REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 2, número 1; 1989.
http://dx.doi.org/10.5209/rev_REMA.1989.v2.n1.18141

On the complicatedness of the pair $(\mathfrak{g}, K)$

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In memoriam Atilio Bauchiero

ABSTRACT. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the complexification of a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and let K be the analytic subgroup of the adjoint group of $\mathfrak{g}$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Let L be an algebraic connected linear reductive complex group acting on a finite dimensional vector space V. In the study of the orbits of this sort of actions, there are some criteria of «non complicatedness»: e.g., «cofreeness» (the ring of all polynomial functions on V is a free module over the ring of all L-invariants), etc. From this viewpoint, we show that the pair $(\mathfrak{g}, K)$ is complicated, at least when $\mathfrak{g}_{\mathbb{R}}$ is not a product of copies of $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$.

1. INTRODUCTION

Let $\mathfrak{g}_{\mathbb{R}}=\mathfrak{k}_{\mathbb{R}}\oplus\mathfrak{p}_{\mathbb{R}}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the corresponding complexification. Let θ be the associated Cartan involution. Also let $\mathfrak{a}_{\mathbb{R}}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{a}$ be its complexification. Now let G be the adjoint group of $\mathfrak{g}$ and let K be analytic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Also let M be the centralizer of $\mathfrak{a}$ in K. This paper is concerned with the action of K in $\mathfrak{g}$ given by the restriction of the Adjoint representation. If $S'(\mathfrak{g})$ denotes the ring of all polynomial functions on $\mathfrak{g}$ then clearly $S'(\mathfrak{g})$ is a G-module and a fortiori a K-module.

If L is a reductive complex linear algebraic group, V is a finite dimensional complex vector space and $\alpha: L\longrightarrow GL(V)$ is a representation then, concerning the classification of the L-orbits in V, there are some criteria of «non-complicatedness». (See [K] or [M 1], p. 160). To state them, let us

[1] This work was supported by CONICET.

1980 Mathematics Subject Classification (1985 revision): 14D25, 14L30.
Editorial de la Universidad Complutense. Madrid, 1989.

recall that V/L is the notation for the affine variety associated to $S'(V)^L$ and $V \xrightarrow{\pi} V/L$ is the projection corresponding to the inclusion of rings. Let $\mathfrak{N} = \mathfrak{N}(V, L)$ be the fiber $\pi^{-1}(\pi(0))$. The criteria are:

A. $\mathfrak{N}$ is a finite union of orbits. Currently (V, L) is visible.
B. All the fibres of π are of the same dimension.
C. $S'(V)$ is a $S'(V)^L$ -free module. Currently (V, L) is cofree.
D. $S'(V)^L$ is a polynomial ring. Currently (V, L) is coregular.
E. The isotropy subgroup L^x is non trivial for every $x \in V$.

In this paper we work out the classification of the pairs $(\mathfrak{g}, K)$ as above for which each criteria is satisfied; see propositions A, B, C, D, E below.

If L_1 and L_2 are groups acting on finite dimensional vector spaces V_1 and V_2 respectively, and if we look $L_1 \times L_2$ acting on $V_1 \times V_2$ in the obvious way then it is trivial that

$$S'(V_1 \times V_2)^{L_1 \times L_2} \simeq S'(V_1)^{L_1} \otimes S'(V_2)^{L_2}$$

so $(V_1 \times V_2, L_1 \times L_2)$ is coregular (resp., cofree) iff (V_1, L_1) and (V_2, L_2) are.

Furthermore, the isotropy subgroup $(L_1 \times L_2)^{(x, y)} \cong L_1{}^x \times L_2{}^y$, the orbit $(L_1 \times L_2)(x, y) \cong L_1 x \times L_2 y$ $(V_1 \times V_2)/(L_1 \times L_2) \cong V_1/L_1 \times V_2/L_2$ and if $\xi_i \in V_i/L_i$, then $\pi^{-1}(\xi_1, \xi_2) = \pi^{-1}(\xi_1) \times \pi^{-1}(\xi_2)$. So $(V_1 \times V_2, L_1 \times L_2)$ satisfies A (resp., B, E) iff (V_1, L_1) and (V_2, L_2) do. Thus we can restrict our attention to the irreducible pairs $(\mathfrak{g}, K)$. As a synthesis, we get for irreducible $\mathfrak{g}_{\mathbb{R}}$:

Theorem: *$(\mathfrak{g}, K)$ never satisfies criteria B nor E; it satisfies criteria A, C, D if and only if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(p, 1)$ or $\mathfrak{su}(p, 1)$.*

We will use the application of the Luna's Slice Etale Theorem to the Invariant theory developped in [KPV] and also used in [Sch 1] to classify all the (V, L) coregular with L simple. Note that we can replace K by any connected algebraic group K' with Lie algebra $\mathfrak{k}$ acting on $\mathfrak{g}$ with the same infinitesimal action as K. Being a case by case analysis, we will follow E. Cartan's list as it appears in [He], chapter IX. Furthermore, it is clear that it suffices to look at the types I and II, see [He] p. 327.

We want to express our thanks to Jorge Vargas, Alejandro Tiraboschi, Oscar Brega and, specially, to Juan Tirao for helpful conversations.

2. PRELIMINARIES

Let V, L be as in the introduction, meaning of course by a representation a morphism of algebraic groups. For $x \in V$, the conjugacy class of the isotropy subgroup L^x is called an isotropy class. If the orbit Lx is closed, L^x is reductive and the representation of L^x in $T_x(V)/T_x(Lx)$ is called the slice representation at x, where T_x notes the tangent space at x. We say that (L^x) is a closed isotropy class.

Lemma 1 ([KPV], [Sch 1]): *Let* $V = V_1 \oplus V_2$ *be a direct sum of finite dimensional L-modules. Then:*

i) *If* (V, L) *is coregulat then* (V_1, L) *and* (V_2, L) *are.*
ii) *If* (V, L) *is coregular then every its slice representation is.*
iii) *If* (H) *is a closed isotropy class of* V_1 *then* (V, L) *coregular implies* (V_2, H) *is.*
iv) *In particular, if the image of H in* $GL(V_2)$ *is a non-trivial finite subgroup of* $SL(V_2)$ *then* (V, L) *is not coregular.*

Proof: *i)* is easy and *ii)* follows from Luna's Theorem (see [KPV]). *iii)* is an application of *i)* and *ii)*; *iv)* is a consequence of the well known Chevalley-Sheppard-Todd Theorem, as it was pointed out in [Sch 1]. ■

The unique minimal closed isotropy class is called the principal isotropy class. For the Adjoint representation, it is a maximal torus. If V has a L-invariant non-degenerate bilinear symmetric form (V is L-orthogonalizable, for short) then the set of those $x \in V$ such that (L^x) is principal contains an open dense subset of V (see [L] and [R]). The hypothesis is certainly fulfilled for the pairs $(\mathfrak{g}, K)$, $(\mathfrak{p}, K)$, $(\mathfrak{g}, G)$ taking the Killing form. It is obvious that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a K-module decomposition.

Lemma 2: *The principal isotropy class of* $(\mathfrak{p}, K)$ *is* (M).

Proof: By Lemma 20, in p. 803, of [K R] and in the notation therein, $M_\theta = (K_\theta)^x$ for all x «regular» in $\mathfrak{a}$. But $M = M_\theta \cap K$, and $K^x = (K_\theta)^x \cap K$, $\forall x$ in the open dense subset of «regular» elements in $\mathfrak{a}$. ■

We denote by $\Pi(V)$ or $\Pi_L(V, \mathfrak{b})$ the set of weights associated to the representation of L in V and a fixed Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{l}$, the Lie algebra of L.

The following result is a well-known consequence of the graded version of the Nakayama Lemma and in the present form is useful to establish that some graded ring is not regular.

Lemma 3: *Let* $A = A_0 \oplus A_1 \oplus \dots$ *be a graded ring with* $A_0 = F$ *a field;* $A_+ = A_1 \oplus \dots$ *is the maximal homogeneous ideal.*

i) *A is regular iff dim Krull* $A = \dim_F A_+/A_+^2$. *In such case, if* $t_1, \dots, t_n$ *are homogeneous elements of A such that their images in* A_+/A_+^2 *form an F-basis, then they are algebraically independent over F.*

ii) *If $A_1=0$ and $t_1, \ldots, t_s$ are F-linearly independent in A_2, then A regular implies $t_1, \ldots, t_s$ are F-a.i.*

iii) *If $A_1=A_3=0$, $t_1, \ldots t_s$ is an F-basis of A_2 and $t_{s+1}, \ldots, t_r$ are F-l.i. in A_4 such that $A_2^2 \cap < t_{s+1}, \ldots, t_r > = 0$ then A regular implies $t_1, \ldots, t_s, t_{s+1}, \ldots, t_r$ are F-a.i.* ■

The non-coregularity of $(\mathfrak{g}, K)$ will follow in some cases from the following fact:

Lemma 4: *Assume that rank $\mathfrak{g}=$ rank $\mathfrak{k}$; that $\mathfrak{k}=\mathfrak{k}_1 \oplus \mathfrak{k}_2$ is a direct sum of Lie algebras where $\mathfrak{k}_2 \cong \mathfrak{sl}(2, \mathbb{C})$; and that as $\mathfrak{k}$-module, $\mathfrak{p}$ is $\rho_1 \otimes \rho_2$ where ρ_2 is the natural representation of $\mathfrak{k}_2$ in $\mathbb{C}^2$ and dim $\rho_1 \geq 4$. Then $(\mathfrak{g}, K)$ is not coregular.*

Proof: It is clear from Lemma 1 that it suffices to show that $(\mathfrak{p}, H)$ is not coregular, where H is a maximal torus of K, whose Lie algebra is isomorphic to $\mathfrak{b}=\mathfrak{b}_1 \oplus \mathfrak{b}_2$, a Cartan subalgebra of $\mathfrak{k}$, $\mathfrak{b}_j$ a Cartan subalgebra of $\mathfrak{k}_j$. Our first task is to descript $\Pi(\mathfrak{p}, \mathfrak{b})$. If σ is the weight of $\mathfrak{k}_2$ such that $\rho_2 \cong V(\sigma)$, then $\Pi(\rho_2)=\{\pm\sigma\}$ Then $\Pi(\mathfrak{p}, \mathfrak{b})=\{\alpha \pm \sigma: \alpha \in \Pi(\rho_1)\}$ by abuse of notation. But $\mathfrak{b}$ is also a Cartan subalgebra of $\mathfrak{g}$ and then if $\lambda \in \Pi(\mathfrak{p}, \mathfrak{b})$, λ is a non-compact root in $\Phi(\mathfrak{g}, \mathfrak{b})$; so $-\lambda \in (\mathfrak{p}, \mathfrak{b})$. Thus if $\alpha \in \Pi(\rho_1)$, $-\alpha$ too.

Next, let $\{t_{\alpha,\sigma}, t_{\alpha,-\sigma}: \alpha \in \Pi(\rho_1)\}$ be a basis of $\mathfrak{p}$ such that $t_{\alpha,\pm\sigma}$ is a vector of weight $\alpha \pm \sigma$ and let $\{T_{\alpha,\sigma}, T_{\alpha,-\sigma}\}$ be the corresponding dual basis. Thus:

$$S'(\mathfrak{p})^H = \bigoplus_{j \geq o} S'(\mathfrak{p})^H_j = \bigoplus_{j \geq o} <\text{monomials in } T_{\alpha,\pm\sigma} \text{ of weight } 0> = \bigoplus_{j \geq o} A_j$$

Clearly, if j is odd then $A_j=0$. Also if $U_\alpha = T_{\alpha,\sigma} \cdot T_{-\alpha,-\sigma}$, then $\{U_\alpha: \alpha \in \Pi(\rho_1)\}$ is a basis of A_2. As dim $\rho_1 \geq 4$, there exist $\alpha, \beta \in \Pi(\rho_1)$ such that $\alpha \neq \pm\beta$. Put $S_{\alpha,\beta} = T_{\alpha,\sigma} T_{-\alpha,\sigma} T_{\beta,-\sigma} T_{-\beta,-\sigma}$. Obviously, $A_2^2 \cap < S_{\alpha,\beta}, S_{\beta,\alpha} > = 0$. But U_α, $S_{\alpha,\beta}$, $S_{\beta,\alpha}$ are not a.i. because $S_{\alpha,\beta} S_{\beta,\alpha} = U_\alpha U_{-\alpha} U_\beta U_{-\beta}$ and Lemma 3 applies. ■

3. THE CASE BY CASE ANALYSIS OF COREGULARITY

Types II, IV: Here $\mathfrak{l}$ is a simple Lie algebra over $\mathbb{C}$, $\mathfrak{g}=\mathfrak{l}\times\mathfrak{l}$ and $\theta(x, y)=(y, x)$. Then it is easy to see that $\mathfrak{k} \simeq \mathfrak{l}$ and as $\mathfrak{k}$-module, $\mathfrak{g}$ is Ad $\oplus$ Ad. Looking at Schwarz tables in [Sch 1], we see that $(\mathfrak{g}, K)$ is coregular iff $\mathfrak{l}=\mathfrak{sl}(2, \mathbb{C})$ (table 1.a.18).

Types I, III: The Classical Structures

Type AI: Here $\mathfrak{g}=\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{k}=\mathfrak{so}(n, \mathbb{C})$ with $n \geq 3$. (For $n=2$, it is isomoprhic to BDI, $p=2$, $q=1$). If $(\mathfrak{g}, K)$ were coregular, then by [Sch 1], table

3a, $\mathfrak{p}$ must be φ_1, the natural action in $\mathbb{C}^n$. We get a contradiction computing $\dim \mathfrak{p} = (n^2+n)/2-1$.

Type AII: Here $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$, $\mathfrak{k} \simeq \mathfrak{sp}(n, \mathbb{C})$ with $n \geq 3$. (For $n=2$, it is isomorphic to BDI, $p=6$, $q=1$). The Schwarz notation for Ad is $\varphi_1{}^2$ so that if $(\mathfrak{g}, K)$ were coregular, by Table 4a, $\mathfrak{p}$ must be φ_1, the natural action on $\mathbb{C}^{2n}$. As $\dim \mathfrak{p} = 2n^2-n-1$, we get a contradiction.

Type AIII: Here $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$, $\mathfrak{k} = \{\begin{pmatrix} A & O \\ O & B \end{pmatrix} \in \mathfrak{g} : A \in \mathbb{C}^{p\times p}\}$, $\mathfrak{p} = \{\begin{pmatrix} O & C \\ D & O \end{pmatrix} \in \mathfrak{g}\}$. When $q=1$, corregularity of $(\mathfrak{g}, K)$ was proved by Cooper in [C]. So, let $q \geq 2$. We can choose $\mathfrak{a}$ as in [He], p. 368. As it was pointed out in the Introduction, we may assume that $K = \{\begin{pmatrix} A & O \\ O & B \end{pmatrix} \in SL(p+q, \mathbb{C}), A \in \mathbb{C}^{p\times p}\}$ and then it is easy to see that $M = \{\begin{pmatrix} A & O \\ O & B \end{pmatrix} \in K : B$ is diagonal, $A = \{\begin{pmatrix} B & O \\ O & C \end{pmatrix}\}$. If we can show that $(\mathfrak{k}, M)$ is non-coregular, we are done.

Now, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3$, where $\mathfrak{k}_1 = \{\begin{pmatrix} A & O \\ O & O \end{pmatrix} \in \mathfrak{g}\} \simeq \mathfrak{sl}(p, \mathbb{C})$, $\mathfrak{k}_2 \simeq \mathfrak{sl}(q, \mathbb{C})$, and $\mathfrak{k}_3 \simeq \mathbb{C}$ is the center of $\mathfrak{k}$. As M-module, $\mathfrak{k}_1$ admits a submodule isomorphic to $\mathfrak{k}_2 : \{\begin{pmatrix} A & O \\ O & O \end{pmatrix} \in \mathfrak{g} : A = \begin{pmatrix} C & O \\ O & O \end{pmatrix}$, with $c \in \mathfrak{sl}(q, \mathbb{C})\}$ and the action of M in $\mathfrak{k}_2$ is given by $B.(a_{ij}) = (b_i b_j^{-1} a_{ij})$ if B is the diagonal $(b_1, ..., b_q)$. Let V be the M-submodule of $\mathfrak{sl}(q, \mathbb{C})$, $V = \{(a_{ij}) : a_{ii} = 0 \, \forall i\}$. Clearly, it suffices to show that $(V \oplus V, M)$ is not coregular. Note that $q \neq 1$ implies $V \neq 0$. Putting $S'(V)^M = A_o \oplus A_1 \oplus ...$, a_{ij}, b_{ij} the canonical coordinates of the first and the second copy of V, respectively, then $A_1 = 0$ and $A_2 = \langle a_{ij} a_{ji}, b_{ij} b_{ji}, a_{ij} b_{ji} \rangle$. Thus Lemma 3 applies.

Type BDI: Here $\mathfrak{g} = \mathfrak{so}(p+q, \mathbb{C})$, $\mathfrak{k} = \mathfrak{so}(p, \mathbb{C}) \oplus \mathfrak{so}(q, \mathbb{C}) = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{p} = \{\begin{pmatrix} O & B \\ C & O \end{pmatrix} : B \in \mathbb{C}^{p\times q}, B + {}^tC = 0\}$. We can choose $\mathfrak{a} = \{\begin{pmatrix} O & B \\ C & O \end{pmatrix} : B$ is «diagonal», i.e. $b_{ij} = 0$ if $i \neq j\}$.

We may assume that $K = SO(p, \mathbb{C}) \times SO(q, \mathbb{C})$ and then it is easy to see that $M = \{(A, B) \in K$: B is the diagonal $(\epsilon_1, ..., \epsilon_q)$ with $\epsilon_i^2 = 1$, $\Pi \epsilon_i = 1$, and $A = \begin{pmatrix} B & O \\ O & C \end{pmatrix}$ with $C \in SO(p-q, \mathbb{C})\}$. $q=1$: Then $(\mathfrak{g}, K)$ is coregular by Cooper [C], Benabdallah [B], or [Sch 1], Table 3 a.2.

$q \geq 3$: It follows from Lemmas 1 and 2 that $(\mathfrak{g}, K)$ coregular implies $(\mathfrak{k}_2, M)$ coregular. Note that the morphism $M \longrightarrow GL(\mathfrak{k}_2)$, say ρ, depends clearly only on $B = (\epsilon_1, ..., \epsilon_q)$ and $\rho(B)(X_{ij}) = (\epsilon_i \epsilon_j X_{ij})$. Then det $\rho(B) = \Pi_{i<j} \epsilon_i \epsilon_j = (\Pi_i \epsilon_i)^{q-1} = 1$. For $B = (1, -1, -1, 1, ..., 1)$, $\rho(B) \neq \mathrm{Id}$; therefore $\rho(M)$ is a finite, non trivial subgroup of $SL(\mathfrak{k}_2)$ and Lemma 1 applies.

$q=2$: Here $M \cong SO(p-2, \mathbb{C}) \times \{\pm I_2\}$, where I_2 is the identity of $GL(2, \mathbb{C})$.

Now, as M-module, $\mathfrak{f}_1 \cong \mathfrak{so}(p-2, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C}) \oplus \mathbb{C}^{p-2} \oplus \mathbb{C}^{p-2}$; where $\mathfrak{so}(p-2, \mathbb{C}) \cong \{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{f}_1 : \quad A \in \mathfrak{so}(p-2, \mathbb{C}) \}$, $\mathfrak{so}(2, \mathbb{C})$ similarly $\mathbb{C}^{p-2} \oplus \mathbb{C}^{p-2} \cong \{ \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathfrak{f}_1 : Y \in \mathbb{C}^{(p-2)\times 2}, x = -{}^tY \} = V$. If $x \in \mathfrak{so}(p-2, \mathbb{C})$ is semisimple regular, then $M^x = T \times \{\pm I_2\}$ where T is a maximal torus of $\mathfrak{so}(p-2, \mathbb{C})$. Thus it suffices to show that $(V, T \times \{\pm I_2\})$ is not coregular. If $(A, \epsilon I) \in M^x$ and $(Y_1, Y_2) \in V$, the action is given by $(\epsilon A Y_1, \epsilon A Y_2)$. Then

$$S'(V)^{TX\{\pm I\}} = \oplus_{j,\,even} S'(V)_j^T = \oplus_{j,\,even} S'(V)_j^t$$

where using an appropiate characterization of $\mathfrak{so}(p-2, \mathbb{C})$, the Cartan subalgebra $\mathfrak{t}$ can be chosen $\{ \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} : D$ is a diagonal $(d_1, \ldots, d_k) \}$, if $p-2 = k$ is even. (The argument when p is odd is similar).

If $v_1, \ldots, v_{2k}, w_1, \ldots, w_{2k}$ is the dual basis associated with $\{ (e_j, 0), (0, e_j) \}$ then $S'(V)_2^T = \langle v_i v_{k+i}, w_i w_{k+i}, v_i w_{k+i}, v_{k+i} w_i \rangle$ and Lemma 3 applies.

Note that k must be ≥ 1, i.e. $p \geq 4$. The remaining cases are (3,2) and (2,2); respectively, $\mathfrak{sp}(2, \mathbb{R})$ (type CI) and $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ (type AI $\times$type AI).

Type DIII: Here $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{f} \simeq \mathfrak{gl}(n, \mathbb{R})$ and as $\mathfrak{f}$-module, $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ where $\mathfrak{p}_i \simeq \mathfrak{so}(n, \mathbb{C})$ with actions $\sigma_1(Z)(U) = ZU + U{}^tZ$, $\sigma_2(Z)\ (U) = -{}^tZU - UZ$.

We can choose $\mathfrak{a} = \{ (V, V) : V = \sum \lambda_j (e_{2j-1,2j} - e_{2j,2j-1}),\ \lambda_j \in \mathbb{C} \}$. We can assume that $K = GL(n, \mathbb{C})$ and then it easy to show that $M \simeq SL(2, \mathbb{C}) \times \ldots \times SL(2, \mathbb{C})$, h times, if $n = 2h$ is even and $M \simeq SL(2, \mathbb{C})^h \times \mathbb{C}^*$ if $n = 2h+1$ is odd. The isomorphism is realized by «blocks in the diagonal». By Lemmas 1 and 2 it suffices to study the pair $(\mathfrak{f}, M)$.

Consider the M-submodule of $\mathfrak{f}$

$$V = \{ Z \in \mathfrak{f} : Z_{ij} = 0 \text{ if } i \geq 4 \text{ or } j \geq 4 \}$$

Obviously $(V, M) \simeq (\mathrm{Ad} \oplus V_1 \oplus V_2,\ \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}))$. Thus we look at $(V_1 \oplus V_2, T)$, where T is a maximal torus of $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ and the action is given by $(t, r)(A, B) = (tAr^{-1}, rBt^{-1})$. Let a_i, b_i be the canonical basis of V_j, $j = 1, 2$.

If $S'(V_1 \oplus V_2)^T = A_0 \oplus A_1 \oplus \ldots$, then $A_1 = 0$, $A_2 = \langle a_1 a_4, a_2 a_3, b_1 b_4, b_2 b_3, a_1 b_1, a_2 b_3, a_3 b_2, a_4 b_4 \rangle$. Thus Lemma 3 applies.

This method works for $n \geq 4$. But for $n = 2, 3$ $\mathfrak{g}_{\mathbb{R}}$ is isomorphic to AIII and AI$\times$AI, respectively.

Type CI: Here $\mathfrak{g}=\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{k}\simeq\mathfrak{gl}(n, \mathbb{C})$ and as $\mathfrak{k}$-module, $\mathfrak{p}\simeq\mathfrak{p}_1\oplus\mathfrak{p}_2$ where $\mathfrak{p}_i=\{A\in\mathfrak{gl}(n, \mathbb{C}) : A={}^tA\}$ with actions $\sigma_1(Z)(A)=ZA+A^tZ$, σ_2 the dual of σ_1. We can choose $\mathfrak{a}=\{(D, D): D$ is diagonal$\}$ and if we assume that $K=GL(n, \mathbb{C})$, it is easy to see that $M=\{X\in K: X$ is a diagonal $(\epsilon_1, ..., \epsilon_n)$ with $\epsilon_i=\pm 1\}$. Looking at the pair $(\mathfrak{k}, M)$ it is immediately that $\det \mathrm{Ad}\, m=\Pi_{i,j}(\epsilon_i\epsilon_j)=1$, if $m=(\epsilon_1, ..., \epsilon_n)\in M$. But $m=(-1, 1, ..., 1)$ acts non trivially so that Lemma 1 iv) applies.

This method works for $n>1$. For $n=1, \mathfrak{sp}(1, \mathbb{C})\simeq\mathfrak{sl}(2, \mathbb{C})$, trivially coregular.

Type CII: Here $\mathfrak{g}=\mathfrak{sp}(p+q, \mathbb{C})$; $\mathfrak{k}\simeq\mathfrak{sp}(p, \mathbb{C})\oplus\mathfrak{sp}(q, \mathbb{C})$ and $\mathfrak{p}\simeq\mathbb{C}^{2p\times 2q}$ with the action $(Z_1, Z_2)X=Z_1X-XZ_2$.

We can choose $\alpha=\{\begin{pmatrix} A & O\\ O & A\end{pmatrix} : A\in\mathbb{C}^{p\times q},\ A=\sum\lambda_i e_{ii}\}$. We can assume that $K=SP(q, \mathbb{C})\times SP(q, \mathbb{C})$ and then it is easy to see that $M=\{(X_1, X_2)\in K$: $X_2=\begin{pmatrix} A^1 & A^2\\ A^3 & A^4\end{pmatrix}$ with A^j diagonal in $GL(q, \mathbb{C})$, $A^1_{ii}A^4_{ii}-A^3_{ii}A^2_{ii}=1$ and $X_1=\begin{pmatrix} B^1 & B^2\\ B^3 & B^4\end{pmatrix}$ with $B^j=\begin{pmatrix} A^j & O\\ O & C^j\end{pmatrix}$, $\begin{pmatrix} C^1 & C^2\\ C^3 & C^4\end{pmatrix}\in SP(p-q, \mathbb{C})\}$.

That is, $M\simeq SL(2, \mathbb{C})^q\times SP(p-q, \mathbb{C})$. Now we can assume $q>1$ because for $q=1, p\geq 2$ we are in the situation of Lemma 4 and $\mathfrak{g}_{\mathbb{R}}^{-}=\mathfrak{sp}(1,1)\simeq\mathfrak{so}(4,1)$, implies $(\mathfrak{g}, K)$ coregular.

It is clear that $\mathfrak{k}_1$ has a M-submodule isomorphic to $\mathfrak{k}_2$, so we are done proving the non coregularity of $(\mathfrak{k}_2\oplus\mathfrak{k}_2, SL(2, \mathbb{C})^q)$.

Put $V_{ij}=\langle e_{i,j}-e_{q+j,q+i}, e_{j,i}-e_{q+i,q+j}, e_{i,q+j}+e_{j,q+i}, e_{q+i,j}+e_{q+j,i}\rangle$ if $i\neq j$ and $W_i=\langle e_{i,i}-e_{q+i,q+i}, e_{q+i,i}, e_{i,q+i}\rangle$; then

$$\mathfrak{k}_2=(\oplus_i W_i)\oplus(\oplus_{i<j}V_{ij}) \text{ and } \oplus_i W_i\simeq \mathbf{Ad}(SL(2, \bar{\mathbb{C}})^q).$$

So we can restrict our attention to the pair $(V_{12}\oplus V_{12}, T)$ where $T=\{(X_1, ..., X_q): X_j$ is a diagonal in $\mathfrak{sl}(2, \mathbb{C})\}$. If α_s, β_r are the dual basis to the descripted above, and $S'(V_{12}\oplus V_{12})^T=A_0\oplus A_1\oplus..$then $A_1=0$, $A_2=\langle\alpha_1\alpha_2, \alpha_3\alpha_4, \beta_1\beta_2, \beta_3\beta_4, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_3\beta_4, \alpha_4\beta_3\rangle$ and Lemma 3 applies.

The Exceptional Structures

Most of the cases follows from Schwarz tables [Sch 1] or from Lemma 4. So we list them. The reference for the K-module structure of $\mathfrak{p}$ is [F de V].

Type	$\mathfrak{g}$	$\mathfrak{f}$	$\mathfrak{p}$	Method
EI	e_6	$\mathfrak{sp}(4,\mathbb{C})$	42	Table 4a.3, $\dim\mathfrak{p}\neq 8$
EIV	e_6	f_4		" 5a.4
EV	e_7	$\mathfrak{sl}(8,\mathbb{C})$	70	" 1a.20, $\dim\mathfrak{p}\neq 8$
EVIII	e_8	$\mathfrak{so}(16,\mathbb{C})$	128	" 3a.2, $\dim\mathfrak{p}\neq 16$
FII	f_4	$\mathfrak{so}(9,\mathbb{C})$	16	" 3a.5, $\dim\mathfrak{p}\neq 9$
EII	e_6	$\mathfrak{sl}(6,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$	$\Lambda^3(\mathbb{C}^6)$	Lemma 4
EVI	e_7	$\mathfrak{so}(12,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$	λ_5(spin)	"
EIX	e_8	$e_7\times\mathfrak{sl}(2,\mathbb{C})$	λ_7	"
FI	f_4	$\mathfrak{sp}(3,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$	λ_3	"
G	g_2	$\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$	$V(3)=3\lambda_1$	"

Note: under «$\mathfrak{p}$» we have listed dim $\mathfrak{p}$ for [Sch 1], ρ_1 for Lemma 4. Here λ_j means the j-fundamental weight, as in [Hu].

There are two remaining cases:

Type EIII: Here $\mathfrak{g}=e_6$, $\mathfrak{f}=\mathfrak{so}(10,\mathbb{C})\oplus\mathbb{C}$, $\mathfrak{p}=\mathfrak{p}_+\oplus\mathfrak{p}_-$. As K-module, $\mathfrak{p}_-$ is dual to $\mathfrak{p}_+$; $\mathfrak{p}_+$ is λ_5(spin) as $[\mathfrak{f},\mathfrak{f}]$-module and $\mathbb{C}$ = center of $\mathfrak{f}$ acts by non-trivial scalars.

Type EVII: Here $\mathfrak{g}=e_7$, $\mathfrak{f}=e_6\oplus\mathbb{C}$; $\mathfrak{p}=\mathfrak{p}_+\oplus\mathfrak{p}_-$, $\mathfrak{p}_+$ is λ_1, etc.

We develope an argument for both of them. Let $\mathfrak{b}=\mathfrak{t}\oplus\mathfrak{d}$ be a Cartan subalgebra of $\mathfrak{f}$, where $\mathfrak{t}$ is a C. s. of $[\mathfrak{f},\mathfrak{f}]$ and $\mathfrak{d}$ is the center; let H be the corresponding maximal torus. The goal is to prove the non-coregularity of $(\mathfrak{p},H)$. Let $\sigma\in\mathfrak{d}^*$ associated to the action on $\mathfrak{p}_+$; $\sigma\neq 0$ because $\mathfrak{g}$ has trivial center. By abuse of notation we call also σ the extension to $\mathfrak{b}$ vanishing on $\mathfrak{t}$; the same convention for $\lambda\in\Pi(\mathfrak{p}_+,\mathfrak{t})$.

Then $\Pi(\mathfrak{p}_+,\mathfrak{b})=\{\lambda+\sigma\colon \lambda\in\Pi(\mathfrak{p}_+,\mathfrak{t})\}$. As usual, let $\{x_\lambda\}$ be the basis of $\mathfrak{p}_+$ where x_λ is a vector of weight $\lambda+\sigma$, $\lambda\in\Pi(\mathfrak{p}_+,\mathfrak{t})$; let $\{y_\lambda\}$ be the basis of $\mathfrak{p}_-$ where y_λ is a vector of weight $-\lambda-\sigma$, and let $\{X_\lambda, Y_\lambda\}$ be the corresponding dual basis. If $S'(\mathfrak{p},H)=\oplus_{i\geq 0} A_i$ then $A_m=<X_{\lambda_1}\dots X_{\lambda_r}, X_{\lambda_{r+1}}\dots X_{\lambda_m}$: $\Sigma_{i\geq r}(\lambda_i+\sigma)+\Sigma_{i>r}(-\lambda_i-\sigma)=0>$. Thus $A_m=0$ if m is odd and $A_2=<X_\lambda\ Y_\lambda>$. Now assume that there are some $\lambda_1,\dots,\lambda_4$ in $\Pi(\mathfrak{p}_+,\mathfrak{t})$ such that $\lambda_1+\lambda_2=\lambda_3+\lambda_4$ and $\lambda_1\neq\lambda_3,\lambda_4$. Then $X_{\lambda_1}X_{\lambda_2}Y_{\lambda_3}Y_{\lambda_4}$, $X_{\lambda_3}X_{\lambda_4}Y_{\lambda_1}Y_{\lambda_2}$ do not belong to A_2^2 and Lemma 3 applies.

The preceding hypothesis is fulfilled in both cases, as we can see easily; note that, as rank $\mathfrak{g}$ = rank $\mathfrak{f}$, we may look at the non-compact roots in $\mathfrak{p}_+$.

From the preceding analysis, we have:

Proposition D: $(\mathfrak{g},K)$ is coregular if and only if it corresponds to $\mathfrak{g}_{\mathbb{R}}=\mathfrak{so}(p,1)$ or $\mathfrak{su}(p,1)$. ∎

4. THE OTHER CRITERIA

Here we assume that L is a semisimple complex algebraic group, and that V is L-orthogonalizable; see section 2.

Proposition E: *i) If every root it in the* $\mathbb{Z}$*-span of* $\Pi(V)$*, then the principal isotropy class of* $(Ad\oplus V, L)$ *is (the class of)*

$$Ker\,(L \longrightarrow GL(Ad\oplus V)).$$

ii) If L is simple and V is non trivial, then every root is in the $\mathbb{Z}$*-span of* $\Pi(V)$.

iii) If $\mathfrak{g}$ *is simple, the principal isotropy class of* ($\mathfrak{g}$, *K*) *is trivial, discarding the trivial case when* $\mathfrak{p}=0$.

iv) ($\mathfrak{g}$, *K*) *never satisfies criteria E.*

Proof: *i)* Let H be the maximal torus of L whose Lie algebra is $\mathfrak{h}$ and pick any element $x\in\mathfrak{h}$ such that $L^x=H$. As $V=\oplus_{\lambda\in\Pi(V)}V_\lambda$, we can choose $y=\sum_\lambda y_\lambda$, $y_\lambda\in V_\lambda-0$. It follows that

$$L^{x+y}=L^x\cap L^y=H\cap L^y=\{A\in H\colon Ay_\lambda=y_\lambda \text{ for all } \lambda\in\Pi(V)\}$$

Now such $A=\exp a$, for some $a\in\mathfrak{h}$, and

$$(\mathrm{Ad}_L\,A)\,y_\lambda=(\exp a)\,y_\lambda=e^{\lambda(a)}y_\lambda=y_\lambda$$

Then $\lambda(a)\in 2\pi i\mathbb{Z}$ for all $\lambda\in\Pi(V)$, because $y_\lambda\neq 0$. By hypothesis, $\mu(a)\in 2\pi i\mathbb{Z}$ for every root μ and then $A\in\mathrm{Ker\ Ad}\ L$.

As V is L-orthogonalizable, the same is true for $\mathrm{Ad}\oplus V$. So, it only remains to show that the set $\{Z\in\mathfrak{l}\oplus V\colon L^Z=\mathrm{Ker\ Ad}\ L\}$ is dense in $\mathfrak{l}\oplus V$.

Let U be a Zariski open non empty subset of $\mathfrak{l}\oplus V$; its image under the projection map $\mathfrak{l}\oplus V\longrightarrow\mathfrak{l}$ is open so it exists x regular semisimple such that for some $y\in V$, $x+y\in U$. Now $\mathfrak{l}^x$, the centralizer of x in $\mathfrak{l}$, is a Cartan subalgebra of $\mathfrak{l}$. From the conjugacy theorem, it follows that $\Phi(\mathfrak{l},\mathfrak{l}^x)$ is contained in the $\mathbb{Z}$-span of $\Pi(V,\mathfrak{l}^x)$. $\{y\in V\colon x+y\in U\}$ and $\{y\in V\colon y_\lambda\neq 0\ \forall\,\lambda\in\Pi(V,\mathfrak{l}^x)\}$ are both open non empty; taking y in the intersection, $x+y\in U$ and $L^{x+y}=\mathrm{Ker\ Ad}$.

ii) Let W be the subgroup of $\mathfrak{h}^*$ generated by $\Pi(V)$ and let $\Phi=\Phi(\mathfrak{l},\mathfrak{h})$. We claim that $\Phi=(\Phi\cap W)\cup(\Phi\cap W^\perp)$. It suffices to show that $\Phi-W^\perp\subset W$. If $\alpha\in\Phi-W^\perp$, there is some $\mu\in\Pi(V)$ such that $(\alpha,\mu)\neq 0$. The α-string through

μ is μ-$r\alpha$, ..., $\mu+q\alpha$ with $r-q=(\alpha,\mu)\neq 0$; thus $\mu\pm\alpha\in\Pi(V)$ and $\alpha\in W$. Since $\mathfrak{l}$ is simple, Φ is irreducible; as $V\neq 0$, $W\neq 0$ and $\Phi=\Phi\cap W$.

iii) Let L be the connected subgroup of K with Lie algebra $\mathfrak{l}=[\mathfrak{k},\mathfrak{k}]$, let $V=\mathfrak{p}$ and let $\mathfrak{h}$, W, Φ be as in the proof of *ii)*. Then $\Phi=(\Phi\cap W)\cup(\Phi\cap W^{\perp})$. Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be the ideals of $\mathfrak{l}$ such that: if $\mathfrak{h}_j$ is a Cartan subalgebra of $\mathfrak{k}_j$ given by $\mathfrak{h}_j=\mathfrak{h}\cap\mathfrak{k}_j$, then the root systems $\Phi(\mathfrak{k}_1,\mathfrak{h}_1)$ and $\Phi(\mathfrak{k}_2,\mathfrak{h}_2)$ are identified with $\Phi\cap W$ and $\Phi\cap W^{\perp}$ respectively. If $\lambda\in\Pi(\mathfrak{p},\mathfrak{h})$, $\lambda(\mathfrak{h}_2)=0$. Thus the action of $\mathfrak{k}_2$ in $\mathfrak{p}$ is trivial. Now Jacobi implies that $[\mathfrak{p},\mathfrak{p}]$ is an ideal of $\mathfrak{k}$ and that $[\mathfrak{k}_2,[\mathfrak{p},\mathfrak{p}]]=0$. Then if $\mathfrak{z}=$center of $\mathfrak{k}$, $[\mathfrak{k}_1+\mathfrak{z}+\mathfrak{p},\mathfrak{k}_2]=0$ and $\mathfrak{k}_1+\mathfrak{z}+\mathfrak{p}$, $\mathfrak{k}_2$ are ideals of $\mathfrak{g}$. By hypothesis $\mathfrak{k}_2=0$ and $\Phi=\Phi\cap W$. Assume here that dim $\mathfrak{z}=1$; as $\mathfrak{k}$-module, $\mathfrak{p}=\mathfrak{p}_+\oplus\mathfrak{p}_-$ and $\mathfrak{z}$ acts in $\mathfrak{p}_+$(in $\mathfrak{p}_-$) via $\sigma\neq 0$ (via $-\sigma$). Also $\Pi(\mathfrak{p}_-,\mathfrak{h})=-\Pi(\mathfrak{p}_+,\mathfrak{h})$. Recalling that $\Phi\cup(\Pi(\mathfrak{p}_+,\mathfrak{h})\times\{\sigma\})\cup(\Pi(\mathfrak{p}_-,\mathfrak{h})\times\{-\sigma\})=\Phi(\mathfrak{g},\mathfrak{h}+\mathfrak{z})$ it is also true that $\{\alpha\in\Phi:(\alpha,\Pi(\mathfrak{p}_+,\mathfrak{h}))>0\}=\Phi_+$ for some choice of a base Δ.

Pick $x\in\mathfrak{k}$, $c\in\mathfrak{z}$, $y\in\mathfrak{p}$ such that $K^{x+c}=H\times Z$ is a maximal torus of K. We want to show that $K^{x+c+y}=K^{x+c}\cap K^{y}=\mathrm{Ker}\,\mathrm{Ad}_{\mathfrak{g}}(K)$. Let $H_1\in\mathfrak{h}$, $H_2\in\mathfrak{z}$ such that $\exp(H_1+H_2)\in K^{y}$. Then $\forall\,\lambda\in\Pi(\mathfrak{p}_+,\mathfrak{h})$ $\lambda(H_1)+\sigma(H_2)\in 2\pi i$. If $\alpha\in\Phi_+$, $\alpha=\lambda_1-\lambda_2$, for some $\lambda_i\in\Pi(\mathfrak{p}_+,\mathfrak{h})$ (look at the α-string). Then $\alpha(H_1)\in 2\pi i\mathbb{Z}$. If $\alpha\in\Phi_+$, $\alpha=\lambda_1-\lambda_2$, for some $\lambda_i\in\Pi(\mathfrak{p}_+,\mathfrak{h})$ (look at the α-string). Then $\alpha(H_1)\in 2\pi i\mathbb{Z}$ and we can follow the line of the proof of *i)*.

iv) For types II-IV it follows from *ii)*; in other case from *iii)*. ∎

Next we will study the dimension of $\mathfrak{g}/K$. We return to the assumption: «L reductive».

From Algebraic Geometry we know, for $\zeta\in V/L$:

$$\dim \pi^{-1}(\zeta)+V/L\geq \dim V. \qquad [1]$$

Furthermore, there exists an open dense subset U of V such that $\forall\,\zeta\in\pi(U)$, the equality in [1] holds.

Lemma 5: *dim $\mathfrak{g}/K=$ dim $\mathfrak{p}$*

Proof: If V/L has generically closed orbits (i.e., the union of the closed orbits contains a non empty open set) then it is follows from[1] that $\dim V/L=\dim V-\dim L+\dim H$, where (H) is a principal isotropy class. Being $\dim H=0$ from Proposition E, $\dim\mathfrak{g}/K=\dim\mathfrak{g}/K=\dim\mathfrak{g}-\dim K=\dim\mathfrak{p}$. ∎

Our following task is to compute the dimension of $\mathfrak{N}$, the cone of unstable points in Mumford's terminology, using the ideas exposed in [Sch 2], via the

Hilbert-Mumford criterion. For convenience, we will summaryze them. See also [M 1], Ch. II or [M 2], p. 41.

Let $\Lambda: \mathbb{C}^* \longrightarrow L$ be a morphism of algebraic groups, briefly a 1-PS. Put $Z_\Lambda = \{v \in V: \Lambda(z)v \longrightarrow 0 \text{ if } z \longrightarrow 0\}$. From the well known characterization $\mathfrak{N}(V, L) = \{v \in V: f(v) = 0 \,\forall\, f \in S'(V)^L$ homogeneous of positive degree$\}$ it follows that $\mathfrak{N}$ contains the various Z_Λ. In fact, the Hilbert-Mumford criterion insures that $\mathfrak{N}(V, L) = \cup_{\Lambda,\, 1-PS} Z_\Lambda$. Now if T is a maximal torus of L and Λ is a 1—PS, IM Λ is conjugated to a subgroup of T and

$$\mathfrak{N}(V, L) = \cup_{\Lambda,\, 1-PS\, in\, T} L.\, Z_\Lambda.$$

Let $\mathfrak{t}$ be the Cartan subalgebra of the Lie algebra of L, $\mathfrak{l}$, corresponding to T. If Λ is a 1—PS in T, note by λ its infinitesimal generator. If $V = \oplus_{\mu \in \Pi(V, \mathfrak{t})} V_\mu$, then $\mu(\lambda) \in \mathbb{Z}$ and $\forall v \in V_\mu$, $z \in \mathbb{C}^*$: $\Lambda(z) v = z^{\mu(\lambda)} v$. So $Z_\Lambda = \oplus_{\mu: \mu(\lambda) > 0} V_\mu$; thus $\mathfrak{N}(V, L)$ is union of a finite number of $L.\, Z_\Lambda$. Call $c_\Lambda = \text{codim } L.\, Z_\Lambda$; then

$$\text{codim } \mathfrak{N} = \inf\{c_\Lambda: \Lambda \text{ is a 1—PS in } T\}.$$

Now let $\mathfrak{p}_\Lambda$ be the (parabolic) subalgebra of $\mathfrak{l}$ that normalizes Z_Λ, $\mathfrak{u}_\Lambda$ the subalgebra of $\mathfrak{l}$ generated by the root vectors not in $\mathfrak{p}_\Lambda$, U_Λ the connected algebraic subgroup of L corresponding to $\mathfrak{u}_\Lambda$. Following [Sch 2] we have $\mathfrak{l} = \mathfrak{p}_\Lambda \oplus \mathfrak{u}_\Lambda$ and

$$c_\Lambda = \dim V - \dim Z_\Lambda - \dim U_\Lambda + e_\Lambda \geq \dim V - \dim Z_\Lambda - \dim U_\Lambda \quad [2]$$
$$\text{where } e_\Lambda = \dim U_\Lambda - \sup\{\dim(T_z(U_\Lambda z) + Z_\Lambda)/Z_\Lambda : z \in Z_\Lambda\}$$

Furthermore, $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{d}$, where $\mathfrak{b}$ is a Cartan subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. Then $\lambda = \lambda_\mathfrak{b} + \lambda_\mathfrak{d}$ (obvious notation). Call φ_λ the unique element in $\mathfrak{b}^*$ such that $\varphi_\lambda(H) = \text{Killing}(\lambda_\mathfrak{b}, H) \,\forall h \in \mathfrak{b}$. Now, $\forall \mu \in \Phi([\mathfrak{l}, \mathfrak{l}], \mathfrak{b}): (\varphi_\lambda, \mu) = \mu(\lambda) \in \mathbb{Z}$ and then $\varphi_\lambda \in E = \mathbb{R}$-span of $\Phi([\mathfrak{l}, \mathfrak{l}], \mathfrak{b})$ in $\mathfrak{b}^*$. (See [Hu], p. 40 and p. 67).

Finally, $\gamma(\mathfrak{t}) = \{\lambda \in \mathfrak{t} : \lambda = d\Lambda(1) \text{ for some } 1-\text{PS } \Lambda \text{ in } \mathfrak{t}\}$ is isomorphic to $\Gamma(T) = \{\Lambda: \lambda\ 1-\text{PS in } T\}$ via $\Lambda \longrightarrow \lambda$; then it is isomorphic to $\mathbb{Z}^d$, $d = \dim \mathfrak{t}$. Moreover, $\gamma(\mathfrak{t})$ is a lattice in $\mathfrak{t}$ and then identifying $\gamma(\mathfrak{b})$ with $\{\varphi_\lambda: \lambda \in \gamma(\mathfrak{b})\}$, $\gamma(\mathfrak{b})$ meets every open cone in E. (See [Ch], 9-06). As usual, rk denotes the rank.

Lemma 6: *codim* $\mathfrak{N} = 1/2\,(\dim \mathfrak{p} + rk\, \mathfrak{g} + rk\, \mathfrak{k})$

Proof: Let $\mathfrak{t}_\mathfrak{g}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t} = \mathfrak{t}_\mathfrak{g} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$. As above, $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{d}$, with $\mathfrak{b}$ a C. s. of $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. Put $\phi = \phi(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$.

i) Let first $L = K$ acting on $V = \mathfrak{k}$ by Ad. Let Λ be a $1-$PS in T. If λ is regular (i.e., φ_λ lies in the interior of some Weyl chamber) then $Z_\Lambda = \mathfrak{k}_+$ for the ordering defined by φ_λ. If not, an easy argument shows that $Z_\Lambda \subsetneqq \mathfrak{f}_+$ for some $\mathfrak{f}_+$. Now the Chevalley Restriction Theorem guarantees that $\dim(\mathfrak{f}/K) = \operatorname{rk}\mathfrak{f}$. For λ regular $\mathfrak{p}_\Lambda = \mathfrak{f}_+ \oplus \mathfrak{t}$, $\mathfrak{u}_\Lambda = \mathfrak{f}_-$. Then from [1] and [2]:

$$\operatorname{rk}\mathfrak{f} \geq \operatorname{codim} \mathfrak{N}(\mathfrak{f}, K) \geq \dim \mathfrak{f} - \dim \mathfrak{f}_+ - \dim \mathfrak{f}_- = \operatorname{rk}\mathfrak{f}.$$

All of this is well known; the profit for us is that $e_\Lambda = 0$; so there exists $Z \in \mathfrak{f}_+$ such that $\dim(T_z(U_\Lambda Z) + \mathfrak{f}_+)/\mathfrak{f}_+ = \dim \mathfrak{f}_-$.

ii) Let now $(V, L) = (\mathfrak{g}, K)$ and let F be the $\mathbb{R}$-span of ϕ. If $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{f}$, then $\mathfrak{t}_\mathfrak{g} = \mathfrak{t}$ it is clear that there are $1-\mathrm{PS}$ in T, regular in both $\mathfrak{t}$ and $\mathfrak{t}_\mathfrak{g}$. We claim that the preceding is true even if $\operatorname{rk}\mathfrak{f} < \operatorname{rk}\mathfrak{g}$.

For $\mu \in \mathfrak{t}^*$, put $\alpha_\mu \in \mathfrak{t}_\mathfrak{g}^*$ as follows: μ in $\mathfrak{t}$, 0 in $\mathfrak{t}_\mathfrak{g} \cap \mathfrak{p}$. θ induces $\phi \longrightarrow \phi$, $\alpha \longrightarrow \alpha\theta$ and hence $F \longrightarrow F$, called also θ. Clearly $\{x \in F\colon \theta x = x\} = \{x \in F\colon x = \alpha_\mu$ for $\mu = x_{|\mathfrak{t}}\}$.

Next for $\alpha \in \phi$, put $\beta = \alpha_{|\mathfrak{t}}$. If $\alpha = \alpha\theta$, $\alpha = \alpha_\beta$ and $\mathfrak{g}_\alpha = \mathfrak{g}_\beta$. If not, put $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha\theta} = \mathfrak{g}_\beta$; it is θ-stable and $\alpha_\beta = 1/2(\alpha + \alpha\theta)$. Under the above map, $\phi(\mathfrak{f}, \mathfrak{t})$ is contained in F, hence E. We identify E with its image.

Now $\{x \in F\colon \theta x = x\}^\perp = \{x \in F\colon \theta x = -x\} = \{x \in F\colon x_{|\mathfrak{t}} = 0\} \supseteq \langle\{1/2(\alpha - \alpha\theta)\colon \alpha \in \phi\}\rangle$. As the Killing form on F is non degenerate, $E = \{x \in F\colon \theta x = x\}$, $E \oplus E^\perp = F$ and the restriction of the Killing form on F to E is still non degenerate.

We must prove that the Zariski open cone in E, $E \cap \{H \in F\colon H$ is regular$\}$ is non empty. If not, putting $P_\alpha = \{H \in F\colon (\alpha, H) = 0\}$, $\alpha \in \phi$, we have $E \subseteq \cup_\alpha P_\alpha$ and by irreducibility, $E \subseteq P_\alpha$ for some α. Now $\alpha = \alpha_1 + \alpha_2$, $\alpha_1 \in E$, $\alpha_2 \in E^\perp$. Thus $(\alpha_1, E) = 0$, hence $\alpha_1 = 0$ and $\alpha \in E^\perp$. That is, $\alpha_{|\mathfrak{t}} = 0$, $\alpha\theta = -\alpha$. Pick $X \in \mathfrak{s}_\alpha \cap \mathfrak{f}$: $X = X_+ = X_-$ with $X_+ \subset \in \mathfrak{g}_\alpha$, $X_- \in \mathfrak{g}_{-\alpha}$. $\forall\ H \in \mathfrak{t}[H, X] = [H, X+] + [H, X_-] = 0$; then $\mathfrak{s}_\alpha \cap \mathfrak{f} \subseteq \mathfrak{s}_\alpha \cap \mathfrak{t} \subseteq \mathfrak{s}_\alpha \cap \mathfrak{t}_\mathfrak{g} = 0 \therefore \mathfrak{s}_\alpha \subseteq \mathfrak{p}$. Then $\forall y \in \mathfrak{t}_\mathfrak{g} \cap \mathfrak{p}$, $[y, \mathfrak{s}_\alpha] \subseteq [\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{s}_\alpha \subseteq \mathfrak{f} \cap \mathfrak{p} = 0$. Then $\alpha_2 = 0$, a contradiction.

iii) From the preceeding and as in *i)*, it can be shown that $\mathfrak{N}$ is the union of the various K. Z_Λ with φ_λ in the open cone $\{X \in E\colon X$ is regular in both $\mathfrak{t}$ and $\mathfrak{t}_\mathfrak{g}\}$. Clearly, $\phi \cup 0 \longrightarrow \Pi(\mathfrak{g}, \mathfrak{t})$, $\alpha \longrightarrow \alpha_{|\mathfrak{t}}$ is surjective. Thus $Z_\Lambda = \oplus_{\beta:\beta(\lambda)>0} \mathfrak{g}_\beta = \mathfrak{g}_+$ for the order defined by λ. Even more,
$Z_\Lambda \supseteq \oplus_{\beta:\beta(\lambda)>0} \mathfrak{f}_\beta = \mathfrak{f}_+$, $\mathfrak{p}_\Lambda = \mathfrak{f}_+ \oplus \mathfrak{t}$ and $\mathfrak{u}_\Lambda = \mathfrak{f}_-$.

Pick $Z \in \mathfrak{f}_+$ such that $\dim(T_Z(U_\Lambda Z) + \mathfrak{f}_+)/\mathfrak{f}_+ = \dim \mathfrak{f}_-$; then $\dim(T_Z(U_\Lambda Z) + \mathfrak{g}_+)/\mathfrak{g}_+ = \dim \mathfrak{f}_-$; so $e_\lambda = 0$ and $\operatorname{codim} \mathfrak{N} = \dim \mathfrak{g} - \dim \mathfrak{g}_+ - \dim \mathfrak{f}_- = 1/2(\dim \mathfrak{p} + \operatorname{rk}\mathfrak{g} + \operatorname{rk}\mathfrak{f})$. ∎

Lemma 7 ([Sch 3], p. 129): *(V, L) is cofree ⟺ (V, L) is coregular and codim* $\mathfrak{N}(V/L) = \dim V/L$. ∎

Proposition B: *$(\mathfrak{g}, K)$ satisfies criteria B* $\Leftrightarrow$ $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ *or* $\mathfrak{su}(n, 1)$.

Proposition C: *$(\mathfrak{g}, K)$ is cofree* $\Leftrightarrow$ $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ *or* $\mathfrak{su}(n, 1)$.

Proofs: If (V, L) is visible, $\mathfrak{N}\,(V, L)$ is the closure of an orbit and then it codim $\mathfrak{N} = \dim(\mathfrak{g}/K)$ holds iff $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(\mathrm{n}, 1)$ or $\mathfrak{su}(\mathrm{n}, 1)$. In view of Lemma 7 and Proposition D, this implies Proposition C. Also, we have $\Rightarrow$ in Proposition B. But cofreeness implies flatness and then all the fibres have the same dimension. ■■

Remark: The cofreeness in case $\mathfrak{so}(\mathrm{n}, 1)$ is also proved in [Sch 2].

Proposition A: *$(\mathfrak{g}, K)$ is never visible.*

Proof: If (V, L) is visible, $\mathfrak{N}\,(V, L)$ is the closure of an orbit and then it follows easily that codim $\mathfrak{N} = \dim\,(V/L)$. (See [K], Lemma 3.5). Furthermore, for L linear reductive (V/L) visible implies that the multiplicity of any non-zero weight is at most 1. ([K], 3.4).

These two facts show the non-visibility of $(\mathfrak{g}, K)$ in most the cases, in view of Lemma 6 and the following well known fact:

If $\mathrm{rk}\,\mathfrak{g} > \mathrm{rk}\,\mathfrak{k}$, then there is some $\alpha \in \Pi_{\mathfrak{k}}(\mathfrak{g})$ with multiplicity greather than one. (In the notation of Lemma 6, we must pick $\alpha \in \phi(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$ such that $\alpha_{|\mathfrak{t}_{\mathfrak{g}} \cap \mathfrak{p}} \neq 0$).

There are two remaining cases:

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(\mathrm{n}, 1)$: Here $\mathfrak{g} = \mathfrak{sl}(\mathrm{n}+1, \mathbb{C})$, $\mathfrak{k} = \{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} \in \mathfrak{g}\colon A \in \mathfrak{gl}(\mathrm{n}, \mathbb{C}) \}$ and hence we may assume that $K = \{ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \in SL(\mathrm{n}+1, \mathbb{C})\colon X \in GL(\mathrm{n}, \mathbb{C}) \}$. Choosing as usual $\mathfrak{b} = \{ \sum_{i=1}^{n+1} H_i e_{i,i}\colon \sum H_i = 0 \}$ as Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$, it is well known that $\phi(\mathfrak{g}, \mathfrak{b}) = \{\alpha_{i,j}\colon \alpha_{i,j}(H) = H_i - H_j$ if $H = \sum H_i e_{i,i},\ i \neq j\}$. Take $\phi_+ = \{\alpha_{i,j}\colon (i < j$ and $j \leq n$ or $i < n)$ or $(i = n+1,\ j = n)\}$. It corresponds to the 1–PS Λ given by $\Lambda(z) =$ the diagonal $(z, z^2, z^3, \ldots, z^{n+1}, z^n)$. Thus, $\mathfrak{g}_+ \subseteq \mathfrak{N}$. Put for $c \in \mathbb{C}$: $y_c = \begin{pmatrix} T & u \\ v & 0 \end{pmatrix}$ where $T = \sum_i e_{i,i+1}$, $u = e_{n-1}$ and $v = c\,e_n$. We claim that: $y_c \in K y_d \Longrightarrow c = d$.

Let $\begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \in K$ such that $\begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} y_c = y_d$. Then $XT = TX$, $Xe_{n-1} = xe_{n-1}$, $cxe_n = \mathrm{d}e_n X$. Now it is easy to show that $X = xI_{n+1} + be_{1,n+1}$ and thus $c = d$.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2n, 1)$: Here $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$; we will follow the notation of [*Hu*], p. 3. Then $\mathfrak{k} = \{ x \in \mathfrak{g}\colon b_1 = b_2 = 0 \}$, $\mathfrak{p} = \{ x \in \mathfrak{g}\colon m = n = p = 0 \}$ and we assume that $K = \{ \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \in SL(2n+1, \mathbb{C})\colon {}^tXsX = s \}$

Choosing $\mathfrak{b}=\{H\in\mathfrak{g}\colon H \text{ is diagonal}\}$ as Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{f}$, it is known that there is some ordering for which $\mathfrak{g}_+=\{x\in\mathfrak{g}\colon b_1=0, p=0$ and m is upper triangular $\}\subseteq\mathfrak{N}$.

Put for $c\in\mathbb{C}$: $y_c=(b_2=e_n, m=T$ as above, $n=c(e_{n-1,n}-e_{n,n-1}))$. Then it is not so difficult to prove that $y_c=y_d$ iff $y_c\in Ky_d$.

5. SOME REMARKS ON THE UNSTABLE CONE

As a corollary of the proof of Lemma 6, we can state: $\mathfrak{N}(\mathfrak{g}, K)$ *is the union of the various* $K.\mathfrak{g}_+$. *Furthermore, codim* $(\mathfrak{N}(\mathfrak{g}, K))=$ *codim* $K.\mathfrak{g}_+$ *for every such* $\mathfrak{g}_+$. This suggests us that the irreducible components of $\mathfrak{N}$ are those $K.\mathfrak{g}_+$. Actually, this follows from a general fact (as in [*G*], Corollary 2, p. 142). Let (V, L) be as above, P a parabolic subgroup of L, W a linear subspace of V such that $P.W\subseteq W$. Then $L.W$ is closed (because of the completeness of L/P).

The following step is to compute $c_{\mathfrak{N}}$, the number of irreducible components of $\mathfrak{N}$. Assume first that rank $\mathfrak{f}=$ rank $\mathfrak{g}$; then $\mathfrak{N}(\mathfrak{g}, K)=U_{\text{every}\,\mathfrak{g}_+}K.\mathfrak{g}_+$. From ([*G*], Corollary 2) we also know that $K.\mathfrak{g}_+=K.\mathfrak{g}_+'$ if and only if there is some $\sigma\in W(\mathfrak{f}, \mathfrak{t})$ such that $\sigma(\mathfrak{g}_+)=\mathfrak{g}_+'$. (Use Bruhat decomposition). Thus.

$$c_{\mathfrak{N}}=|W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})|/|W(\mathfrak{f}, \mathfrak{t})|$$

Assume now rank $\mathfrak{f}<$rank $\mathfrak{g}$. We prove now some easy facts in order to compute $c_{\mathfrak{N}}$. As usual $N_L(S)$ (resp. $C_L(S)$) is the normalizer (resp., the centralizer) of S in L.

i) $N_G(\mathfrak{t})\subseteq N_G(\mathfrak{t}_{\mathfrak{g}})$

Proof: Let $Z\in N_G(\mathfrak{t})$, $\beta\in\Pi(\mathfrak{g}, \mathfrak{t})$. Then $Z.\mathfrak{g}_\beta\subseteq\mathfrak{g}_{\beta Z^{-1}}$. In particular, $Z.\mathfrak{g}_o=Z.\mathfrak{t}_{\mathfrak{g}}\subseteq\mathfrak{t}_{\mathfrak{g}}$.

ii) $C_K(\mathfrak{t})=N_K(\mathfrak{t})\cap C_G(\mathfrak{t}_{\mathfrak{g}})$

Proof: We only need to show $C_K(\mathfrak{t})\subseteq C_G(\mathfrak{t}_{\mathfrak{g}})$. By *i)*, $C_K(\mathfrak{t})\subseteq N_G(\mathfrak{t}_{\mathfrak{g}})$. Let $Z\in C_K(\mathfrak{t})$ and call ζ its class in $N_G(\mathfrak{t}_{\mathfrak{g}})/C_G(\mathfrak{t}_{\mathfrak{g}})=W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$. As ζ fixes every λ in E, regular in $\mathfrak{g}$, then $\zeta=\text{id}$; i.e. $Z\in C_G(\mathfrak{t}_{\mathfrak{g}})$.

From the preceding, we get the following injections of finite groups:

$$W(\mathfrak{f}, \mathfrak{t})=N_K(\mathfrak{t})/C_K(\mathfrak{t})\rightarrow N_G(\mathfrak{t})/(N_G(\mathfrak{t})\cap C_G(\mathfrak{t}_{\mathfrak{g}}))\rightarrow N_G(\mathfrak{t}_{\mathfrak{g}})/C_G(\mathfrak{t}_{\mathfrak{g}})=W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$$

Call W_1 the group in the middle. (Note that all of this can be done if rank $\mathfrak{f}=$rank $\mathfrak{g}$; then $W_1=W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$).

Pick λ, μ in the open cone of regular elements both in $\mathfrak{g}$ and in $\mathfrak{k}$, included in E; call $\mathfrak{g}_+{}^\lambda$, $\mathfrak{g}_+{}^\mu$ the respective maximal nilpotent subalgebras of $\mathfrak{g}$. If $\sigma \in W_1$, $\sigma\mathfrak{g}_+{}^\lambda = \mathfrak{g}_+{}^{\sigma\lambda}$.

iii) If $\sigma \in W(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$ *sends* $\mathfrak{g}_+{}^\lambda$ *to* $\mathfrak{g}_+{}^\mu$ *then* $\sigma \in W_1$.

Proof: Pick $w \in W(\mathfrak{k}, \mathfrak{t})$ such that $w(\mathfrak{k}_+{}^\lambda) = \mathfrak{k}_+{}^\mu$; then σw^{-1} sends $\mathfrak{g}_+{}^{w\lambda}$ to $\mathfrak{g}_+{}^\mu$ so we can replace λ by $w\lambda$ and assume that $\mathfrak{k}_+{}^\lambda = \mathfrak{k}_+{}^\mu$ I.e., $\sigma(\Phi_+{}^\lambda(\mathfrak{k}, \mathfrak{t})) = \Phi_+{}^\mu(\mathfrak{k}, \mathfrak{t})$. But then $\sigma(\Phi^\lambda(\mathfrak{k}, \mathfrak{t})) = \Phi^\mu(\mathfrak{k}, \mathfrak{t})$ and σ normalizes $\mathfrak{t} = \sum_{\alpha \in \Phi^\lambda} [\mathfrak{k}_\alpha, \mathfrak{k}_{-\alpha}]$; i.e. $\sigma \in W_1$.

We summarize the preceding in:

Lemma 8: *The irreducible components of* $\mathfrak{N}(\mathfrak{g}, K)$ *are the* $K.\mathfrak{g}_+$ *where* $\mathfrak{g}_+$ *corresponds to some* $\lambda \in E$ *regular both in* $\mathfrak{g}$ *and in* $\mathfrak{k}$. *The number of components is* $c_\mathfrak{N} = |W_1| / |W(\mathfrak{k}, \mathfrak{t})|$. ∎

Finally, we list some information about W_1 and $c_\mathfrak{N}$ for those $(\mathfrak{g}, K)$ satisfying rank $\mathfrak{k} <$ rank $\mathfrak{g}$. We left to the reader the task to verify it.

Type	$\mathfrak{g}$	$\mathfrak{k}$	W_1	$c_\mathfrak{N}$
AI, $n = 2k$	$\mathfrak{sl}(n+1, \mathbb{C})$	$\mathfrak{so}(n+1, \mathbb{C})$	$\mathbb{Z}_2^k \times_{sd} \mathfrak{S}_k$	1
AI, $n = 2k+1$	$\mathfrak{sl}(n+1, \mathbb{C})$	$\mathfrak{so}(n+1, \mathbb{C})$	$\mathbb{Z}_2^k \times_{sd} \mathfrak{S}_k$	2
AII	$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sp}(2n, \mathbb{C})$	$\mathbb{Z}_2^n \times_{sd} \mathfrak{S}_n$	1
BDI, $p = 2r+1$, $q = 2s+1$	$\mathfrak{so}(p+q, \mathbb{C})$	$\mathfrak{so}(p, \mathbb{C}) X$ $\mathfrak{so}(q, \mathbb{C})$	$\mathbb{Z}_2^{r+s} \times_{sd} \mathfrak{S}_{r+s}$	$\binom{r+s}{s}$
EI	e_6	$\mathfrak{sp}(8, \mathbb{C})$		3
EIV	e_6	f_4		1
II	$\mathfrak{l} \times \mathfrak{l}$, $\mathfrak{l}$ simple	diag $(\mathfrak{l})$	$W(\mathfrak{l})$	1

References

[B] BENABDALLAH, A. I.: *Generateurs de l'algebre* $\mathfrak{U}(G)^K$ *avec* $G = SO(m)$ *ou* $SO_0(1, m-1)$ *et* $K = SO(m-1)$. Bull. Soc. Math. France, 111, 1983, p. 303-326.

[C] COOPER, A.: *The classifying ring of groups whose classifying ring is commutative.* Doctoral Thesis, MIT. (Unpublished).

[Ch] CHEVALLEY, C.: *Seminaire 1.* Paris 1956/1958.

[F de V] FREUDHENTAL, H. and de VRIES, H.: *Linear Lie Groups.* Academic Press, 1969.

[G] GROSSHANS, F. D.: *The variety of points which are not semi-stable.* Illinois J. of Math. 26 (1982), p. 138-148.

[He] HELGASON, S.: *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, 1978.

[Hu] HUMPHREYS, J.: *Introduction to Lie algebras and Representation Theory.* Springer-Verlag, 1980.

[K] KAC, V. G.: *Some Remarks on Nilpotent Orbits.* Journal of Algebra, 64 (1980), p. 190-213.

[KPV] KAC, V. G., POPOV, V. L., VINBERG, E. B.: *Sur les groupes lineaires algebraiques dont l'algebre des invariantes est libre.* C. R. Acad. Sci. Paris 283 (1976), p. 875-878.

[K R] KOSTANT, B. and RALLIS, S.: *Orbits and representations associated with symmetric spaces.* Amer. J. of Math. 93 (1971), p. 753-809.

[L] LUNA, D.: *Sur les orbites fermees des groupes algebriques reductifs.* Inventiones Math. 16 (1972), p. 1-5.

[M 1] MUMFORD, D.: *Geometric Invariant Theory.* Springer-Verlag, 1980. 2nd. edition.

[M 2] MUMFORD, D.: *Stability of projective varieties.* L'Ens. Math. XXIII (1977), p. 39-110.

[R] RICHARDSON, R. W.: *Principal Orbit Types for algebraic transformation groups in characteristic zero.* Inventiones Math. 16 (1972), p. 6-14.

[Sch 1] SCHWARZ, G. W.: *Representations of Simple Lie Groups with Regular Rings of Invariants.* Inventiones Math. 49 (1978), p. 167-191.

[Sch 2] SCHWARZ, G. W.: *Representations of Simple Lie Groups with a free Module of Covariants.* Inventiones Math. 50 (1978), p. 1-12.

[Sch 3] SCHWARZ, G. W.: *Lifting smooth homotopies of orbit spaces.* Publ. Math. IHES 51 (1980), p. 37-136.

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Recibido: 12 de julio de 1988