The $\mathrm{SL}(2,\mathbb{C})$-Character Varieties of Torus Knots

Vicente Muñoz

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM
Consejo Superior de Investigaciones Científicas
Serrano 113 bis
28006 Madrid — Spain
vicente.munoz@imaff.cfmac.csic.es

Facultad de Matemáticas
Universidad Complutense de Madrid
Plaza Ciencias 3
28040 Madrid — Spain

Received: September 29, 2008
Accepted: January 13, 2009

ABSTRACT

Let G be the fundamental group of the complement of the torus knot of type (m,n). This has a presentation $G=\langle x,y \,|\, x^m=y^n\rangle$. We find the geometric description of the character variety $X(G)$ of characters of representations of G into $\mathrm{SL}(2,\mathbb{C})$.

Key words: Torus knot, characters, representations.

2000 Mathematics Subject Classification: Primary: 14D20. Secondary: 57M25, 57M27.

Introduction

Since the foundational work of Culler and Shalen [1], the varieties of $\mathrm{SL}(2,\mathbb{C})$-characters have been extensively studied. Given a manifold M, the variety of representations of $\pi_1(M)$ into $\mathrm{SL}(2,\mathbb{C})$ and the variety of characters of such representations both contain information of the topology of M. This is specially interesting for 3-dimensional manifolds, where the fundamental group and the geometrical properties of the manifold are strongly related.

This can be used to study knots $K\subset S^3$, by analysing the $\mathrm{SL}(2,\mathbb{C})$-character variety of the fundamental group of the knot complement S^3-K. In this paper, we study the case of the torus knots $K_{m,n}$ of any type (m,n). The case $(m,n)=(m,2)$ was analysed in [3] and the general case was recently determined in [2] by a method different from ours.

Partially supported through grant MEC (Spain) MTM2007-63582

1. Character varieties

A *representation* of a group G in $\mathrm{SL}(2,\mathbb{C})$ is a homomorphism $\rho : G \to \mathrm{SL}(2,\mathbb{C})$. Consider a finitely presented group $G = \langle x_1,\dots,x_k | r_1,\dots,r_s\rangle$, and let $\rho : G \to \mathrm{SL}(2,\mathbb{C})$ be a representation. Then ρ is completely determined by the k-tuple $(A_1,\dots,A_k) = (\rho(x_1),\dots,\rho(x_k))$ subject to the relations $r_j(A_1,\dots,A_k) = 0$, $1 \le j \le s$. Using the natural embedding $\mathrm{SL}(2,\mathbb{C}) \subset \mathbb{C}^4$, we can identify the space of representations as

$$\begin{aligned} R(G) &= \mathrm{Hom}(G,\mathrm{SL}(2,\mathbb{C})) \\ &= \{(A_1,\dots,A_k) \in \mathrm{SL}(2,\mathbb{C})^k \,|\, r_j(A_1,\dots,A_k) = 0,\ 1 \le j \le s\} \subset \mathbb{C}^{4k}. \end{aligned}$$

Therefore $R(G)$ is an affine algebraic set.

We say that two representations ρ and ρ' are equivalent if there exists $P \in \mathrm{SL}(2,\mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$, for every $g \in G$. This produces an action of $\mathrm{SL}(2,\mathbb{C})$ in $R(G)$. The moduli space of representations is the GIT quotient

$$M(G) = \mathrm{Hom}(G,\mathrm{SL}(2,\mathbb{C}))/\!/\ \mathrm{SL}(2,\mathbb{C}).$$

A representation ρ is *reducible* if the elements of $\rho(G)$ all share a common eigenvector, otherwise ρ is *irreducible*.

Given a representation $\rho : G \to \mathrm{SL}(2,\mathbb{C})$, we define its *character* as the map $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = \mathrm{tr}\,\rho(g)$. Note that two equivalent representations ρ and ρ' have the same character, and the converse is also true if ρ or ρ' is irreducible [1, Proposition 1.5.2].

There is a character map $\chi : R(G) \to \mathbb{C}^G$, $\rho \mapsto \chi_\rho$, whose image

$$X(G) = \chi(R(G))$$

is called the *character variety of* G. Let us give $X(G)$ the structure of an algebraic variety. By the results of [1], there exists a collection $g_1,\dots,g_a$ of elements of G such that χ_ρ is determined by $\chi_\rho(g_1),\dots,\chi_\rho(g_a)$, for any ρ. Such collection gives a map

$$\Psi : R(G) \to \mathbb{C}^a, \qquad \Psi(\rho) = (\chi_\rho(g_1),\dots,\chi_\rho(g_a)).$$

We have a bijection $X(G) \cong \Psi(R(G))$. This endows $X(G)$ with the structure of an algebraic variety. Moreover, this is independent of the chosen collection as proved in [1].

Lemma 1.1. *The natural algebraic map $M(G) \to X(G)$ is a bijection.*

Proof. The map $R(G) \to X(G)$ is algebraic and $\mathrm{SL}(2,\mathbb{C})$-invariant, hence it descends to an algebraic map $\varphi : M(G) \to X(G)$. Let us see that φ is a bijection.

For ρ an irreducible representation, if $\varphi(\rho) = \varphi(\rho')$ then ρ and ρ' are equivalent representations; so they represent the same point in $M(G)$.

Now suppose that ρ is reducible. Consider $e_1 \in \mathbb{C}^2$ the common eigenvector of all $\rho(g)$. This gives a sub-representation $\rho' : G \to \mathbb{C}^*$ of G. We have a quotient representation $\rho'' = \rho/\rho' : G \to \mathbb{C}^*$, defined as the representation induced by ρ in the quotient space $\mathbb{C}^2/\langle e_1\rangle$. As characters, $\rho'' = \rho'^{-1}$. The representation $\rho' \oplus \rho''$ is the *semisimplification* of ρ. It is in the closure of the $\mathrm{SL}(2,\mathbb{C})$-orbit through ρ. Clearly, $\chi_\rho(g) = \rho'(g) + \rho'(g)^{-1}$. Now if ρ and $\tilde\rho$ are two reducible representations and $\varphi(\rho) = \varphi(\tilde\rho)$, then their semisimplifications have the same character, that is

$$\chi_\rho(g) = \chi_{\tilde\rho}(g) \Rightarrow \rho'(g) + \rho'(g)^{-1} = \tilde\rho'(g) + \tilde\rho'(g)^{-1}\,.$$

Therefore $\rho' = \tilde\rho'$ or $\rho' = \tilde\rho'^{-1}$. In either case ρ and $\tilde\rho$ represent the same point in $M(G)$, which is actually the point represented by $\rho' \oplus \rho'^{-1}$. □

2. Character varieties of torus knots

Let $T^2 = S^1 \times S^1$ be the 2-torus and consider the standard embedding $T^2 \subset S^3$. Let m, n be a pair of coprime positive integers. Identifying T^2 with the quotient $\mathbb{R}^2/\mathbb{Z}^2$, the image of the straight line $y = \frac{m}{n}x$ in T^2 defines the *torus knot* of type (m,n), which we shall denote as $K_{m,n} \subset S^3$ (see [4, Chapter 3]).

For any knot $K \subset S^3$, we denote by $G(K)$ the fundamental group of the exterior $S^3 - K$ of the knot. It is known that

$$G_{m,n} = G(K_{m,n}) \cong \langle x, y \,|\, x^m = y^n \,\rangle\,.$$

The purpose of this paper is to describe the character variety $X(G_{m,n})$.

In [3], the character variety $X(G_{m,2})$ is computed. We want to extend the result to arbitrary m, n, and give a simpler argument than that of [3].

After the completion of this work, we became aware of the paper [2] where the character varieties of $X(G_{m,n})$ are determined (even without the assumption of m, n being coprime). However, our method is more direct than the one presented in [2].

To start with, note that

$$R(G_{m,n}) = \{(A,B) \in \mathrm{SL}(2,\mathbb{C}) \,|\, A^m = B^n\}\,.$$

Therefore we shall identify a representation ρ with a pair of matrices (A,B) satisfying the required relation $A^m = B^n$.

We decompose the character variety

$$X(G_{m,n}) = X_{red} \cup X_{irr}\,,$$

where X_{red} is the subset consisting of the characters of reducible representations (which is a closed subset by [1]), and X_{irr} is the closure of the subset consisting of the characters of irreducible representations.

Proposition 2.1. *There is an isomorphism $X_{red} \cong \mathbb{C}$. The correspondence is defined by*

$$\rho = \left(A = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}, B = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}\right) \mapsto s = t + t^{-1} \in \mathbb{C}\,.$$

Proof. By the discussion in Lemma 1.1, an element in X_{red} is described as the character of a split representations $\rho = \rho' \oplus \rho'^{-1}$. This means that in a suitable basis,

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

The equality $A^m = B^n$ implies $\lambda^m = \mu^n$. Therefore there is a unique $t \in \mathbb{C}$ with $t \neq 0$ such that

$$\begin{cases} \lambda = t^n, \\ \mu = t^m. \end{cases}$$

(Here we use the coprimality of (m,n)). Note that the pair (A,B) is well-defined up to permuting the two vectors in the basis. This corresponds to the change $(\lambda,\mu) \mapsto (\lambda^{-1},\mu^{-1})$, which in turn corresponds to $t \mapsto t^{-1}$. So (A,B) is parametrized by $s = t + t^{-1} \in \mathbb{C}$. □

Lemma 2.2. *Suppose that $\rho = (A,B) \in R(G_{m,n})$. In any of the following cases:*

(a) $A^m = B^n \neq \pm\mathrm{Id}$,

(b) $A = \pm\mathrm{Id}$ or $B = \pm\mathrm{Id}$,

(c) A or B is non-diagonalizable,

the representation ρ is reducible.

Proof. First suppose that A is diagonalizable with eigenvalues λ, λ^{-1}, and suppose that $\lambda^m \neq \pm 1$. Then there is a basis e_1, e_2 in which $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, which is well-determined up to multiplication of the basis vectors by non-zero scalars. Then

$$B^n = A^m = \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix}$$

is a diagonal matrix, different from $\pm\mathrm{Id}$. Therefore B must be diagonal in the same basis, $B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, with $\lambda^m = \mu^n$. This proves the reducibility in case (a).

Now suppose that $A = \lambda\mathrm{Id}$, $\lambda = \pm 1$. Then $B^n = \lambda^m\mathrm{Id}$, so it must be that B is diagonalizable. Using a basis in which B is diagonal, we get the reducibility in case (b).

Finally, suppose that A is not diagonalizable. Then there is a suitable basis on which A takes the form $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, with $\lambda = \pm 1$. Clearly

$$B^n = A^m = \lambda^m \begin{pmatrix} 1 & m\lambda \\ 0 & 1 \end{pmatrix}$$

and so

$$B = \begin{pmatrix} \mu & x \\ 0 & \mu \end{pmatrix},$$

with $\mu = \pm 1$, $\mu^n = \lambda^m$ and $\mu n x = \lambda m$. In this basis, the vector e_1 is an eigenvector for both A and B. Hence the representation (A, B) is reducible, completing the case (c). ☐

Proposition 2.3. *Let X^o_{irr} be the set of irreducible characters, and X_{irr} its closure. Then*

$$\begin{aligned} X^o_{irr} &\cong \{(\lambda,\mu,r) \,|\, \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C} - \{0,1\}\}/\mathbb{Z}_2 \times \mathbb{Z}_2\,, \\ X_{irr} &\cong \{(\lambda,\mu,r) \,|\, \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C}\}/\mathbb{Z}_2 \times \mathbb{Z}_2\,. \end{aligned}$$

where $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts as $(\lambda,\mu,r) \sim (\lambda^{-1},\mu,1-r) \sim (\lambda,\mu^{-1},1-r) \sim (\lambda^{-1},\mu^{-1},r)$.

Proof. Let $\rho = (A, B)$ be an element of $R(G_{m,n})$ which is an irreducible representation. By Lemma 2.2, A is diagonalizable but not equal to $\pm\mathrm{Id}$, and $A^m = \pm\mathrm{Id}$. So the eigenvalues λ, λ^{-1} of A satisfy $\lambda^m = \pm 1$ and $\lambda \neq \pm 1$. Analogously, B is diagonalizable but not equal to $\pm\mathrm{Id}$, with eigenvalues μ, μ^{-1}, with $\mu^n = \pm 1$, $\mu \neq \pm 1$. Moreover,

$$\lambda^m = \mu^n\,.$$

We may choose a basis $\{e_1, e_2\}$ under which A diagonalizes. This is well-defined up to multiplication of e_1 and e_2 by two non-zero scalars. Let $\{f_1, f_2\}$ be a basis under which B diagonalizes, which is well-defined up to multiplication of f_1, f_2 by non-zero scalars. Then $\{[e_1], [e_2], [f_1], [f_2]\}$ are four points of the projective line $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$. Note that the pair (A, B) is irreducible if and only if the four points are different.

The only invariant of four points in $\mathbb{P}^1$ is the double ratio

$$r = ([e_1] : [e_2] : [f_1] : [f_2]) \in \mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}\,.$$

So (A, B) is parametrized, up to the action of $\mathrm{SL}(2,\mathbb{C})$, by (λ, μ, r). Permuting the two basis vectors e_1, e_2 corresponds to $(\lambda,\mu,r) \mapsto (\lambda^{-1},\mu,1-r)$, since

$$([e_2] : [e_1] : [f_1] : [f_2]) = 1 - ([e_1] : [e_2] : [f_1] : [f_2]).$$

Analogously, permuting the two basis vectors f_1, f_2 corresponds to

$$(\lambda,\mu,r) \mapsto (\lambda,\mu^{-1},1-r).$$

Note that this gives an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and X^o_{irr} is the quotient of the set of (λ,μ,r) as above by this action.

To describe the closure of X^o_{irr}, we have to allow f_1 to coincide with e_1. This corresponds to $r = 1$ (the same happens if f_2 coincides with e_2). In this case, e_1 is an eigenvector of both A and B, so the representation (A,B) has the same character as its semisimplification (A',B') given by

$$A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

This means that the point $(\lambda,\mu,1)$ corresponds under the identification $X_{red} \cong \mathbb{C}$ given by Proposition 2.1 to $s_1 = t_1 + t_1^{-1}$, where $t_1 \in \mathbb{C}$ satisfies

$$\begin{cases} \lambda = t_1^n, \\ \mu = t_1^m. \end{cases} \tag{1}$$

Also, we have to allow f_1 to coincide with e_2 (or f_2 to coincide with e_1). This corresponds to $r=0$. The representation (A,B) has semisimplification (A',B') where

$$A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}.$$

So the point $(\lambda,\mu,1)$ corresponds to $s_0 = t_0 + t_0^{-1} \in X_{red} \cong \mathbb{C}$, where $t_0 \in \mathbb{C}$ satisfies

$$\begin{cases} \lambda = t_0^n, \\ \mu^{-1} = t_0^m. \end{cases} \tag{2}$$

$\square$

Proposition 2.3 says that X_{irr} is a collection of $\frac{(m-1)(n-1)}{2}$ lines. A pair (λ,μ) with $\lambda^m = \pm 1$ and $\mu^n = \pm 1$ is given as

$$\lambda = e^{\pi i k/m}, \quad \mu = e^{\pi i k'/n},$$

where $0 \leq k < 2m$, $0 \leq k' < 2n$. The condition $\lambda \neq \pm 1$, $\mu \neq \pm 1$ gives $k \neq 0, m$, $k' \neq 0, n$. Finally, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action allows us to restrict to $0<k<m$, $0<k'<n$. The condition $\lambda^m = \mu^n$ means that

$$k \equiv k' \pmod 2.$$

Denote by $X_{irr}^{k,k'}$ the line of X_{irr} corresponding to the values of k,k'. Then

$$X_{irr} = \bigsqcup_{\substack{0<k<m,\, 0<k'<n \\ k \equiv k' \pmod 2}} X_{irr}^{k,k'}.$$

The line $X_{irr}^{k,k'}$ intersects X_{red} in two points. This gives a collection of $(m-1)(n-1)$ points in X_{red}, which are defined as follows: under the identification $X_{red} \cong \mathbb{C}$, these are the points $s_l = t_l + t_l^{-1}$, where

$$t_l = e^{\pi i l/nm},$$

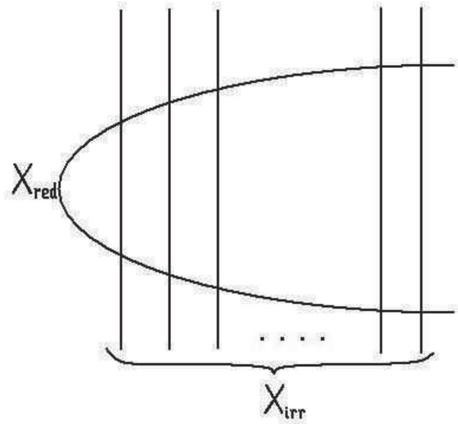

Figure 1 – Picture of $X(G_{m,n})$.

and $0 < l < mn$, $m \nmid l$, $n \nmid l$. Assume that n is odd (note that either m or n should be odd). Then from (1) and (2), the line $X_{irr}^{k,k'}$ intersects at the points $s_{l_0}, s_{l_1} \in X_{red}$ where

$$\begin{aligned} nl_0 &\equiv k \pmod m, & ml_0 &\equiv n - k' \pmod n\,, \\ nl_1 &\equiv k \pmod m, & ml_1 &\equiv k' \pmod n\,. \end{aligned}$$

These two points are different since $k' \not\equiv n - k' \pmod n$, as n is odd.

In the case $(m, n) = (2, n)$, this result coincides with [3, Corollary 4.2].

3. The algebraic structure of $X(G_{m,n})$

We want to give a geometric realization of $X(G_{m,n})$ which shows that the algebraic structure of this variety is that of a collection of rational lines as in Figure 1 intersecting with nodal curve singularities.

The map $R(G_{m,n}) \to \mathbb{C}^3$, $\rho = (A, B) \mapsto (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB))$, defines a map

$$\Psi : X(G_{m,n}) \to \mathbb{C}^3\,.$$

Theorem 3.1. *The map Ψ is an isomorphism with its image $C = \Psi(X(G_{m,n}))$. C is a curve consisting of $\frac{(n-1)(m-1)}{2} + 1$ irreducible components, all of them smooth and isomorphic to $\mathbb{C}$. They intersect with nodal normal crossing singularities following the pattern in Figure* 1*.*

Proof. Let us look first at $\Psi_0 = \Psi|_{X_{red}} : X_{red} \to \mathbb{C}^3$. For a given $\rho = (A, B) \in X_{red}$, with the shape given in Proposition 2.1, we have that

$$\Psi_0 : s = t + t^{-1} \mapsto (t^n + t^{-n}, t^m + t^{-m}, t^{n+m} + t^{-(n+m)}) .$$

This map is clearly injective: the image recovers

$$\{t^n, t^{-n}\},\ \{t^m, t^{-m}\},\ \{t^{n+m}, t^{-(n+m)}\}.$$

From this, we recover $\{(t^n, t^m), (t^{-n}, t^{-m})\}$ and hence the pair t, t^{-1} (since n, m are coprime).

Let us see that Ψ_0 is an immersion. The differential is

$$\frac{d\Psi_0}{dt} = \big(nt^{-n-1}(t^{2n} - 1), mt^{-m-1}(t^{2m} - 1), (n+m)t^{-n-m-1}(t^{2n+2m} - 1)\big) . \tag{3}$$

This is non-zero at all $t \neq \pm 1$. As $\frac{ds}{dt} \neq 0$, we have $\frac{d\Psi_0}{ds} \neq (0, 0, 0)$. For $t = \pm 1$, we note that $\frac{ds}{dt} = t^{-2}(t^2 - 1)$, so

$$\frac{d\Psi_0}{ds} = \left(nt^{-n+1}\frac{t^{2n} - 1}{t^2 - 1}, mt^{-m+1}\frac{t^{2m} - 1}{t^2 - 1}, (n+m)t^{-n-m+1}\frac{t^{2n+2m} - 1}{t^2 - 1}\right),$$

which is non-zero again.

Now, consider a component of X_{irr} corresponding to a pair (λ, μ). Take $r \in \mathbb{C}$. Fix the basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ which is given as the eigenbasis of A. Let $\{f_1, f_2\}$ be the eigenbasis of B. As the double ratio $(0 : \infty : 1 : r/(r-1)) = r$, we can take $f_1 = (1, 1)$ and $f_2 = (r - 1, r)$. This corresponds to the matrices:

$$\begin{aligned} A &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \\ B &= \begin{pmatrix} 1 & r-1 \\ 1 & r \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 1 & r-1 \\ 1 & r \end{pmatrix}^{-1} \\ &= \begin{pmatrix} r(\mu - \mu^{-1}) + \mu^{-1} & (1-r)(\mu - \mu^{-1}) \\ r(\mu - \mu^{-1}) & \mu - r(\mu - \mu^{-1}) \end{pmatrix} . \end{aligned}$$

Therefore:

$$\begin{aligned} \Psi(A, B) &= (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \\ &= (\lambda + \lambda^{-1}, \mu^{-1} + \mu, (\lambda\mu^{-1} + \lambda^{-1}\mu) + r(\lambda - \lambda^{-1})(\mu - \mu^{-1})). \end{aligned}$$

The image of this component is a line in $\mathbb{C}^3$. Its direction vector is $(0, 0, 1)$. At an intersection point with $\Psi_0(X_{red})$, the tangent vector to $\Psi_0(X_{red})$, given in (3), has non-zero first and second component, since $\lambda = t^n$, $\mu = t^m$ and $t \neq 0$, $\lambda^2 \neq 1$, $\mu^2 \neq 1$. So the intersection of these components is a transverse nodal singularity.

Finally, note that the map $\Psi : X(G_{m,n}) \to C$ is an algebraic map, it is a bijection, and C is a nodal curve (the mildest possible type of singularities). Therefore Ψ must be an isomorphism. □

Corollary 3.2. *$M(G) \cong X(G)$, for $G = G_{m,n}$.*

Proof. By Lemma 1.1, $\varphi : M(G) \to X(G)$ is an algebraic map which is a bijection. As the singularities of $X(G)$ are just transverse nodes, φ must be an isomorphism. □

Acknowledgement. The author wishes to thank the referee for useful comments, specially for pointing out the reference [2].

References

[1] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of* 3*-manifolds*, Ann. of Math. (2) **117** (1983), no. 1, 109–146.

[2] J. Martín-Morales and A-M. Oller-Marcén, *On the varieties of representations and characters of a family of one-relator subgroups*, arXiv:0805.4716.

[3] A-M. Oller-Marcén, *The* SL$(2,\mathbb{C})$ *character variety of a class of torus knots*, Extracta Math. **23** (2008), no. 2, 163–172.

[4] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.