjecture of 6.3 were true for all fibered links. This, alas, is not the case. Although, for instance, the iterated torus knots $O\{2,3;2,2k+1\}$ and links $O\{2,3;2,2k\}$, which are not all Hopf-plumbed, can indeed be shown to satisfy (*), for other fibered links this inequality can fail arbitrarily badly. For instance, the link K_n in Figure 6.5 is fibered for every integer n (K_0 is the connected sum of a positive and a negative Hopf link, and K_n is produced from K_0 by repeated Stallings-Harer 0-twists) and can be shown to have enhancement 1 for all n (a proof is given in [N&R]); it falsifies (*) for negative n.



6.5. FIGURE

6.6. It may be possible to salvage something from the Conjecture. The following idea has not been developed, and is put forward here partly for amusement value.

Specialize the oriented polynomial to $R_L(w) = P_L(v, v^2)$, $w = v^2$. This is not unmotivated; in fact, R_L has (and is nearly characterized by) the geometrically interesting property that, though non-trivial, it cannot distinguish the positive Hopf link from the unknot. This suggests that, for fibered K, $\mathrm{ord}_w\, R_K$ might be related (by an inequality) to the enhancement of K.

Indeed, 6.3 appears to generalize; moreover, the behavior of $\mathrm{ord}_w\, R_K$ for the links of 6.5 is just as good as for the Hopf-plumbed examples. Here, the major obstacle to progress is the following inequality (an analogue of 4.2), which I have been unable to establish.

Conjecture. *If* F *is a quasipositive Seifert surface with* $s(F)$ *«split pieces» (i.e.,* $s(F)-1$ *is the rank of the free abelian group* $\pi_2(S^3\backslash F)$*), then* $\mathrm{ord}_w R_{\partial F} \geq 1 - s(F)$.

7. CAN THE SEIFERT FORM BE «ENHANCED»?

Let F be a Seifert surface. As we have seen, the Seifert form of F doesn't determine quasipositivity of F, is insufficient to calculate the oriented, semi-oriented, absolute, and Jones polynomials of the boundary of F, and--should F happen to be a fiber surface--is ignorant of the enhancement of (the boundary of) F. One may wonder whether there is an «enhanced Seifert form» which *does* determine one or more of these--preferably all of them, and in such a way as to advance our understanding of their interrelations.

There are ample hints in the preceding sections that in some way the geometry of two complex variables provides an underlying connection among quasipositivity, the enhancement, and (less clearly) the new link polynomials; perhaps that is the place to look for an enhanced Seifert form. On a different tack, on August 12, 1988, at the American Mathematical Society's Centennial meeting in Providence, Rhode Island, the mathematical physicist Edward Witten anounced a geometric interpretation [Wi] of the Jones polynomial (and at least some of the other new polynomials) in terms of Quantum Field Theory; as the details emerge, they may reveal an enhanced Seifert form as a sidelight.

The speculation which follows has rather a different flavor, and is meant to be suggestive rather than programmatic.

If S is a surface, call a 1-submanifold C of S *full* if C is non-empty and no component of C bounds a disk on S (i.e., the regular neighborhood of C on S is a full subsurface). Write SCC(S) for the set of full (oriented) 1-submanifolds of S modulo ambient isotopy on S (possibly exchanging components). Each embedding $f: S \to S^3$ as a Seifert surface $F=f(S)$ induces a mapping from SCC(S) into *Links*, the set of oriented link types in S^3, and (using a push-off map on the second factor) from SCC(S)X SCC(S) into *Links* X *Links*. In some sense, these mappings give a «universal enhanced Seifert form», and less enhanced Seifert forms result by composition with suitable link invariants. For instance, composing SCC(S)X SCC$(S) \to$ *Links* X *Links* with «linking number» essentially recovers the usual Seifert form.

Of course, this «universal enhanced Seifert form» begs too many questions--for instance, it determines the link type of the boundary of F and therefore all the invariants of that boundary. But one may still wonder whether a useable invariant might be yielded by a mapping of *Links* or *Links* X *Links* which retains (even slightly) more information than linking number.

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