

## *Some remarks on the density of regular mappings in Sobolev classes of $S^M$ -valued functions*

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**ABSTRACT.** Some results are given about the density of continuous maps from  $\Omega$ , a bounded regular domain of  $\mathbb{R}^N$  to  $S^M$  in the Sobolev classes  $W^{s,p}(\Omega, S^M)$ .

### 0. INTRODUCTION

We are interested in the following question: suppose that  $\Omega$  is a regular domain of  $\mathbb{R}^N$  and  $\mathfrak{R}$  is a submanifold of  $\mathbb{R}^{M+1}$ . Consider, for two real numbers  $p \geq 1$ , and  $s \geq 0$ , any vector valued function  $u$  of the Sobolev space  $W^{s,p}(\Omega, \mathbb{R}^{M+1})$  taking its values in  $\mathfrak{R}$ . Is it possible to approximate  $u$  in the space  $W^{s,p}(\Omega, \mathbb{R}^{M+1})$  by regular functions taking also their values in  $\mathfrak{R}$ ? Or equivalently is the following subset of  $C^\infty(\Omega, \mathbb{R}^{M+1})$ :

$$C^\infty(\Omega, \mathfrak{R}) = \{u \in C^\infty(\Omega, \mathbb{R}^{M+1}); \forall x \in \Omega, u(x) \in \mathfrak{R}\}$$

dense in the subset of  $W^{s,p}(\Omega, \mathbb{R}^{M+1})$ :

$$W^{s,p}(\Omega, \mathfrak{R}) = \{u \in W^{s,p}(\Omega, \mathbb{R}^{M+1}); \forall x \in \Omega, u(x) \in \mathfrak{R}\}?$$

First of all observe that none of these sets are vector spaces. In particular smoothing functions of  $W^{s,p}(\Omega, \mathfrak{R})$  by taking mean values on balls or by convolution produces functions whose values do not lie in  $\mathfrak{R}$ .

It is known that when  $p \geq N$ ,  $C(\Omega, S^M) \cap W^{1,p}(\Omega, S^M)$  (where  $S^M$  is the unit sphere of  $\mathbb{R}^{M+1}$ ) is dense in  $W^{1,p}(\Omega, S^M)$  and this for all values of the integer  $M$ . (See [2], [10], [12].) If  $p < N$  this result is no longer true and in fact the relation between  $p$  and  $M$  turns out to be determinant. For example, if  $M$  is less than  $p$  it may be that there is no density as it is shown in [4], [10] and [12]. On the other hand, H. Brezis knew how to prove density result for the

space  $L^p(\Omega, S^M)$  for every  $p \geq 1$  using stereographic projection. Using similar ideas, F. Béthuel and D. Zheng in [5] have proved the density in the space  $W^{1,p}(\Omega, S^M)$  when  $p < M$ . The way to do this is to approximate functions of  $W^{1,p}(\Omega, S^M)$  in two steps. First by functions of  $W^{1,p}(\Omega, S^M)$  not necessarily continuous but taking their values on a segment of the sphere  $S^M$ . Secondly, since any segment of a sphere  $S^M$  is diffeomorphic to  $\mathbb{R}^M$  it is rather simple to approximate any function of  $W^{1,p}(\Omega, S^M)$ , taking values on a segment of sphere, by functions of  $C^\infty(\Omega, S^M)$ .

This is also what we do in the more general case of the sets  $W^{m,p}(\Omega, S^M)$ ,  $m$  integer and  $\Lambda_{p,q}^s(\Omega, S^M)$ ,  $s$  non negative real. Our main results are:

**Theorem 1.** *Given two integers  $M, N$  and three reals  $s > 0$ ,  $\infty > p \geq 1$  and  $\infty > q \geq 1$  such that  $sp \geq N$  the set  $C(\Omega, S^M) \cap \Lambda_{p,q}^s(\Omega, S^M)$  is dense in  $\Lambda_{p,q}^s(\Omega, S^M)$ .*

**Theorem 2.** *Given three reals  $\infty > p \geq 1$ ,  $\infty \geq q \geq 1$ ,  $s \geq 0$  and two integers  $N, M$  such that  $sp < N$ , if:  $\max(1, s) \cdot \max(p, q) < M$ , or  $0 < s < 1/p$  and  $1 + N/p > s + N/q$ , then the set  $C^\infty(\Omega, S^M)$  is dense in  $\Lambda_{p,q}^s(\Omega, S^M)$ .*

**Theorem 3.** *If  $\mathfrak{R}$  is a compact Riemann manifold of dimension  $M$ ,  $B^N(0, 1)$  is the unit ball of  $\mathbb{R}^N$   $s > 0$  and  $p \geq 1$  are two reals such that  $sp < N$  and the  $[sp]$ -th homotopy group of  $\mathfrak{R}$ ,  $\Pi_{[sp]}(\mathfrak{R})$ , is not the trivial one then  $C(B^N(0, 1), \mathfrak{R}) \cap W^{s,p}(B^N(0, 1), \mathfrak{R})$  is not dense in  $W^{s,p}(B^N(0, 1), \mathfrak{R})$ .*

It seems to be quite clear that theorem 2 must be true whenever  $sp < M$  but we do not know how to prove that when  $0 < s < 1$ . On the other hand we shall see that if  $sp > M$  it may be that there is no density even if  $sq < M$ . We do not know if there is density or not when  $sp < M$  and  $sq > M$ .

In order to prove these density results we need some «stability properties» for the sets  $\Lambda_{p,q}^s(\Omega, S^M)$  with  $s \geq 1$ ,  $p \geq 1$ ,  $q \geq 1$  under left-composition by Lipschitz functions. It is well known (see [3] for example) that the spaces  $W^{m,p}(\Omega, \mathbb{R}^{M+1}) \cap L^\infty(\Omega, \mathbb{R}^{M+1})$  are algebras for the pointwise product of functions (for  $p=2$  it is the Schauder algebra). This is a simple consequence of the Gagliardo and Nirenberg's inequalities. It turns out that these spaces are also «stable» under left-composition by Lipschitz functions and not only these but also the spaces  $\Lambda_{p,q}^s(\Omega, \mathbb{R}^{M+1}) \cap L^\infty(\Omega, \mathbb{R}^{M+1})$ . That is to say: for any function  $u$  of  $\Lambda_{p,q}^s(\Omega, \mathbb{R}^{M+1}) \cap L^\infty(\Omega, \mathbb{R}^{M+1})$  and any function  $\Phi$  of  $W^{s', \infty}(\mathbb{R}^{M+1}, \mathbb{R}^L)$ , with  $s'$  any real greater than  $s$  and no less than one, the composed  $\Phi \circ u$  belongs to  $\Lambda_{p,q}^s(\Omega, \mathbb{R}^L) \cap L^\infty(\Omega, \mathbb{R}^L)$ . The conditions  $s' \geq 1$  is necessary as shows a result of J. Simon [13].

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### 1. PRELIMINARY RESULTS

First of all let us recall some notations, definitions and well known facts about the classical functional Banach spaces of Besov  $\Lambda_{p,q}^s$  of potential  $H^{s,p}$  and of Sobolev  $W^{s,p}$ . (For all this part see for example H. Triebel [14].)

For any non negative integer  $m$  and any real  $p \geq 1$ :

$$W^{m,p}(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N); \|f\|_{m,p} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_p < \infty\}$$

$$W^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N) \text{ and } \|f\|_{0,p} = \|f\|_p$$

For  $+\infty > p \geq 1$ ,  $\infty > q \geq 1$  and  $s > 0$ , for any integer  $M$  greater than  $s$ :

$$\Lambda_{p,q}^s(\mathbb{R}^N) = \left\{ f \in L^p(\mathbb{R}^N); \|f\|_{s,p,q} = \|f\|_p + \left( \int_{\mathbb{R}^N} \|\Delta_h^M f\|_p^q \frac{dh}{|h|^{N+sq}} \right)^{1/q} < \infty \right\}$$

$$\Lambda_{p,\infty}^s(\mathbb{R}^N) = \left\{ f \in L^p(\mathbb{R}^N); \|f\|_{s,p,\infty} = \|f\|_p + \sup_{h \in \mathbb{R}^N, h \neq 0} \left\{ \frac{\|\Delta_h^M f\|_p}{|h|^s} \right\} \right\}$$

where

$$\Delta_h f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

and

$$\Delta_h^{M+1} f = \Delta_h(\Delta_h^M f) \quad \forall M \in \mathbb{N}$$

In all the following let  $\Omega$  be any bounded and smooth domain of  $\mathbb{R}^N$ . One can define the corresponding spaces  $W^{m,p}(\Omega)$ ,  $\Lambda_{p,q}^s(\Omega)$  of functions defined on  $\Omega$  as the restriction to  $\Omega$  of the functions of the spaces  $W^{m,p}(\mathbb{R}^N)$ ,  $\Lambda_{p,q}^s(\mathbb{R}^N)$ . These spaces have an inner description. Namely the space  $W^{m,p}(\Omega)$  has the same formal characterisation that  $W^{m,p}(\mathbb{R}^N)$ , changing  $\mathbb{R}^N$  by  $\Omega$ . On the other hand if  $0 \leq k < s$  and  $L > s - k$ ,  $\forall f \in \Lambda_{p,q}^s(\Omega)$ :

$$\|f\|_{s,p,q} = \|f\|_p + \sum_{|\alpha| \leq m} \left( \int_{\mathbb{R}^N} \left( \int_{\Omega_{h,L}} |\Delta_h^\alpha D^\alpha f(x)|^p dx \right)^{p/q} \frac{dh}{|h|^{N+(s-k)q}} \right)^{1/q}$$

where

$$\Omega_{h,L} = \bigcap_{j=-L}^L \left\{ x; x + j \frac{h}{2} \in \Omega \right\}$$

When  $\Omega$  is a bounded regular domain we have the following continuous embeddings between these different spaces. (See H. Triebel [14], page 195.)

**Theorem.** *i) Let  $1 \leq p \leq \infty$ ,  $1 \leq p' \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $1 \leq q' \leq \infty$  and  $0 < s' < s < +\infty$ . Then  $\Lambda_{p,q}^s(\Omega) \subset \Lambda_{p',q'}^{s'}(\Omega)$  if  $s - N/p > s' - N/p'$ .*

*ii) Let  $1 \leq q \leq q' \leq \infty$ , and  $1 \leq p < \infty$  and  $0 < s < \infty$ . Then  $\Lambda_{p,q}^s(\Omega) \subset \Lambda_{p,q'}^s(\Omega)$ .*

The following relations between these spaces are well known:

$$\text{for } 1 \leq p < \infty, \text{ and } s > 0 \text{ non integer } W^{s,p} = \Lambda_{p,q}^s$$

$$\text{for } 1 \leq p < \infty \text{ and } m \in \mathbb{N} \quad W^{m,p} = H^{m,p}.$$

Finally we recall a result about extension of functions of  $\Lambda_{p,q}^s(\Omega)$  (respectively  $H^{s,p}(\Omega)$ ) to functions of  $\Lambda_{p,q}^s(\mathbb{R}^N)$  (respectively  $H^{s,p}(\mathbb{R}^N)$ ):

**Theorem.** *If  $1 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$  (resp.  $1 \leq p < \infty$ ,  $1 \leq q < +\infty$ ) and  $0 < s < +\infty$ , then the restriction operator  $R$  is a retraction for the space  $\Lambda_{p,q}^s(\mathbb{R}^N)$  (resp.  $H^{s,p}(\mathbb{R}^N)$ ). If  $L$  is a natural number then there is a common coretraction from  $\Lambda_{p,q}^s(\Omega)$  (resp.  $H^{s,p}(\Omega)$ ) to  $\Lambda_{p,q}^s(\mathbb{R}^N)$  (resp.  $H^{s,p}(\mathbb{R}^N)$ ) for all  $s$  such that  $|s| < L$  for any  $1 \leq q \leq \infty$  and  $1 \leq p \leq \infty$ .*

**Remark 1.1.** The Besov spaces can be defined in a more general way for  $s \in \mathbb{R}$ ,  $0 \leq p \leq +\infty$  and  $0 \leq q \leq \infty$ . These definitions are equivalent to the ones we give here only for  $s > 0$ ,  $1 \leq p < +\infty$  and  $1 \leq q \leq +\infty$  which are the cases we are interested in. (See [14].)

Let us give now a simple result on the stability of the Sobolev spaces of integer order by left-composition with Lipschitz functions.

**Proposition 1.2.** *Let  $p$  be any real number greater or equal than one and  $m$  any non negative integer. Consider any vector valued function  $\Phi$  belonging to the space  $W^{m,\infty}(\mathbb{R}^M, \mathbb{R}^L) \cap C^0(\mathbb{R}^M)$  such that  $\Phi(0) = 0$ . Then, for any function  $u$  of  $H^{m,p}(\mathbb{R}^N, \mathbb{R}^L) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^M)$ ,  $\Phi \circ u$  belongs to  $H^{m,p}(\mathbb{R}^N, \mathbb{R}^L) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^L)$  and  $\|\Phi \circ u\|_{m,p} \leq C \|\Phi\|_{m,\infty} \|u\|_{m,p} (1 + \|u\|_\infty^{m-1})$ .*

**Proof.** The proof is based on the inequalities of Gagliardo and Nirenberg (see [8]). Let  $u$  and  $\Phi$  be such as in the hypothesis. We have to estimate the following norm:

$$\|\Phi \circ u\|_{m,p} = \|\Phi \circ u\|_p + \sum_{j=1}^m \|D_j^m \Phi \circ u\|_p$$

By the hypothesis on  $\Phi$  it is clear that  $\Phi \circ u$  belongs to  $L^\infty(\mathbb{R}^N, \mathbb{R}^L)$ .

On the other hand, almost everywhere in  $\mathbb{R}^N$ :

$$D_j^m(\Phi \circ u)(x) = \sum_{n=1}^m \sum_{k_1=1}^M \dots \sum_{k_n=1}^M D_{k_1 \dots k_n} \Phi(u(x)) \sum_{L_1 + \dots + L_n = m} a_{L_1 \dots L_n} D_{j_1} u_{k_1}(x) \dots D_{j_n} u_{k_n}(x)$$

where the  $a_{L_1, L_2, \dots, L_n}$  are non negative integers depending only of  $m$  and  $N$ .

By Holder's inequality:

$$\|D_j^m(\Phi \circ u)\|_p \leq \sum_{n=1}^m \|\Phi\|_{n, \infty} \sum_{L_1 + \dots + L_n = m} a_{L_1 \dots L_n} \left[ \sum_{k_1=1}^M \|D_{i_1}^{L_1} u_{k_1}\|_m^p \right] \dots \left[ \sum_{k_n=1}^M \|D_{i_n}^{L_n} u_{k_n}\|_m^p \right]$$

Finally, using the inequalities of Gagliardo and Nirenberg (see [7]) we get:

$$\|D_j^m(\Phi \circ u)\|_p \leq C(N, m) \|\Phi\|_{m, \infty} \|u\|_{m, p} (1 + \|u\|_{\infty}^{m-1})$$

**Remark 1.3.** By the same method and using the dominated convergence theorem one can prove the following: If  $\{u^n\}$  is a sequence of functions of  $H^{m, p}(\mathbb{R}^N, \mathbb{R}^M) \cap L^{\infty}(\mathbb{R}^N, \mathbb{R}^M)$  such that:

- i)  $u^n \rightarrow u$  in  $H^{m, p}(\mathbb{R}^N, \mathbb{R}^M)$ .
- ii)  $\exists C > 0; \|u^n\|_{\infty} \leq C \forall n \in \mathbb{N}$  and  $\|u\|_{\infty} \leq C$ .
- iii)  $\Phi \in C^m(\mathbb{R}^M, \mathbb{R}^L)$ .

then there is a subsequence of  $\{\Phi \circ u^n\}$  converging to  $\Phi \circ u$  in  $H^{m, p}(\mathbb{R}^N, \mathbb{R}^L)$ .

**Remark 1.4.** Proposition 1.2 and remark 1.3 remain true if we consider  $u$  belonging to  $W^{m, p}(\Omega_1, \mathbb{R}^M)$  and  $\Phi$  of  $W^{m, \infty}(\Omega_2, \mathbb{R}^L)$  where  $\Omega_1, \Omega_2$  are regular domains of  $\mathbb{R}^N$  and  $\mathbb{R}^L$  respectively and  $Im(u) \subset \Omega_2$ . If  $\Omega_1$  is bounded the condition  $\Phi(0) = 0$  is not necessary (this condition is only needed in order to prove that  $\Phi \circ u$  belongs to  $L^p$  but if  $\Omega$  is bounded that is true as soon as  $\Phi \circ u$  belongs to  $L^{\infty}$ ).

In order to extend this simple result to more general spaces we shall need the following lemma.

**Lemma 1.5.** Let  $s$  be any non negative real. Then for any  $1 \leq p < \infty$  and  $1 \leq q \leq \infty$ :

$$\Lambda_{p, q}^s \cap L^{\infty} \subset \Lambda_{r, p, r, q}^{s/r} \quad \forall r \geq 1$$

and

$$\forall u \in \Lambda_{p, q}^s \cap L^{\infty} : \|u\|_{s/r, r, p, r, q} \leq C(s, r) \|u\|_{s, p, q}^{1/r} \|u\|_{\infty}^{1-1/r}.$$

**Proof.** Let  $M$  be any integer such that  $M > s \geq s/r$  for any  $r \geq 1$ . Then

$$u \in \Lambda_{p,q}^s \Leftrightarrow \|u\|_p + \left\{ \int \frac{\|\Delta_h^M u\|_q^p}{|h|^{N+sq}} dh \right\}^{\frac{1}{q}} < \infty$$

$$u \in \Lambda_{r,p,rq}^{s/r} \Leftrightarrow \|u\|_{rp} + \left\{ \int \frac{\|\Delta_h^M u\|_{rp}^{rq}}{|h|^{N+sq}} dh \right\}^{\frac{1}{rq}} < \infty$$

but:

$$\|\Delta_h^M u\|_{rp}^{rq} = \left\{ \int_{\mathbb{R}^N} |\Delta_h^M u(x)|^{rp} dx \right\}^{\frac{q}{p}} \leq ((M+1)\|u\|_{\infty}^{r-1})^q \left\{ \int_{\mathbb{R}^N} |\Delta_h^M u(x)|^p dx \right\}^{\frac{q}{p}}$$

and that gives the inclusion and the inequality for  $q < \infty$ . For  $q = \infty$  the proof is similar.

We can prove now the following:

**Proposition 1.6.** *Let  $s$  be any non negative real,  $1 \leq p < \infty$ ,  $1 \leq q \leq \infty$  and  $m$  the integer part of  $s$ . Consider any vector valued function  $\Phi$  of  $W^{s',\infty}(\mathbb{R}^N, \mathbb{R}^L) \cap C^0$ ,  $\max(1,s) < s' < m+1$ , such that  $\Phi(0)=0$ . Then for any function  $u$  of the space  $\Lambda_{p,q}^s(\mathbb{R}^N, \mathbb{R}^M) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^M)$  the function  $\Phi u$  belongs to  $\Lambda_{p,q}^s(\mathbb{R}^N, \mathbb{R}^L) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^L)$  and  $\|\Phi u\|_{s,p,q} \leq C \|\Phi\|_{s',\infty} \|u\|_{s,p,q} (\|u\|_{\infty}^m + 1)$ .*

**Proof.** We shall consider on the Besov space  $\Lambda_{p,q}^s(\mathbb{R}^N, \mathbb{R}^M)$  the norm

$$\|u\|_{s,p,q} = \|u\|_p + \left\{ \int \frac{\|\Delta_y^L u\|_q^p}{|y|^{N+sq}} dy \right\}^{\frac{1}{q}} \quad \text{where } L = m+1$$

As  $\Phi(0)=0$  and  $\Phi$  is Lipschitz it is clear that  $\Phi u$  belongs to  $L^p$ . On the other hand it is simple (but tedious) to see by induction on  $m$  that:

$$\begin{aligned} \Delta_y^m \Phi u &= \sum_{j=0}^{m-1} \sum_{r=0}^j \sum_{k_1+\dots+k_r=j} \int_0^1 \dots \int_0^1 b_{k_1 \dots k_r} d^{r+1} \Phi(G_{-y \cdot (m-r-1)} \delta_{ys_1} \dots \delta_{ys_{r+1}} u) \\ &G_{y \cdot j} \Delta_y^{m-j} u G_{-y \cdot (m+k_1-2j-1)} \Delta_y^{k_1} \delta_{ys_1} u \dots G_{-y \cdot (m+2k_1+\dots+2k_r+k_{r+1}-2j-i+1)} \Delta_y^{k_{r+1}} \delta_{ys_1} \dots \\ &\dots \delta_{ys_{r+1}} u \dots G_{-y \cdot (m+2k_1+\dots+2k_{r-1}+k_r-2j-r)} \Delta_y^k \delta_{ys_1} \dots \delta_{ys_r} u ds_1 \dots ds_{r+1} \end{aligned}$$

where:

$$b_{k_1 \dots k_r} = \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix} \cdot \begin{bmatrix} j-1 \\ k_1-1 \end{bmatrix} \cdot \begin{bmatrix} j-k_1-1 \\ k_2-1 \end{bmatrix} \cdot \begin{bmatrix} j-k_1-k_2-\dots-k_{r-1}-1 \\ k_r-1 \end{bmatrix}$$

for any  $r \in [0, j]$ ,  $j \in [0, m-1]$

with:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a!}{b!(a-b)!}$$

$\delta_{ys} u(x) = su(x+y/2) + (1-s)u(x-y/2)$  for any  $s \in (0,1)$ ,  $x \in \mathbb{R}^N$ ,  $y \in \mathbb{R}^N$  and  $G_y u(x) = u(x+y/2)$ .

By this:

$$\Delta_y^{m+1} \Phi u = \Delta_y (\Delta_y^m \Phi u) = A_1 + A_2 + A_3 \quad (1.7)$$

where:

$$A_1 = \sum_{j=0}^{m-1} \sum_{r=0}^j \sum_{k_1 + \dots + k_r = j} b_{k_1 \dots k_r j} \int_0^1 \dots \int_0^1 d^{r+1} \Phi(G_{-y \cdot (m-r-1)} \delta_{ys_1} \dots \delta_{ys_{r+1}} u) G_{y \cdot (j+1)} \Delta_y^{m-j} u \\ \dots G_{-y \cdot (m+2k_1 + \dots + 2k_{r-1} - 2j - r + 1)} \Delta_y^k \delta_{ys_1} \dots \delta_{ys_r} u \, ds_1 \dots ds_{r+1}$$

$$A_2 = \sum_{j=0}^{m-1} \sum_{r=0}^j \sum_{k_1 + \dots + k_r = j} b_{k_1 \dots k_r j} \int_0^1 \dots \int_0^1 d^{r+1} \Phi(G_{-y \cdot (m-r-1)} \delta_{ys_1} \dots \delta_{ys_{r+1}} u) G_{y \cdot j} \Delta_y^{m-j+1} u \\ G_{-y \cdot (m+k_1-2j)} \Delta_y^{k_1} \delta_{ys_1} u \dots G_{-y \cdot (m-k_r-r+1)} \Delta_y^{k_r} \delta_{ys_1} \dots \delta_{ys_r} u \, ds_1 \dots ds_{r+1}$$

$$A_3 = \sum_{j=0}^{m-1} \sum_{r=0}^j \sum_{k_1 + \dots + k_r = j} b_{k_1 \dots k_r j} \int_0^1 \dots \int_0^1 \Delta_y (d^{r+1} \Phi(G_{-y \cdot (m-r-1)} \delta_{ys_1} \dots \delta_{ys_{r+1}} u)) \\ G_{y \cdot (j-1)} \Delta_y^{m-j} u \dots G_{-y \cdot (m-k_r-r+1)} \Delta_y^{k_r} \delta_{ys_1} \dots \delta_{ys_r} u \, ds_1 \dots ds_{r+1}$$

Using Holder's inequalities, the translation invariance of  $L^p$  norms and lemma 1.5 one gets:

$$\|A_i\|_{s,p;q} \leq C \|\Phi\|_{m,\infty} \|u\|_{s,p;q} \sum_{r=0}^m \|u\|_{\infty}^r \quad \text{for } i=1,2$$

In order to obtain the same estimate for  $A_3$  we have to use also that, by the hypothesis on  $\Phi$ :

$$\forall \xi, \zeta \in \mathbb{R}^M, \forall r \leq m-1, \forall v_i \in \mathbb{R}^M \quad i=1, \dots, r+1:$$

$$| [d^{r+1} \Phi(\xi) - d^{r+1} \Phi(\zeta)] v_1 \dots v_{r+1} | \leq C \|\Phi\|_{s',\infty} |\xi - \zeta|^{s'-m} |v_1| \dots |v_{r+1}|$$

**Remark 1.7.** One can prove a result similar to the one of Remark 1.3 for the spaces  $\Lambda_{p,q}^s(\mathbb{R}^N, \mathbb{R}^M) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^M)$ .

**Remark 1.8.** Proposition (1.6) remains true if we consider  $u$  belonging to the space  $\Lambda_{p,q}^s(\Omega_1, \mathbb{R}^M) \cap L^\infty(\Omega_1, \mathbb{R}^M)$  of functions defined on a smooth domain of  $\mathbb{R}^N$  and  $\Phi$  a function of  $W^{s',\infty}(\Omega_2, \mathbb{R}^L)$  where  $\Omega_2$  is a smooth domain of  $\mathbb{R}^M$  containing  $Im(u)$ . Just using the extension operators from  $\Lambda_{p,q}^s(\Omega_1, \mathbb{R}^M)$  to  $\Lambda_{p,q}^s(\mathbb{R}^N, \mathbb{R}^M)$ , from  $W^{s',\infty}(\Omega_2, \mathbb{R}^L)$  to  $W^{s',\infty}(\mathbb{R}^M, \mathbb{R}^L)$  and their inverse.

As in remark (1.4) if  $\Omega_1$  is bounded we do not need no more the condition  $\Phi(0)=0$  in order to have the results of proposition (1.6).

**Remark 1.9.** The condition  $s' \geq 1$  is necessary as shows the following result of J. Simon [13]:

**Theorem.** (J. Simon). *Given  $p \in (0,1)$ ,  $\forall s \in (0,1)$  and  $\forall \varepsilon > 0$ ,  $\exists w \in W^{s,r}(\Omega) \forall r \in [1, \infty]$ ;  $|w|^{p-1}w \notin W^{s+p\varepsilon,r'}(\Omega) \forall r' \in [1, \infty]$ .*

In fact, J. Simon gives a counterexample where the function  $w$  is Lipschitz of order  $s$  on  $\Omega$ .

**Remark 1.10.** It is well known that for any non negative integer  $m$  the space  $W^{m,p} \cap L^\infty$  is an algebra (for  $p=2$  it is the Schauder algebra). The proof is a simple consequence of the formula of Leibnitz and the Gagliardo and Nirenberg's inequalities.

This result remains true for any Besov space  $\Lambda_{p,q}^s$  with  $s > 0$ ,  $p \geq 1$  and  $q \geq 1$ . We only have to show this for the homogeneous spaces  $\dot{\Lambda}_{p,q}^s$ . But this is very easy using the characterisation of these spaces given by J. Dorrnsoro in [6]:  $u \in \dot{\Lambda}_{p,q}^s$  iff there is an integer  $M > s$  such that, if  $Q$  is any cube in  $\mathbb{R}^N$  and  $P_Q^M(f)$  is the unique polynomial in  $P_M$  (the space of polynomials of degree less or equal than  $M$ ) such that:

$$\int_Q (f - P_Q^M(f)) x^\alpha dx = 0 \quad \forall \alpha \in \mathbb{N}^N; |\alpha| < M$$

and if

$$\Omega_{f,M}(x,t) = \sup \left\{ |Q|^{-1} \int_Q |f - P_Q^M(f)| dz; x \in Q, |Q| = t^N \right\}$$

one has:

$$\left( \int_Q (t^{-x} \|\Omega_{f,M}(\cdot, t)\|_p)^q t^{-1} dt \right)^{\frac{1}{q}} < \infty$$



and the fact that  $P_Q^M$  satisfies:

$$\|P_Q^M(v)\|_\infty \leq C|Q|^{-1} \int_Q |v| dz \leq C\|v\|_\infty \quad (\text{see [6]})$$

The same result is true for the space  $H^{s,p} \cap L^\infty$  using similar results of J. Dorronsoro about Bessel-potential spaces (see [7]). The author is grateful to J. R. Dorronsoro for fruitful conversations about this Remark.

## 2. DENSITY RESULTS

Let  $\Omega$  be a bounded and regular domain of  $\mathbb{R}^N$  and  $S^M$  the unit sphere of  $\mathbb{R}^{M+1}$ . As it has been said in the introduction we prove in this section the density of the set of regular functions defined on  $\Omega$  and taking values on  $S^M$ , on the sets  $\Lambda_{p,q}^s(\Omega, S^M)$ , when some relations hold between the coefficients  $s, p, q, M$ , and  $N$ .

First we give a simple extension of a density result in [2] and [12]:

**Theorem 2.1.** *Let  $M, N$  and  $m$  three positive integers and  $p$  a real no less than one such that  $mp \geq N$ . Then  $C(\Omega, S^M) \cap W^{m,p}(\Omega, S^M)$  is dense in  $W^{m,p}(\Omega, S^M)$ .*

**Proof.** Any function  $u$  of  $W^{m,p}(\Omega, S^M)$  is bounded and then belongs to  $W^{1,mp}(\Omega, S^M)$  by the Gagliardo and Nirenberg's inequalities. Now using the method of [2] we obtain, by taking averages of  $u$  over balls, a sequence of continuous functions  $\{u_\varepsilon\}$  converging to  $u$  in  $W^{1,mp}(\Omega, S^M)$  when  $\varepsilon$  tends to zero and such that  $\text{dist}(u_\varepsilon(x), S^M)$  tends to zero uniformly on  $\Omega$  with  $\varepsilon$  (using that  $mp \geq N$ ). The result follows as in [2].

**Theorem 2.2.** *Let  $M$  and  $N$  be two positive integers. For  $p$  and  $q$  reals greater than one and any non negative real  $s$  such that  $sp \geq N$ , the set  $C(\Omega, S^M) \cap \Lambda_{p,q}^s(\Omega, S^M)$  is dense in  $\Lambda_{p,q}^s(\Omega, S^M)$ .*

**Proof.** Here again the idea of the proof is the same as in [2].

Consider any function  $u$  of  $\Lambda_{p,q}^s(\Omega, S^M)$  and its extension to  $\mathbb{R}^N$ ,  $U$  of  $\Lambda_{p,q}^s(\mathbb{R}^N, \mathbb{R}^{M+1})$ . Define for any  $\varepsilon > 0$ :

$$U_\varepsilon(x) = |B_\varepsilon(x)|^{-1} \int_{B_\varepsilon(x)} U(z) dz \quad \text{where} \quad B_\varepsilon(x) = \{z \in \mathbb{R}^N; |z-x| \leq \varepsilon\}$$

These functions belong to  $C(\mathbb{R}^N, S^N)$  and if  $u_\varepsilon = U_{\varepsilon \cap \Omega}$  then, as  $\varepsilon \rightarrow 0$ ,  $u_\varepsilon \rightarrow u$  in  $\Lambda_{p,q}^s(\Omega, \mathbb{R}^{M+1})$ . On the other hand consider any real  $r \geq 1$  such that  $1/p < s/r \leq 1$ . For any  $x \in \Omega$

$$[\text{dist}(u_\varepsilon(x), S^M)]^{\frac{sp}{r}} \leq |B_\varepsilon|^{-1} \int_{B_\varepsilon(x)} |U(y) - u_\varepsilon(x)|^{\frac{sp}{r}} dy \leq C \int_{B_{2\varepsilon}(0)} \int_{B_\varepsilon(x)} \frac{|U(z) - U(z-h)|^{\frac{sp}{r}}}{|B_\varepsilon|^2} dz dh$$

Now if  $q > p$ :

$$C \int_{B_{2\varepsilon}(0)} \int_{B_\varepsilon(x)} \frac{|U(z) - U(z-h)|^{\frac{sp}{r}}}{|B_\varepsilon|^2} dz dh \leq \\ C \left\{ \int_{B_{2\varepsilon}(0)} \left( \int_{B_\varepsilon(x)} |U(z) - U(z-h)|^{\frac{sp}{r}} dz \right)^{\frac{q}{p}} \frac{dh}{|h|^{N+sq}} \right\}^{\frac{p}{q}}$$

using that  $sp \geq N$ .

Therefore  $[\text{dist}(u_\varepsilon(x), S^M)]^{sp/r} \rightarrow 0$  when  $\varepsilon \rightarrow \infty$  uniformly for  $x \in \Omega$ .

If  $q \leq p$ ,  $\Lambda_{p,q}^s \subset \Lambda_{p,p}^s$  and then:

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} [|U(z) - U(y)|^{sp/r} / |z - y|^{N+sp}] dy dz < \infty$$

Using again that:

$$[\text{dist}(u_\varepsilon(x), S^M)]^{sp/r} \leq \int_{B_\varepsilon} \int_{B_\varepsilon} [|U(z) - U(y)|^{sp/r} / |z - y|^{N+sp}] dy dz$$

we get the same conclusion.

Taking now  $v_\varepsilon = \text{Proj}_{S^M} u_\varepsilon$  for  $\varepsilon$  small enough we obtain the result.

So we shall consider in all the following that  $sp < N$ . In this case our first purpose is to approximate any function of  $\Lambda_{p,q}^s(\Omega, S^M)$  by functions, not necessarily continuous taking values only on a segment of sphere. In order to do this we shall need the following deformation lemma.

**Lemma 2.3.** For any  $\varepsilon > 0$  and  $x^0 \in S^M$  let  $V_{x^0, \varepsilon/2} = S^M \cap B^{M+1}(x^0, \varepsilon/2)$  and  $W_{x^0, \varepsilon/2} = S^M - \text{Int}(V_{x^0, \varepsilon/2})$ . There is a  $C^\infty$  map  $\Phi_\varepsilon$  from  $S^M$  to  $W_{x^0, \varepsilon/2}$  such that:

- i)  $\Phi_\varepsilon|_{W_{x^0, \varepsilon/2}} = \text{Id}|_{W_{x^0, \varepsilon/2}}$ .
- ii)  $\forall L \in \mathbb{N}, \exists C > 0; \forall \alpha \in \mathbb{N}^{M+1} |\alpha| \leq L; \|\mathcal{D}^\alpha \Phi\|_\infty \leq C\varepsilon^{-|\alpha|}$ .

**Proof.** This is a «regular version» of a lemma proved in [5] (where  $\Phi$  has only to be Lipschitz). The proof is essentially the same.

We can prove now our first density result for the particular case of Sobolev spaces  $W^{m,p}(\Omega, S^M)$  with  $m$  an integer.

**Theorem 2.4.** For any real  $p$ , greater or equal than one, for any positive integer  $M$  and any non negative integer  $m$  such that  $mp < M$  the space  $C^\infty(\Omega, S^M)$  is dense in  $W^{m,p}(\Omega, S^M)$ .

**Proof.** Let  $\varepsilon > 0$  be fixed. By lemma 2.1, for any  $x^0 \in S^M$  there is a  $C^\infty$  function  $\Phi_\varepsilon$  from  $S^M$  to  $W_{x^0, \varepsilon/2}$  such that:

i)  $\Phi_\varepsilon|_{W_{x^0, \varepsilon/2}} = Id|_{W_{x^0, \varepsilon/2}}$ .

ii)  $\forall L \in \mathbb{N}, \exists C > 0; \forall \alpha \in \mathbb{N}^{M+1} \quad |\alpha| \leq L; \|\mathcal{D}^\alpha \Phi_\varepsilon\|_\infty \leq C\varepsilon^{-|\alpha|}$ .

In order to apply the results of section 1 we extend this function to all of  $\mathbb{R}^{M+1}$ . For this purpose consider any  $C^\infty$  function  $h$  from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  such that  $h(1) = 1$ ,  $\text{supp } h \subset (1/2, 3/2)$  and define  $\Psi_{x^0, \varepsilon} = \Phi_{x^0, \varepsilon}(x/|x|) \cdot h(x)$ . This function is  $C^\infty$  and  $\mathcal{D}^\alpha \Psi(x) \leq C \cdot \varepsilon^{-|\alpha|}$  for any  $x$  of  $\mathbb{R}^{M+1}$  and  $\alpha$  such that  $|\alpha| \leq L$ .

If  $P_\varepsilon$  is the maximal number of disjoint sets of the form  $V_{x^0, \varepsilon}$  contained in  $S^M$  there is a constant  $K$  such that  $P_\varepsilon \geq K \cdot \varepsilon^{-M}$ . Let  $\{V_{x^0, \varepsilon}\}_{i=1, \dots, P_\varepsilon}$  such a family of sets and define

$$\forall i \in \{1, \dots, P_\varepsilon\} \quad u_{i, \varepsilon} = \Psi_{x^0, \varepsilon} \circ u \equiv \Phi_{x^0, \varepsilon} \circ u.$$

By construction  $u_{i, \varepsilon} = u$  on  $W_{x^0, \varepsilon/2}$  and  $u_{i, \varepsilon}$  goes from  $\Omega$  to  $W_{x^0, \varepsilon/2}$ .  
On the other hand:

$$\begin{aligned} \sum_{i=1}^{P_\varepsilon} \|u - u_{i, \varepsilon}\|_{m, p}^p &= \left[ \left\| \sum_{i=1}^{P_\varepsilon} (u - u_{i, \varepsilon}) \right\|_{m, p} \right]^p = \\ &= \left[ \left\| \sum_{i=1}^{P_\varepsilon} (Id - \Psi_{i, \varepsilon}) \circ u \right\|_{m, p} \right]^p \leq C \|\Phi_\varepsilon \circ u\|_{m, p}^p \end{aligned} \quad (2.5)$$

with  $\Phi_\varepsilon(x) = \sum (x - \Psi_{i, \varepsilon}(x))$  for any  $x$  of  $\mathbb{R}^{M+1}$ .  $\Phi_\varepsilon$  is a  $C^\infty$  function such that  $\Phi_\varepsilon(0) = 0$  and then by proposition (1.2):

$$\sum_{i=1}^{P_\varepsilon} \|u - u_{i, \varepsilon}\|_{m, p}^p \leq C \|\Phi_\varepsilon\|_{m, \infty}^p \|u\|_{m, p}^p.$$

But, by definition of  $\Phi_\varepsilon$  and using that the supports of the functions  $(Id - \Psi_{i, \varepsilon})$  and  $(Id - \Psi_{j, \varepsilon})$  are disjoint for  $i \neq j$  we have:

$$\|\Phi_\varepsilon\|_{m, \infty} \leq \max\{\|Id - \Psi_{i, \varepsilon}\|_{m, \infty}; i = 1, \dots, P_\varepsilon\} \leq C\varepsilon^{-m}$$

from this we deduce:

$$\sum_{i=1}^{P_\varepsilon} \|u - u_{i, \varepsilon}\|_{m, p}^p \leq C\varepsilon^{-mp} \|u\|_{m, p}^p$$

It follows that there is at least one  $i \in \{1, \dots, P_\varepsilon\}$  such that:

$$\|u - u_{i, \varepsilon}\|_{m, p}^p \leq C\varepsilon^{-mp} (P_\varepsilon)^{-1} \cdot \|u\|_{m, p}^p \leq C\varepsilon^{M-mp} \cdot \|u\|_{m, p}^p$$

This inequality gives us an approximation of any function  $u$  of  $W^{m,p}$  taking values on  $S^M$  by functions  $u_{i,\varepsilon}$  of  $W^{m,p}$  with values only in a segment of sphere  $W_{x^i,\varepsilon}$ . We have to approximate now  $u_{i,\varepsilon}$  by a smooth map from  $\Omega$  to  $S^M$ . There is no loss of generality if we suppose that  $x^i = (1, 0, \dots, 0)$ . Consider the stereographic projection  $\mathfrak{P}$  of  $S^M$  on  $\mathbb{R}^M$  with pole  $x^i$ . Its restriction  $P$  to  $W_{x^i,\varepsilon}$  is a  $C^\infty$  diffeomorphism from  $W_{x^i,\varepsilon}$  into  $P(W_{x^i,\varepsilon})$ .

Let us define now:  $\forall z \in B^{M+1}(0,1) \setminus B^{M+1}(x^i,\varepsilon)$ ,  $z \neq 0$ :  $\dot{P}^E(z) = h(|z|) \cdot P(z/|z|)$  and  $P^E(0) = 0$  (where  $h$  is as in 2.4). It is a  $C^\infty$  extension of  $P$  to  $B^{M+1}(0,1) \setminus B^{M+1}(x^i,\varepsilon)$ . On the other hand the image of  $[B^{M+1}(0,1) \setminus B^{M+1}(x^i,\varepsilon)]$  by  $P^E$  is contained in  $P(W_{x^i,\varepsilon})$ .

By the results of section 1  $P^E(u_{i,\varepsilon})$  is now a function of  $W^{m,p}(\Omega, \mathbb{R}^M) \cap L^\infty(\Omega, \mathbb{R}^M)$  and can be approximated by smooth functions  $v_{i,\varepsilon,n}$  of  $C^\infty(\Omega, \mathbb{R}^M) \cap L^\infty(\Omega, \mathbb{R}^M)$ . Define now  $u_{i,\varepsilon,n} = P^{-1}(v_{i,\varepsilon,n})$ . These functions belongs to  $C^\infty(\Omega, \mathbb{R}^M)$ . We want to show that there is a subsequence tending to  $u_{i,\varepsilon}$  in  $W^{m,p}$ . By remark 1.3 this holds because:

- i)  $v_{i,\varepsilon,n} \rightarrow P^E(u_{i,\varepsilon})$  in  $W^{m,p}(\Omega, \mathbb{R}^M)$ .
- ii)  $\exists C > 0$ ;  $\forall n$   $\|v_{i,\varepsilon,n}\|_\infty \leq C$ .

and then  $u_{i,\varepsilon,k} \rightarrow P^{-1}(P^E(u_{i,\varepsilon})) \equiv u_{i,\varepsilon}$  in  $W^{m,p}$  for a subsequence of  $u_{i,\varepsilon,n}$ .

In order to prove a similar result for the space  $\Lambda_{pq}^s$  we would like to apply the same method. Unfortunately formula (2.5) is not true for these spaces. Nevertheless one can, with a slight modification prove the following:

**Theorem 2.6.** *For any bounded smooth domain  $\Omega$  of  $\mathbb{R}^N$  and three reals  $1 \leq p < \infty$ ,  $1 \leq q \leq \infty$ ,  $s \geq 1$  such that  $s \cdot \max(p,q) < M$ , the space  $C^\infty(\Omega, S^M)$  is dense in  $\Lambda_{pq}^s(\Omega, S^M)$ .*

**Proof.** As in 2.4 consider, for any  $\varepsilon > 0$  fixed, a family  $\{V_{x^i,\varepsilon}\}_{i=1,\dots,p_\varepsilon}$  of disjoint subsets of  $S^M$  such that  $P_\varepsilon \geq K\varepsilon^{-M}$  and  $\Phi_{x^i,\varepsilon}$  the corresponding function from  $S^M$  to  $W_{x^i,\varepsilon/2}$  equals to the identity on  $W_{x^i,\varepsilon/2}$ . Define now  $\Psi_{x^i,\varepsilon}(x) = h(|x|) \cdot \Phi_{x^i,\varepsilon}(x/|x|)$ ,  $\Psi_{x^i,\varepsilon}(0) = 0$  where  $h$  is as in 2.4. Let  $U$  be the image of  $u$  by the extension operator from  $\Lambda_{pq}^s(\Omega, S^M)$  to  $\Lambda_{pq}^s(\mathbb{R}^N, \mathbb{R}^{M+1})$ . Define  $U_{i,\varepsilon} = \Psi_{x^i,\varepsilon} \circ U$  and consider the function  $D_{i,\varepsilon} = F_{i,\varepsilon} \circ U$  where we have posed  $F_{i,\varepsilon}(z) = h(|z|) \cdot z/|z| - \Psi_{x^i,\varepsilon}(z)$ .

Observe first of all that

$$\text{supp } D_{i,\varepsilon} \subset Q_{i,\varepsilon} = \{x \in \mathbb{R}^N; U(x)/|U(x)| \in V_{x^i,\varepsilon}\} \text{ and } Q_{i,\varepsilon} \cap Q_{j,\varepsilon} = \emptyset \text{ if } i \neq j$$

and therefore for any integer  $L$ :

$$\text{supp } \Delta_y^L D_{i,\varepsilon} \subset \bigcup_{k=-L}^{k=L} \tau_{y^k} Q_{i,\varepsilon} \equiv A^{i,\varepsilon}(y) \quad (2.7)$$

with:

$$\tau_{yk}Q_{i,\varepsilon} = \{x \in \mathbb{R}^N; x + yk \in Q_{i,\varepsilon}\}$$

Let be  $m$  the integer part of  $s$ ; arguing as in (1.7), for any  $i \in \{1, \dots, P_\varepsilon\}$ :

$$\Delta_y^{m+1} D_{i,\varepsilon} = S_{1,y}^{i,\varepsilon} + S_{2,y}^{i,\varepsilon} + S_{3,y}^{i,\varepsilon}$$

Now we have to estimate the sum  $\sum_{i=1}^{P_\varepsilon} \|D_{i,\varepsilon}\|_{s,p,q}^q$ . The way to do this is the same as for  $A_i (i=1,2,3)$  in (1.7) with a slight modification. As the sets  $A^{i,\varepsilon}(y)$  are not translation invariants we have to use the properties of the supports of the functions  $D_{i,\varepsilon}$  in the following way:

$$\begin{aligned} & \sum_{i=1}^{P_\varepsilon} \int_{\mathbb{R}^n} \|G_{yr} \Delta_y^L U\|_{L^q(A^{i,\varepsilon}(y))}^q \frac{dy}{|y|^{N+sq}} \leq \\ & \leq \sum_{i=1}^{P_\varepsilon} \sum_{k=-m}^L \int_{\mathbb{R}^n} \|G_{yr} \Delta_y^L U\|_{L^q(\tau_{y,k}Q_{i,\varepsilon})}^q \frac{dy}{|y|^{N+sq}} \quad (2.8) \\ & \leq [P_\varepsilon] \left(1 - \frac{a}{b}\right)^+ C(L) \int_{\mathbb{R}^n} \|G_{yr} \Delta_y^L U\|_{L^q(\mathbb{R}^N)}^q \frac{dy}{|y|^{N+sq}} \end{aligned}$$

for any integers  $r$  and  $L$  and two any finite reals  $a \geq 1, b \geq 1$ .

And we obtain for  $s' > s$ :

$$\left(\sum_{i=1}^{P_\varepsilon} \|D_{i,\varepsilon}\|_{s,p,q}^q\right)^{1/q} \leq C\varepsilon^{-s'} P^{(\frac{1}{q} - \frac{1}{p})^+} \|U\|_{s,p,q} \quad (2.9)$$

We choose  $s'$  such that  $s' \max(p,q) < M$ .

From (2.9) we deduce that, for at least one  $i$  of  $\{1, \dots, P_\varepsilon\}$  we have:

$$\|D_{i,\varepsilon}\|_{s,p,q} \leq C\varepsilon^{-s'} [P_\varepsilon]^{-\min(1/p, 1/q)} \|U\|_{s,p,q} \leq C\varepsilon^{-s' + M \min(1/p, 1/q)} \|U\|_{s,p,q}$$

Now call  $u_{i,\varepsilon} = U_{i,\varepsilon}|_\Omega$ . Then  $u_{i,\varepsilon} \in \Lambda_{p,q}^s(\Omega, W_{x',\varepsilon})$  and  $D_{i,\varepsilon}$  is an extension of  $u - u_{i,\varepsilon}$  to all of  $\mathbb{R}^N$ . Therefore:

$$\|u - u_{i,\varepsilon}\|_{s,p,q} \leq \|D_{i,\varepsilon}\|_{s,p,q} \leq C\varepsilon^{[M \min(1/p, 1/q) - s']} \|u\|_{s,p,q}$$

The proof now follows as above using stereographic projection.

**Remark 2.10.** If  $sp > M$ , the conclusion can fail even if  $sq < M$ . For example, it is known (see [2], [5] and [12]) that the function  $x/|x|$  belongs to

$H^1(B^3, S^2)$  and can not be approximated by functions of  $C^\infty(B^3, S^2)$  in  $H^1(B^3, S^2)$ . On the other hand  $x/|x|$  belongs to  $W^{2,r}(B^3, S^2)$  for any  $r < 3/2$ .

Now, using the inclusion properties recalled in the first section:

$W^{2,r}(B^3, S^2) \subset \Lambda_{r1}^{2-\varepsilon}(B^3, S^2) \forall \varepsilon > 0$  because  $2 - \varepsilon/r \geq (2 - \varepsilon) - \varepsilon/r$   
 $\Lambda_{r1}^{2-\varepsilon}(B^3, S^2) \subset H^{-1}(B^3, S^2)$  for  $r > 1$  and  $\varepsilon > 0$  such that  $(2 - \varepsilon) - r/3 \geq 1/2$ .  
 (All the inclusions are continuous).

If  $r = 3/(2 + \delta)$  with  $1 > \delta > 0$  and  $\varepsilon + \delta < 1/2$  in order to have  $(2 - \varepsilon) - r/3 \geq -1/2$ , this condition implies also that  $r \geq 2/(2 - \varepsilon)$ . Therefore suppose that  $\{u_n\}$  is a sequence of  $C^\infty(B^3, S^2)$  converging to  $x/|x|$  in  $\Lambda_{r1}^{2-\varepsilon}(B^3, S^2)$ . By continuous injection from this last space on  $H^1(B^3, S^2)$  the sequence  $\{u_n\}$  should be convergent to  $x/|x|$  also in this last space what is imposible.

As in the proof of proposition 1.5, for proving theorem 2.6 we have needed  $M$  greater than  $\max(s, 1/p)$ . So we can not give a general density result as (2.6) when  $0 < s < 1$ . (In that case with exactly the same proof as in (2.6), we can prove the density only when  $1 \leq p < M$ ). Nevertheless we can prove the following:

**Theorem 2.11.** *If  $p$  and  $q$  are no less than one, for any integers  $N \geq 1$  and  $M \geq 1$  and any  $s$  in  $(0, 1/p)$  such that  $1 + N/p > s + N/q$  the set  $C(\Omega, S^M)$  is dense in  $\Lambda_{pq}^s(\Omega, S^M)$ .*

We shall prove this result in two steps. First showing that the set of step functions on  $\Omega$  taking their values on  $S^M$  is dense in  $\Lambda_{pq}^s(\Omega, S^M)$ . We conclude using stereographic projection as above.

The first step will be done with three simple lemmas.

**Lemma 2.12.** *If  $s \in (0, 1/p)$ , for any  $N \geq 1$  and  $M \geq 1$ , the characteristic function of a cube  $Q$  of  $\mathbb{R}^N$  belongs to  $\Lambda_{pq}^s(\mathbb{R}^N)$ .*

**Proof.** Using the characterisation of  $\Lambda_{pq}^s$  for  $0 < s < 1$  given in [9] it is an elementary calculus to see that if  $Q = I_1 \times I_2 \times \dots \times I_N$  where the  $I_j$  are intervals of  $\mathbb{R}$  with Lebesgue measure  $L_j$

$$\|\chi_Q\|_{s,p,q} \leq C(N,s,p,q) \sum_{i=1}^N L_1^{1/p}, L_2^{1/p} \dots L_i^{-s+(1/p)} \dots L_N^{1/p}$$

where  $\chi_Q$  is the characteristic function of  $Q$ .

**Lemma 2.13.** *Under the hypothesis of theorem 2.11 the set of finite and linear combinations of characteristic functions of cubes contained in  $\Omega$ , with vectorial coefficients belonging to  $\mathbb{R}^{M+1}$  is dense in  $\Lambda_{pq}^s(\Omega, S^M)$ .*

**Proof.** Given any function  $u$  of  $\Lambda_{p,q}^s(\Omega, S^M)$  it is well known that for any  $\varepsilon > 0$  fixed there is a function  $v$  of  $C^\infty(\Omega, B^M(0,1))$  such that:  $\|u - v\|_{s,p,q} \leq \varepsilon$ .

Now it is easy, using diadic cubes in  $\mathbb{R}^N$ , to construct a sequence  $\{v_j\}$  of step functions of the form:

$$v_j(x) = \sum_{k \in K_j} v(x_{j,k}) \chi_{j,k}$$

(where  $\chi_{j,k}$  is the characteristic function of  $Q(j,k)$  the  $k$ -th cube of the  $j$ -th generation,  $x_{j,k}$  belongs to  $Q(j,k)$  and  $\#K_j$  is of order  $2^{Nj}$ ,  $\Omega \subset Q(0,0)$ ) such that:

$$\|v_j - v_{j+1}\|_{s,p,q}^q \leq C(\Omega, v, s, p, q) 2^{-j \cdot (-qs + \frac{qN}{p} + q - N)}$$

We deduce from this inequality and the hypothesis that  $\{v_j\}$  is a Cauchy sequence in  $\Lambda_{p,q}^s(\text{int } Q(0,0))$ . Then defining  $u_j = v_j|_\Omega$ ,  $\{u_j\}$  is a Cauchy sequence in  $\Lambda_{p,q}^s(\Omega)$ . As for almost every  $x$  of  $\Omega$ :  $u_j(x) \rightarrow u(x)$ ,  $\{u_j\}$  converges to  $v$  in  $\Lambda_{p,q}^s(\Omega)$  and this ends the proof.

**Lemma 2.14.** *Under the same hypothesis that in theorem 2.12 the set of step functions defined on  $\Omega$  and taking their values on  $S^M$  is dense in  $\Lambda_{p,q}^s(\Omega, S^M)$ .*

**Proof.** Let  $u$  be any function of  $\Lambda_{p,q}^s(\Omega, S^M)$ . By the above lemma we know that there is a sequence of step functions  $\{v_n\}$  from  $\Omega$  to  $B(0,1)$  converging to  $u$  in  $\Lambda_{p,q}^s(\Omega, S^M)$ . We can write

$$v_n(x) = \sum_{k \in K_n} \xi_{n,k} \chi_{n,k}(x)$$

where  $\xi_{n,k} \in B(0,1)$ . Let  $\Phi$  be a  $C^1$  non negative function from  $B(0,1)$  into itself such that  $\Phi(x) = |x|$  for any  $x$  of  $B(0,1)$  such that  $|x| \geq 1/4$  and  $|\Phi(x)| \leq 1/4$  if  $|x| \leq 1/4$ . Define now:

$$w_n = \sum_{k' \in K'} \xi_{n,k'} \chi_{n,k'} + \sum_{k'' \in K''} \zeta_0 \chi_{n,k''} \quad \text{where } \zeta_0 \in S^M \text{ is fixed}$$

With  $K' = \{k'; |1 - |\xi_{n,k'}|| < 1/2\}$  and  $K'' = \{k''; |1 - |\xi_{n,k''}|| < 1/2\}$

Let us prove that  $\{w_n\}$  still tends to  $u$  in  $\Lambda_{p,q}^s(\Omega, B(0,1))$ :

$$\begin{aligned} \|u - w_n\|_{s,p,q}^q &\leq C \|u - v_n\|_{s,p,q}^q + \|v_n - w_n\|_{s,p,q}^q \\ &\leq C \|u - v_n\|_{s,p,q}^q + C \left\| \sum_{k'' \in K''} (\xi_{n,k''} - \zeta_0) \chi_{n,k''} \right\|_{s,p,q}^q \\ &\leq C \|u - v_n\|_{s,p,q}^q + C \sum_{k'' \in K''} \|\chi_{n,k''}\|_{s,p,q}^q \end{aligned}$$

Using now the results of section 1 it is clear that  $\Phi_0 w_n$  tends to one in  $\Lambda_{p,q}^s(\Omega, B(0,1))$ . So we have:

$$\begin{aligned} \|1 - \Phi_0 w_n\|_{s,p,q}^q &\geq C \sum_{k' \in K''} |1 - \Phi(\xi_{n,k'})|^q \|\chi_{n,k'}\|_{s,p,q}^q \geq \\ &\geq C 2^{-q} \sum_{k' \in K''} \|\chi_{n,k'}\|_{s,p,q}^q \end{aligned}$$

Therefore:

$$\|u - w_n\|_{s,p,q}^q \leq C \|u - v_n\|_{s,p,q}^q + C 2^q \|1 - \Phi_0 w_n\|_{s,p,q}^q$$

and  $\{w_n\}$  tends to  $u$  as  $n$  goes to  $+\infty$ . Observe that all the functions  $w_n$  take their values on  $B(0,1) \setminus B(0,1/2)$  because by construction of  $\Phi$ ,  $\Phi(x) \geq 1/2$  implies  $|x| \geq 1/2$ . Letting now  $u_n = \text{Proj } s^2 w_n$  we have the result.

### 3. NON DENSITY RESULTS

As we said in the introduction, F. Bethuel and X. Zheng have proved that if  $\mathfrak{R}$  is a compact Riemann manifold of dimension  $M$  imbedded in  $\mathbb{R}^N$ ,  $p < N$  and  $\pi_{[p]}(\mathfrak{R}) \neq \{0\}$  (where  $\pi_{[p]}(\mathfrak{R})$  is the  $[p]$ -homotopy group of  $\mathfrak{R}$ ) then  $C(B^N(0,1), \mathfrak{R})$  is not dense in  $W^{1,p}(B^N(0,1), \mathfrak{R})$  (see [4] and [5]). We extend this result to the following cases:

**Theorem 3.1.** *i) Let  $1 \leq p < \infty$ ,  $1 \leq q < \infty$ ,  $m \in \mathbb{N}$  such that  $mp < N$  and  $\pi_{[mp]}(\mathfrak{R}) \neq \{0\}$ . Then  $C(B^N(0,1), \mathfrak{R}) \cap W^{m,p}(B^N(0,1), \mathfrak{R})$  is not dense in  $W^{m,p}(B^N(0,1), \mathfrak{R})$ .*

*ii) If  $1 \leq p < \infty$ ,  $1 \leq q < \infty$ ,  $s > 0$  are such that  $sp < N$  and there are  $s' > 0$ ,  $p' \geq 1, q' \in [1, p']$  for which  $[sp] = [s'p']$ ,  $\Lambda_{p,q}^s(B^N(0,1), \mathfrak{R}) \subset \Lambda_{p',q'}^{s'}(B^N(0,1), \mathfrak{R})$  and  $\pi_{[sp]}(\mathfrak{R}) \neq \{0\}$  then  $C(B^N(0,1), \mathfrak{R}) \cap \Lambda_{p,q}^s(B^N(0,1), \mathfrak{R})$  is not dense in  $\Lambda_{p,q}^s(B^N(0,1), \mathfrak{R})$ .*

In order to prove this theorem we give before two propositions relating convergence in Sobolev or Besov spaces and homotopy properties.

**Proposition 3.2.** *Let  $\mathfrak{M}$  be a compact Riemann manifold of dimension  $N$ ,  $\mathfrak{R}$  a compact Riemann submanifold of  $\mathbb{R}^{M+1}$  and  $g: \mathfrak{M} \rightarrow \mathbb{R}^{M+1}$  a  $W^{m,p}$  map such that  $g(x)$  belongs to  $\mathfrak{R}$  for almost every  $x$  of  $\mathfrak{M}$  and  $mp < N$ . There is an  $\varepsilon > 0$  such that, if  $f_1$  and  $f_2$  from  $\mathfrak{M}$  to  $\mathfrak{R}$  are functions of  $W^{m,p}$  and  $\|f_i - g\|_{m,p} < \varepsilon$  ( $i = 1, 2$ ) then  $f_1$  and  $f_2$  are  $[mp]$ -homotopic.*



**Proof.** Since  $\mathfrak{R}$  is bounded in  $\mathbb{R}^{M+1}$ , the maps  $g, f_1$  and  $f_2$  belong to  $L^\infty$ . Then by the Gagliardo and Nirenberg's inequalities:

$$\|f_i - g\|_{1,mp} \leq C \|f_i - g\|_{m,p} \|f_i - g\|_\infty \leq C \|f_i - g\|_{m,p} \text{ for } i = 1, 2.$$

By Theorem 2 of [15], there is an  $\varepsilon > 0$  such that  $\|f_i - g\|_{1,mp} \leq \varepsilon$  ( $i = 1, 2$ ) implies that  $f_1$  and  $f_2$  are  $[mp]$ -homotopic.

**Proposition 3.3.** *Let  $\mathfrak{R}$  be a compact Riemann submanifold of  $\mathbb{R}^{M+1}$ . Let  $g$  be a  $\Lambda_{pq}^s$  map from  $B^N(0,1)$  into  $\mathfrak{R}$  with  $sp < N$  and  $q \leq p$ . There is an  $\varepsilon > 0$  such that if  $f_1$  and  $f_2$ , from  $B^N(0,1)$  to  $\mathfrak{R}$  are in  $\Lambda_{pq}^s$  and  $\|f_i - g\|_{s,p,q} < \varepsilon$  ( $i = 1, 2$ ) then  $f_1$  and  $f_2$  are  $[sp]$ -homotopic.*

**Proof.** Let  $\{f_i\}$  be a sequence of  $\Lambda_{pq}^s$  maps from  $B^N(0,1)$  into  $\mathfrak{R}$  such that  $\|f_i - g\|_{s,p,q} \leq 2^{-i}$ . Since  $q \leq p$ ,  $\Lambda_{pq}^s \subset \Lambda_{pp}^s$  with continuity. On the other hand since  $\mathfrak{R}$  is bounded the functions  $f, f_1$  and  $f_2$  are bounded and then for  $r = \min(s, s/([s] + 1))$ :

$$\|f_i - g\|_{r,sp/r:sp/r} \leq C \|f_i - g\|_{s,p,p} \leq C \|f_i - g\|_{s,p,q} \leq C 2^{-i}$$

Using elementary properties of Lebesgue integral on  $\mathbb{R}^N$  and the same arguments as in [15] we obtain that there are  $r_0 \in (0, 1/2)$ ,  $t_1 \in (-1, 1)$  and for  $i = 2, \dots, m$  (where  $m = N - [sp] - 1$ ),

$$t_i \in (-\sqrt{1-t_1^2} \dots \sqrt{1-t_{i-1}^2}, \sqrt{1+t_1^2} \dots \sqrt{1+t_{i-1}^2})$$

such that, if

$$X = \left\{ x' \in \mathbb{R}^{[sp]+1}; \sum_{i=1}^{[sp]+1} x_i^2 + \sum_{i=1}^m t_i^2 = r_0 \right\} \quad \mathbf{t} = (t_1, \dots, t_m)$$

$$\exists C > 0; \forall k \in \mathbb{N} \int_X \left( \int_{B^N(0,1)} |f_k(x', \mathbf{t}) - f_k(y)|^{sp/r} \frac{dy}{|(x', \mathbf{t}) - y|^{N+sp}} \right) d\sigma(x') \leq C \quad (3.4)$$

$$\lim_{k \rightarrow \infty} \int_X |f_{k+1}(x', \mathbf{t}) - f_k(x', \mathbf{t})|^{sp/r} d\sigma(x') = 0 \quad (3.5)$$

We conclude with the same tools that in [15] using that the  $[sp]$ -skeleton of  $B^N(0,1)$  is  $S^{[sp]}$ .

**Proof of 3.1.** i) Let  $P_{[mp]+1}$  be the projection  $\mathbb{R}^N$  to  $\mathbb{R}^{[mp]+1}$  and  $\pi$  radial projection from  $\mathbb{R}^{[mp]+1}$  to  $S^{[mp]}$ . Define the function  $g = \pi \circ P_{[mp]+1}$ . We have that  $g \in W^{m,p}(B^N(0,1), S^{[mp]})$  and  $g|_{S^{[mp]}} = Id$ .

On the other hand, since  $\Pi_{[mp]}(\mathfrak{R}) \neq \{0\}$  there is a  $C^{[mp]+1}$  map,  $\Phi$  from  $S^{[mp]}$  to  $\mathfrak{R}$  which can not be extended continuously to  $B^{[mp]+1}$ . Let us define

$f = \Phi \circ g$ . By the results of section 1,  $f \in W^{m,p}((B^N(0,1), \mathfrak{R}))$ . Now suppose there is a sequence  $\{f_n\}$  of functions of  $C((B^N(0,1), \mathfrak{R}))$  converging to  $f$  in  $W^{m,p}((B^N(0,1), \mathfrak{R}))$ . By proposition (3.2)  $f_n$  and  $f$  are  $[mp]$ -homotopic for  $n$  large enough. Since  $f_n$  is smooth on  $B^N(0,1)$ , by the homotopy extension theorem we may extend  $f$  to  $B^N(0,1)$  continuously but that is impossible by construction.

ii) Let us define the function  $f$  in the same way as above, with  $[sp]$  instead of  $[mp]$ . Suppose again that there is a sequence of continuous functions  $\{f_n\}$  converging to  $f$  in  $\Lambda_{p,q}^s(B^N(0,1), \mathfrak{R})$ . By the hypothesis  $\{f_n\}$  converges to  $f$  in  $\Lambda_{p,q}^s(B^N(0,1), \mathfrak{R})$ . The proof follows now as above.

**Remark 3.4.** Using the inclusions recalled in the first section it is very simple to see that the conditions in ii) of Theorem (3.1) are satisfied if  $s \cdot p$  is not an integer and  $\Pi_{[sp]}(\mathfrak{R}) \neq \{0\}$ . If  $sp$  is an integer we must have  $q \leq p$ .

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