REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 1, números 1, 2, 3; 1988.
http://dx.doi.org/10.5209/rev_REMA.1988.v1.n1.18174

On embedding l_1 as a complemented subspace of Orlicz vector valued function spaces

FERNANDO BOMBAL

ABSTRACT. Several conditions are given under which l_1 embeds as a complemented subspace of a Banach space E if it embeds as a complemented subspace of an Orlicz space of E-valued functions. Previous results in [7] and [1] are extended in this way.

INTRODUCTION AND TERMINOLOGY

Pisier proved in [7] that if a Banach space E contains no copy of l_1, then the space $L_p(\mu,E)$ does not contain it either, for $1<p<\infty$. In [1] the result is extended to the case of Orlicz spaces $L_\Phi(\mu,E)$ and we study also the problem of embedding l_1 as a complemented subspace of $L_\Phi(\mu,E)$. A complete characterization is obtained when E is a Banach lattices, getting only partial results in the general case. The aim of this note is to give some new different conditions under which $L_\Phi(\mu,E)$ contains a complemented copy of l_1 if and only if so does either $L_\Phi(\mu)$ or E.

As for notations, E will denote a Banach space, E^* its topological dual and (Ω,Σ,μ) a finite, complete measure space. A series Σx_n in E is said to be *weakly unconditionally Cauchy* (w.u.c. in short) if $\Sigma|x^*(x_n)|<\infty$ for every $x^*\in E^*$. A subset B of E is called *weakly conditionally compact* if every sequence in B has a weakly Cauchy subsequence. Given a Young's function Φ with conjugate function Ψ (see [10], p. 77 and ff.), for every strongly measurable function $\Omega\to E$ we shall write

$$M_\Phi(f)=\int\Phi(\|f\|)\,d\mu.$$

The Orlicz space $L_\Phi(\mu,E)$ is the vector space of all (classes of) strongly measurable functions f from Ω into E such that $M_\Phi(kf)<\infty$ for some $k>0$ (if

---

Supported in part by CAICYT grant 0338/84.
1980 Mathematics Subject Classification (1985 revision): 46E40, 46B25.
Editorial de la Universidad Complutense. Madrid, 1988.

$\Phi(t)=t^p$, $1\leqslant p<\infty$, $L_\Phi(\mu,E)$ is the usual Lebesgue space $L_p(\mu,E)$). $L_\Phi(\mu,E)$ coincides with the set of all strongly measurable functions $f:\Omega\to E$ such that

$$\|f\|_\Phi=\mathrm{Sup}\{\int\|f\|\varphi\, d\mu:\varphi\in L_\Psi(\mu,\mathbb{K}),\ M_\Psi(\varphi)\leqslant 1\}<\infty.$$

This expression defines a Banach space norm in $L_\Phi(\mu,E)$. We have

$$L_\infty(\mu,E)\subset L_\Phi(\mu,E)\subset L_1(\mu,E),$$

with continuous inclusions. Recall that Φ is said to verify the (Δ_2)-condition if it is everywhere finite and

$$\limsup_{t\to\infty}\frac{\Phi(2t)}{\Phi(t)}<\infty.$$

In this case, the simple functions are dense in $L_\Phi(\mu,E)$. Finally, we shall use the name «l_1-sequence» to denote a sequence equivalent to the usual basis of l_1. A complemented l_1 sequence will be an l_1-sequence which spans a complemented subspace.

For notations and terminology used and not defined, we refer to [4] and [5].

THE RESULTS

Recall that a subset A of a Banach space E is called a (V^*) set ([6]) if for every w.u.c. series Σx_n^* in E^*, the following holds:

$$\lim_{n\to\infty}\sup\{|x_n^*(x)|:x\in A\}=0$$

It is evident that every (V^*) set is bounded. Also, every weakly conditionally compact set is a (V^*) set ([2], cor. 1.3). E is said to have *property weak* (V^*) if, conversely, every (V^*) set is weakly conditionally compact. Spaces not containing copies of l_1 and closed subspaces of order continuous Banach lattices, have property weak (V^*) (see [2]). Property weak (V^*) appears as a weakening of the so called *property* (V^*), introduced by Pelczynski in [6] and extensively studied.

To proceed any further, we shall need the following results:

Lemma A. ([2], *prop.* 1.1) *A bounded subset of a Banach space is a* (V^*) *set if and only if it does not contain a complemented* l_1 *sequence.*

Lemma B. ([2], *Th.* 3.2) *Let $K \subset L_1(\mu,E)$ be uniformly integrable. If K is not a (V^*) set, there exists $B \in \Sigma$ with $\mu(B)>0$, such that $\{f(\omega): f \in K\}$ is not a (V^*) set for every $\omega \in B$.*

Lemma C. ([2], *Cor.* 1.7) *Let $A \subset E$ be bounded. If for every $\varepsilon>0$ there exists a (V^*) set $A_\varepsilon \subset E$ such that*

$$A \subset A_\varepsilon + \varepsilon B(E),$$

where $B(E)$ is the unit closed ball of E, then A is a (V^*) set.

The first result is a characterization of property weak (V^*):

Theorem 1. *A Banach space has property weak (V^*) if and only if any l_1 sequence has a complemented l_1 subsequence.*

Proof. Suppose E has property weak (V^*) and let $(x_n) \subset E$ be a l_1 sequence. Then $A = \{x_n : n \in \mathbb{N}\}$ is not weakly conditionally compact and so it is not a (V^*) set. An appeal to lemma A yields a complemented l_1 subsequence of (x_n).

Conversely, if E does not have property weak (V^*), there exists a (V^*) set K that is not weakly conditionally compact. Rosenthal's l_1 theorem ([4], th. 2.e.5) produces a l_1 sequence (x_n) in K that, by lemma A, can not have a complemented l_1 subsequence.

EXAMPLES

a) The James space J (see, f.i., [4], example 1.d.2) is a non reflexive separable Banach space that does not contain copies either of c_0 or l_1. In particular, it has property weak (V^*), but it is neither a Banach lattice nor a subspace of an order continuous Banach lattice ([5], th. 1.c.5).

b) The space $E = J \oplus l_1$ has property weak (V^*), as a direct sum of spaces having it. Besides, it does not contain a copy of c_0 ([8], th. 1) and it is not weakly sequentially complete (because its closed subspace J is not). Hence, it is not a Banach lattice by [5], th. 1.c.4. This proves that there are spaces with the weak (V^*) property, containing l_1, and such that they are not Banach lattices.

The general question of wether the embedding of l_1 as a complemented subspace of $L_\Phi(\mu,E)$ implies necessarily that either $L_\Phi(\mu,\mathbb{K}) = L_\Phi(\mu)$ or E contains a complemented copy of l_1, is still open, as far as we know. The answer, is affirmative if E is a Banach lattice and μ a non-purely atomic probability measure, or $L_\Phi(\mu,E)$ contains an uniformly bounded complemented l_1 sequence ([1], Th 5 and 6). The result is also true when μ is purely atomic and $\Phi(t) = t^p$ $(1<p<\infty)$. Next result gives also a positive answer when E has property weak (V^*):

Proposition 2. *Let E be a Banach space with the weak (V^*) property and Φ a Young's function satisfying the (Δ_2)-condition. Then $L_\Phi(\mu,E)$ contains a complemented copy of l_1 if and only if either $L_\Phi(\mu)$ or E contains a complemented copy of l_1.*

Proof. Suppose $L_\Phi(\mu)$ does not contain a complemented copy of l_1. As $L_\Phi(\mu)$ is an order continuous Banach lattice, it follows from [9], th. 16 that l_1 does not embed in $L_\Phi(\mu)$. Theorem 4 of [1] proves then that E contains a copy of l_1 and, by theorem 1, also a complemented copy of l_1.

The examples given after theorem 1 show that the scope of theorem 3 is different from that of proposition 2 in [1].

The following is an extension of a result of Maurey and Pisier ([7], Th. 2) for complemented l_1 sequences:

Theorem 3. *Let E be a Banach space and $K=\{f_n:n\in\mathbb{N}\}\subset L_1(\mu,E)$ an uniformly integrable sequence. If for almost all ω the sequence $\{f_n(\omega):n\in\mathbb{N}\}$ does not have a complemented l_1 subsequence, then K does not contain a complemented l_1 subsequence.*

Proof. Suppose on the contrary that K contains a complemented l_1 subsequence. Then, by lemma A, K is not a (V^*) set. For every n, $m\in\mathbb{N}$, let us write

$$A_{nm}=\{\omega\in\Omega:|f_n(\omega)|\leqslant m\},\qquad f_{nm}=f_n\chi_{A_{nm}}\qquad\text{and}\qquad K_m=\{f_{nm}:n\in\mathbb{N}\}.$$

By the uniform integrability of K,

$$K\subset K_m+\varepsilon_m B(L_1(\mu,E)),$$

where $B(L_1(\mu,E))$ denotes the closed unit ball and (ε_m) is a null sequence of positive numbers. Because of lemma C, there is an $m\in\mathbb{N}$ such that K_m is not a (V^*) set. Lemma B provides a set $B\in\Sigma$ of positive measure, such that for every ω in B, $\{f_{nm}(\omega):n\in\mathbb{N}\}$ is bounded and not a (V^*) set. Lemma A assures then that it contains a complemented l_1 sequence.

In general, it is not clear that a bounded subset of $L_\Phi(\mu,E)$ which is not a (V^*) set, can not be a (V^*) subset of $L_1(\mu,E)$ (see [2], Prop. 1.10). This is the main reason why the above theorem is not automatically verified when K is a complemented l_1 sequence in $L_\Phi(\mu,E)$ (under mild conditions on Φ, this implies K uniformly integrable). In order to assure it is true, at least in some cases, let us call a subset $K\subset L_\Phi(\mu,E)$ *equi-Φ-integrable if*

$$\lim_{m\to\infty}\sup\{\|f\chi_{\{w:|f(w)|>m\}}\|_\Phi:f\in K\}=0.$$

With this notation, we have:

Theorem 4. *Let E be a Banach space and Φ a Young's function satisfying the (Δ_2)-condition. If $K=\{f_n : n \in \mathbb{N}\}$ is a complemented equi-Φ-integrable l_1 sequence in $L^\Phi(E)$, then E contains a complemented copy of l_1.*

Proof. Reasoning as in theorem 3 we get an uniformly bounded subset K_m which is not a (V^*) set. Lemma A produces then a uniformly bounded complemented l_1 sequence. Theorem 5 of [1] yields the result.

References

[1] F. BOMBAL, On l_1 subspaces of Orlicz vector valued function spaces. *Mah. Proc. Cambridge Phil. Soc.* (1987), 107-112.
[2] F. BOMBAL, On (V^*) sets and Pelczynski's property (V^*). To appear.
[3] F. BOMBAL and C. FIERRO, Compacidad débil en espacios de Orlicz de funciones vectoriales. *Rev. Acad. Ci. de Madrid*, 78 (1984), 157-163.
[4] J. LINDENSTRAUSS and L. TZAFRIRI, *Classical Banach Spaces*, vol. I. Springer, 1977.
[5] J. LINDENSTRAUSS and L. TZAFRIRI, *Classical Banach spaces*, vol. II. Springer, 1979.
[6] A. PELCZYNSKI, On Banach spaces on which every unconditionally converging operator is weakly compact. *Bull. Acad. Pol. Sci.*, 10 (1962), 641-648.
[7] G. PISIER, Une proprieté de stabilité de la classe des espaces ne contenant pas l_1. *C. R. Acad. Sci. Paris, Ser. A*, 285 (1978), 747-749.
[8] C. SAMUEL, *Sur la reproducibilite des espaces l^p*. Seminaire d'analyse fonctionnelle. Ecole Polytechnique, Paris (1978-1979). Exp. XXVI.
[9] L. TZAFRIRI, Reflexivity in Banach lattices and their subspaces. *J. of Funct. Analysis*, 10 (1972), 1-18.
[10] A. C. ZAANEN, *Linear Analysis*. North Holland, 1953.

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense
28040 Madrid (Spain)

Recibido: 15 de abril de 1988