About the Banach Envelope of $l_{1,\infty}$

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ABSTRACT

We study the Banach envelope of the quasi-Banach space $l_{1,\infty}$ consisting of all sequences $x=(\xi_k)$ with $s_n(x)=O\left(\frac{1}{n}\right)$, where $\left(s_n(x)\right)$ denotes the non-increasing rearrangement of $x=(\xi_k)$. The situation turns out to be much more complicated than that in the well-known case of the separable subspace $l^\circ_{1,\infty}$, whose members are characterized by $s_n(x)=o\left(\frac{1}{n}\right)$.

Key words: Banach envelope, Marcinkiewicz space $l_{1,\infty}$, weak l_1-space.

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1. Notation and terminology

We use standard notation and terminology of Banach space theory; see [21], for example.

Throughout, X and Y are real quasi-normed linear spaces. The symbol $|\!|\!|\cdot|\!|\!|$ denotes quasi-norms, while $\|\cdot\|$ is reserved for norms. By an embedding we mean a one-to-one continuous linear map $J : X \to Y$. Let $\mathbb{N} := \{1, 2, \dots\ \}$. The cardinality of a set $\mathbb{S}$ is denoted by $|\mathbb{S}|$.

2. Sequence spaces

The n^{th} *approximation number* of a bounded real sequence $x=(\xi_k)$ is defined by

$$s_n(x) := \inf\left\{\sup_{k\notin\mathbb{F}}|\xi_k| \,:\, |\mathbb{F}|<n\right\}$$

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or, equivalently, by

$$s_n(x) := \inf\{t > 0 \, : \, |\mathbb{S}(x,t)| < n\}, \quad \text{where} \quad \mathbb{S}(x,t) := \{k \, : \, |\xi_k| > t\}.$$

We refer to $\mathbb{S}(x) := \bigcup_{t>0} \mathbb{S}(x,t) = \{k : \xi_k \neq 0\}$ as the *support* of $x=(\xi_k)$. A sequence x is called *finite* if it has a finite support. The finite sequences form a linear space, denoted by c_{00}.

A *rearrangement* of a sequence $x=(\xi_k)$ is obtained by permutating its coordinates ξ_k, and adding or deleting 0's. In the case of a null sequence, $\big(s_n(x)\big)$ is the non-increasing rearrangement of $|x|=(|\xi_k|)$.

The Banach space of all bounded real sequences $x=(\xi_k)$ equipped with the norm

$$\|x|l_\infty\| := \sup_{1\le k<\infty} |\xi_k|$$

is denoted by l_∞, and c_0 stands for the closed linear subspace of all null sequences.

Let $0<p<\infty$. The *Marcinkiewicz space* or *weak l_p space*, denoted by $l_{p,\infty}$, consists of all $x=(\xi_k)$ such that

$$|||x|l_{p,\infty}||| := \sup_{1\le n<\infty} n^{1/p} s_n(x) = \sup_{t>0} t\,|\mathbb{S}(x,t)|^{1/p}$$

is finite. It easily turns out that $|||\cdot|l_{p,\infty}|||$ is a quasi-norm, which satisfies the quasi-triangle inequality

$$|||x+y|l_{p,\infty}||| \le c_p\big(|||x|l_{p,\infty}||| + |||y|l_{p,\infty}|||\big).$$

In passing, we note that $c_p=2^{1/p}$ is the best possible constant. Indeed, substituting

$$x = \Big(\quad 1 \quad , \ \dots \ , \tfrac{1}{n^{1/p}}, \ \dots \ , \tfrac{1}{(2n-1)^{1/p}}, 0, 0, \dots \ \Big)$$

and

$$y = \Big(\tfrac{1}{(2n-1)^{1/p}}, \ \dots \ , \tfrac{1}{n^{1/p}}, \ \dots \ , \quad 1 \quad , 0, 0, \dots \ \Big),$$

we may infer from $|||x|l_{p,\infty}||| = |||y|l_{p,\infty}||| = 1$ and $s_{2n-1}(x+y) = \frac{2}{n^{1/p}}$ that $(2-\frac{1}{n})^{1/p} \le c_p$.

The *little Marcinkiewicz space* or *little weak l_p space*, denoted by $l^\circ_{p,\infty}$, is defined to be the closed hull of c_{00} in $l_{p,\infty}$. The members of $l^\circ_{p,\infty}$ are characterized by the condition that $\lim\limits_{n\to\infty} n^{1/p} s_n(x) = 0$.

In analogy to an observation of Hunt [9, pp. 259–260], it turns out that $l_{p,\infty}$ fails to be normable for $0<p\le 1$, whereas there exists equivalent norms in the case $1<p<\infty$. Take, for example,

$$\|x|l_{p,\infty}\| := \sup_{1\le n<\infty} n^{1/p} \Big[\frac{1}{n}\sum_{k=1}^n s_k(x)\Big].$$

From now on, we concentrate on the case $p=1$.

The *Sargent space* $m_{1,\infty}$ is the collection of all $x=(\xi_k)$ for which

$$\|x|m_{1,\infty}\| := \sup_{1\le n<\infty} \frac{s_1(x)+\ldots+s_n(x)}{\frac{1}{1}+\ldots+\frac{1}{n}}$$

is finite. Note that $m_{1,\infty}$ is a Banach space. Using the abbreviation

$$\mu_n(x) := \frac{1}{L_n}\sum_{k=1}^{n} s_k(x) \quad \text{with} \quad L_n := \sum_{k=1}^{n}\frac{1}{k},$$

we get

$$\|x|m_{1,\infty}\| = \sup_{1\le n<\infty} \mu_n(x).$$

The *little Sargent space* $m_{1,\infty}^{\circ}$ is defined to be the closed hull of c_{00} in $m_{1,\infty}$, while $m_{1,\infty}^{\bullet}$ denotes the closed hull of $l_{1,\infty}$ in $m_{1,\infty}$. The members of $m_{1,\infty}^{\circ}$ are characterized by $\lim\limits_{n\to\infty}\mu_n(x)=0$. We have the strict inclusions

$$l_{1,\infty}^{\circ} \subset l_{1,\infty} \subset m_{1,\infty}^{\bullet}, \quad l_{1,\infty}^{\circ} \subset m_{1,\infty}^{\circ} \subset m_{1,\infty}^{\bullet}, \quad \text{and} \quad m_{1,\infty}^{\bullet} \subset m_{1,\infty}.$$

Obviously $(\frac{1}{k})\in l_{1,\infty}\setminus l_{1,\infty}^{\circ}$ and $(\frac{1}{k})\in m_{1,\infty}^{\bullet}\setminus m_{1,\infty}^{\circ}$. Define $a^{(\lambda)}=(\alpha_k^{(\lambda)})$ with $0<\lambda\le 1$ by

$$\alpha_k^{(\lambda)} := m^{\lambda}2^{-m^2} \quad \text{if} \quad 2^{(m-1)^2}<k\le 2^{m^2}.$$

Then $a^{(\lambda)}\in m_{1,\infty}^{\circ}\setminus l_{1,\infty}$ for $0<\lambda<1$ and $a^{(1)}\in m_{1,\infty}\setminus m_{1,\infty}^{\bullet}$. The latter example is due to Russu [22].

The *Matsaev space* $l_{\infty,1}$ is the collection of all $x=(\xi_k)$ such that

$$\|x|l_{\infty,1}\| := \sum_{k=1}^{\infty}\frac{1}{k}s_k(x)$$

is finite. It turns out that $l_{\infty,1}$ is a separable Banach space. By means of the standard duality

$$\langle x,y\rangle := \sum_{k=1}^{\infty}\xi_k\eta_k \quad \text{with } x=(\xi_k) \text{ and } y=(\eta_k),$$

we obtain

$$\big(l_{1,\infty}^{\circ}\big)^{*} = \big(m_{1,\infty}^{\circ}\big)^{*} = l_{\infty,1} \quad \text{and} \quad \big(l_{1,\infty}^{\circ}\big)^{**} = \big(m_{1,\infty}^{\circ}\big)^{**} = \big(l_{\infty,1}\big)^{*} = m_{1,\infty}.$$

The linear space $b_{1,\infty}$ consists of all $x=(\xi_k)$ that admit a representation

$$x=\sum_{k=1}^{\infty}x_k \text{ (coordinatewise), where } x_1,x_2,\ldots\in l_{1,\infty} \text{ and } \sum_{k=1}^{\infty}|||x_k|l_{1,\infty}|||<\infty.$$

Obviously, $l_{1,\infty}\subseteq b_{1,\infty}$. This inclusion is strict. Indeed, a result of Markus, Mityagin, and Ariño mentioned at the beginning of section 4 implies that $m_{1,\infty}^{\circ}\subseteq b_{1,\infty}$. Therefore $a^{(\lambda)}\in b_{1,\infty}\setminus l_{1,\infty}$ for $0<\lambda<1$. It follows from $\|x_k|m_{1,\infty}\|\le|||x_k|l_{1,\infty}|||$ and $\sum_{k=1}^{\infty}|||x_k|l_{1,\infty}|||<\infty$ that $\sum_{k=1}^{\infty}x_k$ converges in $m_{1,\infty}$. Hence $b_{1,\infty}$ is a dense subset of $m_{1,\infty}^{\bullet}$.

We can easily check that $b_{1,\infty}$ becomes a Banach space under the norm

$$\|x|b_{1,\infty}\| := \inf \sum_{k=1}^{\infty} |||x_k|l_{1,\infty}|||,$$

the infimum being taken over all representations described above.

Observe that the closed unit balls

$$B_l := \{x \in l_{1,\infty} : |||x|l_{1,\infty}||| \leq 1\} \quad \text{and} \quad B_m := \{x \in m_{1,\infty} : \|x|m_{1,\infty}\| \leq 1\}$$

have the same extreme points x^{ex}. The collection of these extreme points, which are characterized by $s_n(x^{\text{ex}}) = \frac{1}{n}$ for $n = 1, 2, \dots$, is denoted by B^{ex}.

Since $m_{1,\infty}$ can be identified with the dual of $l_{\infty,1}$, we know from the classical Kreĭn-Milman theorem that B_m is the weakly* closed convex hull of B^{ex}.

Recall that the extreme points of the closed unit ball $B_\infty := \{x \in l_\infty : \|x|l_\infty\| \leq 1\}$ have the form $x^{\text{ex}}_\infty = (\pm 1)$.

The *σ-convex hull* of a bounded subset B of a Banach space is defined by

$$\sigma\text{-conv}(B) := \Big\{ \sum_{k=1}^{\infty} \lambda_k x_k \ : \ x_1, x_2, \dots \in B, \ \ \lambda_1, \lambda_2, \dots \geq 0, \ \ \sum_{k=1}^{\infty} \lambda_k = 1 \Big\}.$$

In the following, we need a trivial case of the Choquet theorem; see [2, § 27] and [19, chap. 3].

Lemma. *The closed unit ball B_∞ coincides with the σ-convex hull of its extreme points.*

Proof. For every sequence $x = (\xi_k) \in B_\infty$, let

$$a = (\alpha_k), \quad \text{where} \quad \alpha_k := \begin{cases} +1 & \text{if } \xi_k \geq 0, \\ -1 & \text{if } \xi_k < 0. \end{cases}$$

Then $\|2x - a|l_\infty\| \leq 1$. Therefore, beginning with $x_1 := x$, we inductively find a_1, $x_2 := 2x_1 - a_1$, a_2, $x_3 := 2x_2 - a_2$, a_3, $\dots$ such that

$$\|x_i|l_\infty\| \leq 1 \quad \text{and} \quad x_{i+1} := 2x_i - a_i.$$

This construction yields

$$x = \sum_{i=1}^{n} 2^{-i} a_i + 2^{-n} x_{n+1} = \sum_{i=1}^{\infty} 2^{-i} a_i. \qquad \square$$

Using the preceding lemma and the formula $B_l = B_\infty \cdot B^{\text{ex}}$, we can easily show that the norm of $b_{1,\infty}$ is just the *Minkowski functional* of the σ-convex hull of B^{ex}. In particular, it follows that $b_{1,\infty}$ consists of all sequences that admit a representation

$$x = \sum_{k=1}^{\infty} \xi_k x_k^{\text{ex}} \quad \text{(coordinatewise), where} \quad s_n(x_k^{\text{ex}}) = \tfrac{1}{n} \ \text{ and } \ \sum_{k=1}^{\infty} |\xi_k| < \infty.$$

Historical remarks. The definition of $L_{p,\infty}[0,1]$ goes back to the famous interpolation theorem of Marcinkiewicz [14, p. 1272], who considered functions U such that $A[\mathrm{mes}E(|U(x)| > A)]^{1/p}$ is uniformly bounded for all $A > 0$. Function spaces that are non-atomic forerunners of $l_{\infty,1}$ and $m_{1,\infty}$ were invented by Lorentz [13, pp. 418–420].

Later on, Sargent [23, p. 162; 24, pp. 64–65] and Garling [5, p. 96–99] introduced the sequence spaces $l_{\infty,1}$, $m_{1,\infty}$, and $m^{\circ}_{1,\infty}$.

Almost simultaneously, the Soviet school presented a third approach that was created for treating non-self-adjoint operators on Hilbert space. The ideals $\mathfrak{S}_\omega$, $\mathfrak{S}_\Omega$, and $\mathfrak{S}^{(\circ)}_\Omega$ are the operator-theoretical counterparts of $l_{\infty,1}$, $m_{1,\infty}$, and $m^{\circ}_{1,\infty}$; see [7, pp. 139–150] as well as the original papers of Gokhberg/Kreĭn [6], Matsaev [16], Markus [15], Mityagin [17], and Russu [22], written in the early 1960's.

Curiously enough, the three lines of development took place without reference to each other. Thus it is impossible to decide who was the first inventor of the *sequence spaces* under consideration. In any case, all of them could be called *Lorentz spaces*.

3. The Banach envelope of a quasi-normed linear space

The following definition is made in the spirit of category theory. The objects are the real quasi-normed linear spaces and, since we are interested in the *isometric* theory, the morphisms are the *contractions*, $|||T : X \to Y||| \le 1$.

The *Banach envelope* of a quasi-normed linear space X is a Banach space X^{ban} together with a contraction E from X into X^{ban} (not necessarily one-to-one) such that for every contraction T from X into an arbitrary Banach space Y there exists a unique 'extension' T^{ban}:

$$\begin{tikzcd} X^{\mathrm{ban}} \arrow[dr, "T^{\mathrm{ban}}"] & \\ X \arrow[u, "E"] \arrow[r, "T"'] & Y \end{tikzcd}$$

Note that X^{ban} is unique up to an isometry; see [25, pp. 170–171].

The Banach envelope X^{ban} can be obtained as the closed hull of $K(X)$ in X^{**}, where K denotes the canonical map from X into its bidual.

Another construction goes as follows. Letting

$$\pi^{\max}(x) := \inf\Big\{ \sum_{k=1}^{n} |||x_k||| \,:\, x_1,\dots,x_n \in X,\ \ \sum_{k=1}^{n} x_k = x,\ n=1,2,\dots \Big\}$$

yields a semi-norm on X that coincides with the *Minkowski functional* of the convex hull of $B_X := \{x \in X \,:\, |||x||| \le 1\}$,

$$\pi^{\max}(x) := \inf\{\varrho \ge 0 \,:\, x \in \varrho\,\mathrm{conv}(B_X)\}.$$

Obviously, $\pi^{\max}$ is the largest semi-norm π on X for which $\pi(x) \le |||x|||$. Passing to the quotient space X/N with $N := \{x \in X : \pi^{\max}(x) = 0\}$, we obtain a normed linear space, whose completion is the required Banach envelope. The underlying norm is denoted by $\| \cdot | X^{\text{ban}} \|$.

Since $L_p[0,1]$ with $0 < p < 1$ supports only the functional o, the Banach envelope of $L_p[0,1]$ is trivial. To exclude such uninteresting cases, we assume that X carries sufficiently many functionals. In other words, X should have a separating dual X^*:

For every $x \neq \mathsf{o}$ in X there exists an x^* in X^* such that $\langle x^*, x\rangle \neq 0$.

The *separation property* is equivalent to the fact that X admits an embedding into a Banach space Y. In particular, we may use the canonical map $K : X \to X^{**}$. Then the contraction $E : X \to X^{\text{ban}}$ (appearing in the diagram on the preceding page) is an embedding as well.

The concept of the Banach envelope was independently introduced by Peetre [18, p. 125] and Shapiro [26, p. 116]. The latter considered only the case that X has a separating dual. Further information can be found in [10, pp. 27–28].

The rest of this section is concerned with a 'philosophical' question.

Let $J : X \to Y$ be an embedding of a normed linear space X into a Banach space Y, which means that X can be viewed as a linear subspace of Y. If the unique continuous extension $\widetilde{J} : \widetilde{X} \to Y$ is one-to-one as well, then we may say that the character of the members of X is preserved in passing to the completion $\widetilde{X}$. For instance, we think of the case that the members of Y are sequences, functions, or operators.

Unfortunately, as shown by a trivial example, $\widetilde{J}$ need not be one-to-one. Note that

$$\|x\| := \Big(\Big| \sum_{k=1}^{\infty} \xi_k \Big|^2 + \sum_{k=1}^{\infty} |\xi_k|^2 \Big)^{1/2}$$

is a norm on c_{00}, the linear space of all finite sequences $x = (\xi_k)$. Let J be the embedding of c_{00} into l_∞, and consider the sequences

$$x_n := (\overbrace{\tfrac{1}{n}, \dots, \tfrac{1}{n}}^{n\text{-times}}, 0, \dots).$$

Then

$$\|x_m - x_n\| = \sqrt{\tfrac{1}{m} - \tfrac{1}{n}} \quad \text{for } m < n \quad \text{and} \quad \|x_n\| = \sqrt{1 + \tfrac{1}{n}}.$$

In view of

$$\lim_{n\to\infty} \|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|Jx_n | l_\infty\| = \lim_{n\to\infty} \tfrac{1}{n} = 0,$$

the (abstract!) element $\tilde{x} \in \widetilde{c}_{00}$ generated by the Cauchy sequence (x_n) is different from o, while $\widetilde{J}\tilde{x} = \mathsf{o}$. This means that the complete hull $\widetilde{c}_{00}$ cannot be *canonically* identified with a sequence space.

The most famous example occurs in the theory of operator ideals and tensor products. Let $\mathfrak{F}(X) = X^* \otimes X$ denote the ideal of finite rank operators on a Banach space X, and define the norm

$$\nu^\circ(T) := \inf \sum_{k=1}^{n} \|x_k^*\| \|x_k\|,$$

the infimum being taken over all *finite* representations

$$Tx = \sum_{k=1}^{n} \langle x, x_k^* \rangle x_k \quad \text{for } x \in X,$$

where $x_1^*, \ldots, x_n^* \in X^*$ and $x_1, \ldots, x_n \in X$. The completion of $\mathfrak{F}(X)$ with respect to ν° can be identified with the projective tensor product $X^* \widehat{\otimes} X$.

We know from Grothendieck [8, chap. I, p. 165] that the continuous extension $\widetilde{J}$ of the embedding $J : \mathfrak{F}(X) \to \mathfrak{L}(X)$ is one-to-one if and only if X has the *approximation property*; see also [21, p. 280]. Another necessary and sufficient condition requires that ν° coincides with the nuclear norm

$$\nu(T) := \inf \sum_{k=1}^{\infty} \|x_k^*\| \|x_k\|.$$

In the latter case the infimum ranges over all *infinite* representations

$$Tx = \sum_{k=1}^{\infty} \langle x, x_k^* \rangle x_k \quad \text{for } x \in X,$$

where $x_1^*, x_2^*, \ldots \in X^*$ and $x_1, x_2, \ldots \in X$.

4. Old and new results, open problems

For $0 < p < 1$, the Banach envelope of $l_{p,\infty}$ is just l_1; see [10, p. 28]. If $1 < p < \infty$, then $l_{p,\infty}$ admits an equivalent norm, which means that $l_{p,\infty}^{\text{ban}}$ coincides *isomorphically* with $l_{p,\infty}$. Thus $p = 1$ is the only non-trivial case. In fact, this case seems to be highly sophisticated, and *I am unable to give a final solution.*

First of all, we observe that $l_{1,\infty}$ and $l_{1,\infty}^\circ$ are separated by their duals. Since the bidual of $l_{1,\infty}^\circ$ can be identified with $m_{1,\infty}$, the situation turns out to be especially nice for the little Marcinkiewicz space whose Banach envelope is just $m_{1,\infty}^\circ$. This observation is implicitly contained in papers of Markus [15, pp. 95, 108] and Mityagin [17, p. 823]; see also [20, p. 186]. Not aware of the work of the Soviet mathematicians, Ariño [1] rediscovered this result 25 years later.

It is hoped that the diagram

$$\begin{tikzcd}
l_{1,\infty}^{**} \arrow[rrrr, "J_m^{**}"] & & & & m_{1,\infty}^{**} \\
& l_{1,\infty}^{\mathrm{ban}} \arrow[ul] \arrow[rr, "J_{m^\bullet}^{\mathrm{ban}}"] \arrow[dr, "J_b^{\mathrm{ban}}"'] & & m_{1,\infty}^{\bullet} \arrow[ur] \arrow[ddr] & \\
& & b_{1,\infty} \arrow[ur, "J"'] \arrow[drr] & & \\
l_{1,\infty} \arrow[uuu] \arrow[uur] \arrow[urr] \arrow[rrrr, "J_m"] & & & & m_{1,\infty} \arrow[uuu]
\end{tikzcd}$$

helps to understand the complications that occur in the case of $l_{1,\infty}^{\mathrm{ban}}$. Almost all arrows denote canonical embeddings. Possible exceptions could be

$$J_b^{\mathrm{ban}} : l_{1,\infty}^{\mathrm{ban}} \to b_{1,\infty}, \quad J_{m^\bullet}^{\mathrm{ban}} : l_{1,\infty}^{\mathrm{ban}} \to m_{1,\infty}^{\bullet}, \quad \text{and} \quad J_m^{**} : l_{1,\infty}^{**} \to m_{1,\infty}^{**}.$$

These maps are the continuous extensions and the bidual of the canonical embeddings

$$J_b : l_{1,\infty} \to b_{1,\infty}, \qquad J_{m^\bullet} : l_{1,\infty} \to m_{1,\infty}^{\bullet}, \quad \text{and} \quad J_m : l_{1,\infty} \to m_{1,\infty},$$

respectively.

As shown in section 6, the norms $\| \cdot | l_{1,\infty}^{\mathrm{ban}} \|$ and $\| \cdot | m_{1,\infty} \|$ are not equivalent on $l_{1,\infty}$. Therefore $J_{m^\bullet}^{\mathrm{ban}}$ fails to be an isomorphism. This implies that at least one of the conjecture below must be true.

Conjecture 1. *The norms $\| \cdot | l_{1,\infty}^{\mathrm{ban}} \|$ and $\| \cdot | b_{1,\infty} \|$ are not equivalent on $l_{1,\infty}$. In other words, the map $J_b^{\mathrm{ban}} : l_{1,\infty}^{\mathrm{ban}} \to b_{1,\infty}$ is onto but not one-to-one.*

Conjecture 2. *The norms $\| \cdot | b_{1,\infty} \|$ and $\| \cdot | m_{1,\infty} \|$ are not equivalent on $l_{1,\infty}$. In other words, the map $J : b_{1,\infty} \to m_{1,\infty}^{\bullet}$ is one-to-one but not onto.*

Recall that

$$\|x | l_{1,\infty}^{\mathrm{ban}} \| := \inf \sum_{k=1}^{n} |||x_k | l_{1,\infty}||| \quad \text{and} \quad \|x | b_{1,\infty} \| := \inf \sum_{k=1}^{\infty} |||x_k | l_{1,\infty}|||,$$

where the infima range over all *finite* and *infinite* representations

$$x = \sum_{k=1}^{n} x_k \;\; (n = 1, 2, \dots) \quad \text{and} \quad x = \sum_{k=1}^{\infty} x_k \;\; \text{(coordinatewise)},$$

respectively. Comparing these definitions with those of $\nu^\circ(T)$ and $\nu(T)$ shows that Conjecture 1 is not unimaginable.

In summary, we notice a surprising (formal?) analogy:

$l_{1,\infty}^{\mathrm{ban}}$	$\leftrightarrow$	$X^* \widehat{\otimes} X$	projective tensor product,
$b_{1,\infty}$	$\leftrightarrow$	$\mathfrak{N}(X)$	ideal of nuclear operators,
$m_{1,\infty}$	$\leftrightarrow$	$\mathfrak{I}(X)$	ideal of integral operators.

5. The Kalton-Sukochev norm

Explicit expressions for the norms $\big\| \cdot |L_{1,\infty}[0,1]^{\rm ban}\big\|$ and $\big\| \cdot |L_{1,\infty}[0,\infty)^{\rm ban}\big\|$ were discovered by Cwikel/Fefferman [3, p. 150] and [4, p. 277]; see also Kupka/Peck [12, p. 237]. Unfortunately, their approach was strictly limited to the *non-atomic* case. This gap could be closed only recently by Kalton/Sukochev [11].

Following the latter authors, for every bounded sequence $x=(\xi_k)$ and $m=1,2,\dots$, we define the finite or infinite quantity

$$\beta_m(x) := \sup_{ma<n} \frac{\sum\limits_{k=ma+1}^{n} s_k(x)}{\sum\limits_{k=a+1}^{n} \frac{1}{k}},$$

where the supremum ranges over all $a = 0,1,2,\dots$ and $n = 1,2,\dots$ such that $ma < n$.

Obviously,

$$|||x|l_{1,\infty}||| = \beta_1(x) \geq \beta_2(x) \geq \dots \quad \text{whenever} \quad x\in l_{1,\infty}.$$

On the other hand, if $\beta_m(x)$ is finite for some m, then $x\in l_{1,\infty}$. Indeed, by A.1, we get

$$ma\, s_{2ma}(x) \leq \sum_{k=ma+1}^{2ma} s_k(x) \leq \beta_m(x) \sum_{k=a+1}^{2ma} \tfrac{1}{k} \leq (\log 2m)\, \beta_m(x) \quad \text{for } a=1,2,\dots .$$

Moreover,

$$ns_n(x) \leq \sum_{k=1}^{n} s_k(x) \leq \beta_m(x) \sum_{k=1}^{n} \tfrac{1}{k} \leq (1+\log 2m)\, \beta_m(x) \quad \text{for} \quad n \leq 2m.$$

Thus

$$|||x|l_{1,\infty}||| = \sup_{1\leq k<\infty} ks_k(x) \leq (4\log 2m)\, \beta_m(x).$$

The results presented in the remaining part of this section are essentially borrowed from Kalton/Sukochev [11]. For the convenience of the reader, I have added the elementary proofs.

Sublemma. $\sum\limits_{k=a+1}^{n} s_k(x) = \inf \Big\{ \sum\limits_{h=1}^{n-a} s_h(x-u) \,:\, |\mathbb{S}(u)| \leq a \Big\}$ *for* $x\in c_0$, $a=0,1,2,\dots$ *and* $n=1,2,\dots$ *such that* $a < n$. *The infimum is attained.*

Proof. If $|\mathbb{S}(u)| \leq a$, then $s_{a+1}(u)=0$. Hence

$$\sum_{k=a+1}^{n} s_k(x) = \sum_{h=1}^{n-a} s_{a+h}(x) \leq \sum_{h=1}^{n-a} \big[s_{a+1}(u)+s_h(x-u)\big] = \sum_{h=1}^{n-a} s_h(x-u),$$

which implies that

$$\sum_{k=a+1}^{n} s_k(x) \leq \inf \Big\{ \sum_{h=1}^{n-a} s_h(x-u) \,:\, |\mathbb{S}(u)| \leq a \Big\}.$$

In order to get equality, we may assume that $|\xi_1|\geq|\xi_2|\geq\dots$. Putting

$$u := (\xi_1,\dots,\xi_a,\ 0\ ,\dots,\ 0\ ,\dots)$$

yields

$$x-u = (\ 0\ ,\dots,\ 0\ ,\xi_{a+1},\dots,\xi_n,\dots).$$

Thus $s_h(x-u) = |\xi_{a+h}|=s_{a+h}(x)$. □

Lemma. *If $a,b=0,1,2,\dots$ and $n=1,2,\dots$ such that $a+b<n$, then*

$$\sum_{k=a+b+1}^{n} s_k(x+y) \leq \sum_{k=a+1}^{n} s_k(x) + \sum_{k=b+1}^{n} s_k(y) \quad \text{for} \quad x,y\in c_0.$$

Proof. By the preceding sublemma, we may choose $u,v\in c_{00}$ such that

$$\sum_{k=a+1}^{n} s_k(x) = \sum_{h=1}^{n-a} s_h(x-u) \quad \text{and} \quad |\mathbb{S}(u)| \leq a,$$

$$\sum_{k=b+1}^{n} s_k(y) = \sum_{h=1}^{n-b} s_h(y-v) \quad \text{and} \quad |\mathbb{S}(v)| \leq b.$$

Then $|\mathbb{S}(u+v)| \leq |\mathbb{S}(u)|+|\mathbb{S}(v)| \leq a+b$ and, therefore,

$$\begin{aligned}\sum_{k=a+b+1}^{n} s_k(x+y) &\leq \sum_{h=1}^{n-a-b} s_h(x+y-u-v) \leq \sum_{h=1}^{n-a-b} s_h(x-u) + \sum_{h=1}^{n-a-b} s_h(y-v)\\ &\leq \sum_{h=1}^{n-a} s_h(x-u) + \sum_{h=1}^{n-b} s_h(y-v) = \sum_{k=a+1}^{n} s_k(x) + \sum_{k=b+1}^{n} s_k(y).\end{aligned}$$ □

Proposition 1. *$\beta(x):=\lim\limits_{m\to\infty}\beta_m(x)$ defines a norm on $l_{1,\infty}$.*

Proof. We conclude from

$$\sum_{k=2ma+1}^{n} s_k(x+y) \leq \sum_{k=ma+1}^{n} s_k(x) + \sum_{k=ma+1}^{n} s_k(y) \quad \text{for} \quad x,y\in c_0$$

that

$$\beta_{2m}(x+y) \leq \beta_m(x) + \beta_m(y) \quad \text{for} \quad x,y\in c_0,$$

which implies the triangle inequality, $\beta(x+y) \leq \beta(x)+\beta(y)$. □

Proposition 2. *$\beta(x) \geq \|x|m_{1,\infty}\|$ for $x\in l_{1,\infty}$.*

Proof. Letting $a=0$ in the definition of $\beta_m(x)$, we obtain

$$\beta_m(x) \geq \sup_{1\leq n<\infty} \frac{\sum_{k=1}^{n} s_k(x)}{\sum_{k=1}^{n} \frac{1}{k}} = \|x|m_{1,\infty}\|.$$ □

Proposition 3. $\beta(x) = \|x|m_{1,\infty}\|$ *for* $x \in l^{\circ}_{1,\infty}$.

Proof. Given any finite sequence x, we let $m_0 := |\mathbb{S}(x)|$. Then it follows from $s_{m_0+1}(x) = 0$ that

$$\sum_{k=ma+1}^{n} s_k(x) = 0 \quad \text{whenever } n > ma \geq m_0.$$

Thus

$$\beta_m(x) = \sup_{1 \leq n < \infty} \frac{\sum_{k=1}^{n} s_k(x)}{\sum_{k=1}^{n} \frac{1}{k}} = \|x|m_{1,\infty}\| \quad \text{if} \quad m \geq m_0,$$

which proves the required equality for $x \in c_{00}$. Since $\beta(x)$ and $\|x|m_{1,\infty}\|$ are continuous with respect to quasi-norm $|||\cdot|l_{1,\infty}|||$, the result extents to all $x \in l^{\circ}_{1,\infty}$. $\square$

A *theorem of Kalton-Sukochev* says that

$$\beta(x) = \|x|l^{\text{ban}}_{1,\infty}\| \quad \text{for} \quad x \in l_{1,\infty}.$$

The proof of this fundamental achievement is rather cumbersome. Luckily, we need only its elementary part:

Proposition 4. $\beta(x) \leq \|x|l^{\text{ban}}_{1,\infty}\|$ *for* $x \in l_{1,\infty}$.

Proof. In view of $\beta(x) \leq |||x|l_{1,\infty}|||$ for $x \in l_{1,\infty}$, the inequality follows from the fact that $\|\cdot|l^{\text{ban}}_{1,\infty}\|$ is the greatest norm on $l_{1,\infty}$ majorized by $|||\cdot|l_{1,\infty}|||$. $\square$

6. Tricky sequences

The following theorem, which is the main result of this paper, belongs to the *'frustrating'* part of Banach space theory.

Theorem. *The norms* $\|\cdot|m_{1,\infty}\|$ *and* $\|\cdot|l^{\text{ban}}_{1,\infty}\|$ *fail to be equivalent on* $l_{1,\infty}$.

Proof. Obviously, $\|x|m_{1,\infty}\| \leq \|x|l^{\text{ban}}_{1,\infty}\|$ for $x \in l_{1,\infty}$. To see that there holds no converse estimate $\|x|l^{\text{ban}}_{1,\infty}\| \leq c\|x|m_{1,\infty}\|$, a family of *'tricky'* sequences will be constructed. The symbols A.1, … ,A.4 refer to some elementary facts from calculus that are summarized in the appendix.

(A) Given natural numbers $i_1, i_2, \dots$ such that

$$2^h i_h \leq i_{h+1}, \tag{1}$$

we define $n_0 := 0$,

$$n_h := \sum_{m=1}^{h} (2^m - 1) i_m \quad \text{for } h = 1, 2, \dots,$$

and

$$n_{h,p} := n_{h-1} + (2^p - 1) i_h \quad \text{for } p = 0, \dots, h. \tag{2}$$

Then

$$n_{h-1} = n_{h,0} < n_{h,1} < \cdots < n_{h,h-1} < n_{h,h} = n_h$$

and

$$n_h = n_{h-1} + (2^h - 1)i_h.$$

It follows by induction that

$$n_h \le 2^h i_h - i_1. \tag{3}$$

Indeed, the case $h=1$ is trivial, since $n_1 = i_1$. Assume that $n_{h-1} \le 2^{h-1} i_{h-1} - i_1$ is true for some $h \ge 2$. Using (1), we obtain

$$n_h = n_{h-1} + (2^h - 1)i_h \le 2^{h-1} i_{h-1} - i_1 + (2^h - 1)i_h \le i_h + (2^h - 1)i_h - i_1 = 2^h i_h - i_1.$$

Combining (1) and (3) yields

$$n_h \le i_{h+1} - i_1. \tag{4}$$

In view of (2), we get

$$n_{h,p} \le 2^p i_h - i_1. \tag{5}$$

(B) Let

$$\mathbb{N}_{h,p} := \{n : n_{h,p-1} < n \le n_{h,p}\} \quad \text{for } p = 1, \ldots, h$$

and

$$\mathbb{N}_h := \bigcup_{p=1}^{h} \mathbb{N}_{h,p} = \{n : n_{h-1} < n \le n_h\} \quad \text{for } h = 1, 2, \ldots\ .$$

Note that $|\mathbb{N}_{h,p}| = 2^{p-1} i_h$ and $|\mathbb{N}_h| = (2^h - 1)i_h$. Consequently, the size of the sets $\mathbb{N}_h$ and $\mathbb{N}_{h,p}$ increases rapidly:

$$|\mathbb{N}_{h+1}| > 2^{h+1}|\mathbb{N}_h| \quad \text{and} \quad |\mathbb{N}_{h,p+1}| = 2|\mathbb{N}_{h,p}|.$$

(C) Define the sequence $x = (\xi_n)$ by

$$\xi_n := \tfrac{1}{2^{p-1} i_h} = \tfrac{1}{|\mathbb{N}_{h,p}|} \quad \text{for} \quad n \in \mathbb{N}_{h,p}.$$

The condition (1) guarantees that (ξ_n) does not increase:

$$\xi_{n_h} = \tfrac{1}{2^{h-1} i_h} \quad \text{is greater than} \quad \xi_{n_h+1} = \tfrac{1}{i_{h+1}}.$$

Thus $s_n(x) = \xi_n$. We conclude from (5) that

$$n s_n(x) \le n_{h,p} \tfrac{1}{2^{p-1} i_h} < 2 \quad \text{for} \quad n \in \mathbb{N}_{h,p}.$$

Hence $x \in l_{1,\infty}$.

(D) In the next step, the norm $\beta(x)$ will be estimated from below.
Obviously,

$$\sum_{k\in\mathbb{N}_{h,p}} \xi_k = 1.$$

Because of (2) and (3), we have $(2^q-1)i_h \le n_{h,q}$ and $n_h < 2^h i_h$. Therefore

$$\sum_{k=(2^q-1)i_h+1}^{2^h i_h} s_k(x) \ \ge \sum_{k=n_{h,q}+1}^{n_h} s_k(x) = \sum_{p=q+1}^{h} \sum_{k\in\mathbb{N}_{h,p}} \xi_k = h-q.$$

Moreover, by A.1,

$$\sum_{k=i_h+1}^{2^h i_h} \tfrac{1}{k} \le \int_{i_h}^{2^h i_h} \tfrac{1}{t}\, dt = \log 2^h i_h - \log i_h = h \log 2.$$

Hence

$$\beta_{2^q-1}(x) \ge \frac{\displaystyle\sum_{k=(2^q-1)i_h+1}^{2^h i_h} s_k(x)}{\displaystyle\sum_{k=i_h+1}^{2^h i_h} \tfrac{1}{k}} \ge \tfrac{h-q}{h\log 2}.$$

If $h{:=}qr$, then

$$\beta_{2^q-1}(x) \ge \tfrac{r-1}{r\log 2}.$$

Letting $q\to\infty$ yields

$$\beta(x) \ge \tfrac{r-1}{r\log 2}.$$

Consequently, $r\to\infty$ gives

$$\beta(x) \ge \tfrac{1}{\log 2}. \tag{6}$$

(E) To estimate $\|x|m_{1,\infty}\|$ from above, we use the quantity

$$\lambda_n(x) := \tfrac{1}{1+\log n} \sum_{k=1}^{n} s_k(x)$$

instead of

$$\mu_n(x) := \tfrac{1}{L_n} \sum_{k=1}^{n} s_k(x) \quad \text{with} \quad L_n := \sum_{k=1}^{n} \tfrac{1}{k}.$$

By A.1,

$$\|x|m_{1,\infty}\|_{\log} := \sup_{1\le n<\infty} \lambda_n(x)$$

defines an equivalent norm: $\|x|m_{1,\infty}\|_{\log} \le \|x|m_{1,\infty}\| \le \frac{3}{2}\|x|m_{1,\infty}\|_{\log}$.

(F) For the special indexes n_h, we get

$$\lambda_{n_h}(x) = \frac{\sum_{m=1}^{h} \sum_{k\in\mathbb{N}_m} \xi_k}{1+\log n_h} = \frac{\frac{1}{2}h(h+1)}{1+\log n_h}.$$

Thus, given ε with $0<\varepsilon\leq 1$, we can successively find $i_1, i_2, \dots$ such that

$$2^h i_h \leq i_{h+1} \quad \text{and} \quad \lambda_{n_h}(x) \leq \tfrac{1}{\log 2}\,\varepsilon.$$

In the rest of this section, it will be shown that $\lambda_n(x)\leq \frac{1}{\log 2}\,\varepsilon$ for *all* n's.

(G) As an intermediate step, we treat the indexes $n_{h,p}$ with $p=1,\dots,h-1$ and $h\geq 2$. In this case,

$$\lambda_{n_{h,p}}(x) = \frac{\sum_{m=1}^{h-1} \sum_{k\in\mathbb{N}_m} \xi_k + \sum_{q=1}^{p} \sum_{k\in\mathbb{N}_{h,q}} \xi_k}{1+\log n_{h,p}} = \frac{\frac{1}{2}(h-1)h+p}{1+\log n_{h,p}}.$$

Let

$$F_h(t) := \frac{a_h + \log[t+b_h]}{1+\log t},$$

where

$$a_h := \tfrac{1}{2}(h-1)h\log 2 - \log i_h \quad \text{and} \quad b_h := i_h - n_{h-1}.$$

We deduce from (1) that $n_{h,p}+b_h = 2^p i_h$. Therefore

$$a_h + \log[n_{h,p}+b_h] = \left[\tfrac{1}{2}(h-1)h+p\right]\log 2,$$

which in turn yields

$$\lambda_{n_{h,p}}(x) = \tfrac{1}{\log 2}\,F_h(n_{h,p}).$$

In particular,

$$\lambda_{n_{h-1}}(x) = \tfrac{1}{\log 2}\,F_h(n_{h-1}) \quad \text{and} \quad \lambda_{n_h}(x) = \tfrac{1}{\log 2}\,F_h(n_h).$$

We infer from (4) that

$$b_h = i_h - n_{h-1} \geq i_1 \geq 1.$$

According to the choice of the i_h's, we have $F_h(n_{h-1})\leq\varepsilon\leq 1$. Thus, with $t_0 = n_{h-1}$, all conditions posed in A.2 are satisfied, and by A.4 we get

$$\lambda_{n_{h,p}}(x) = \tfrac{1}{\log 2}\,F_h(n_{h,p}) \leq \tfrac{1}{\log 2}\max\{F_h(n_{h-1}), F_h(n_h)\} \leq \tfrac{1}{\log 2}\,\varepsilon.$$

(H) What remains is to consider those $\lambda_n(x)$'s whose index n differs from the $n_{h,p}$'s. First of all, we observe that

$$\lambda_n(x) = \tfrac{1}{i_1}\,\tfrac{n}{1+\log n} < \tfrac{1}{1+\log i_1} = \lambda_{n_1}(x) \le \tfrac{1}{\log 2}\,\varepsilon \quad \text{for} \quad 1 \le n < n_{1,1} = n_1 = i_1.$$

From now on, let $n_{h,p-1} < n < n_{h,p}$ with $p = 1, \dots, h$ and $h \ge 2$. Then

$$\lambda_n(x) = \frac{\frac{1}{2}(h-1)h + p - 1 + \frac{n - n_{h,p-1}}{2^{p-1} i_h}}{1 + \log n} = \frac{1}{2^{p-1} i_h}\,\frac{c_{h,p} + n}{1 + \log n},$$

where

$$c_{h,p} := 2^{p-1} i_h [\tfrac{1}{2}(h-1)h + p - 1] - n_{h,p-1}.$$

Because of (2) and (4), we obtain

$$c_{h,p} = 2^{p-1} i_h [\tfrac{1}{2}(h-1)h + p - 2] + i_h - n_{h-1} \ge i_1 \ge 1.$$

Thus the observations stated in A.3 and A.4 tell us that

$$\lambda_n(x) \le \max\{\lambda_{n_{h,p-1}}(x), \lambda_{n_{h,p}}(x)\} \le \tfrac{1}{\log 2}\,\varepsilon.$$

Consequently,

$$\|x|m_{1,\infty}\|_{\log} = \sup_{1 \le n < \infty} \lambda_n(x) \le \tfrac{1}{\log 2}\,\varepsilon. \tag{7}$$

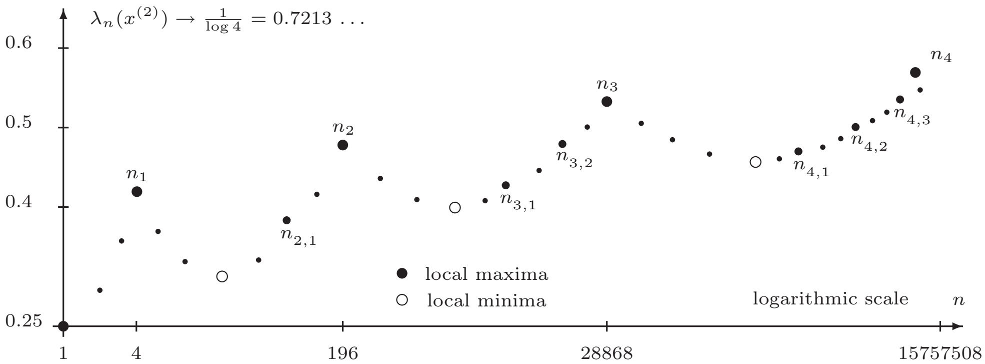
Finally, we infer from (6) and (7) that $\|x|l_{1,\infty}^{\mathrm{ban}}\| \le c\|x|m_{1,\infty}\|_{\log}$ cannot hold for all $x \in l_{1,\infty}$ uniformly. $\square$

Remarks. Let $x^{(N)}$ be the sequence associated with $i_h := 2^{\frac{1}{2}h(h+1)N}$, where N is any natural number. Then, it follows from $(2^h - 1) i_h < n_h < 2^h i_h$ that

$$\lim_{h\to\infty} \lambda_{n_h}(x^{(N)}) = \sup_{1 \le h < \infty} \lambda_{n_h}(x^{(N)}) = \tfrac{1}{N \log 2}.$$

Moreover, the sequence $\big(\lambda_{n_h}(x^{(N)})\big)$ is increasing as $h \to \infty$.

The following diagram indicates the behavior of $\lambda_n(x^{(2)})$:



The sequence $\big(\lambda_n(x^{(2)})\big)$ has *one* local minimum on every interval $\mathbb{N}_{h,1}$. Curiously enough, beginning with $h = 6$, there is also *one* local minimum on every interval $\mathbb{N}_{h,2}$. However, this phenomenon appears only for $N = 2$. In other words, if $N > 2$, then $\big(\lambda_n(x^{(N)})\big)$ has no other local maxima than those attained at the n_h's.

7. Appendix: Some elementary facts from calculus

A.1 We have

$$\sum_{a+1}^{b} \tfrac{1}{k} < \int_a^b \tfrac{1}{t}\,dt < \sum_{a}^{b-1} \tfrac{1}{k}.$$

The asymptotic behavior of the quantity $L_n := \sum_{k=1}^{n} \frac{1}{k}$ is described by

$$\tfrac{2}{3}(1+\log n) < \tfrac{1}{n} + \log n \le \sum_{k=1}^{n} \tfrac{1}{k} \le 1 + \log n.$$

A.2 Let

$$F(t) := \frac{a + \log[t+b]}{1+\log t} \quad \text{for} \quad t \ge 1.$$

If $b > 0$, $t_0 \ge 1$, and $F(t_0) \le 1$, then either $F(t)$ increases on $[t_0, \infty)$ or there exists some point $t_{\min} > t_0$ such that $F(t)$ decreases on $[t_0, t_{\min}]$ and increases on $[t_{\min}, \infty)$.

Proof. If $t \ge 1$, then the sign of

$$F'(t) = \frac{t(1+\log t) - (t+b)(a+\log[t+b])}{t(t+b)(1+\log t)^2}$$

is determined by that of the nominator

$$N(t) := t(1+\log t) - (t+b)(a+\log[t+b]).$$

Regarding $N(t)$ as a function on $(0, \infty)$, we have

$$N'(t) = 1 - a + \log t - \log[t+b] = 1 - a - \log[1+\tfrac{b}{t}] \quad \text{and} \quad N''(t) = \tfrac{1}{t} - \tfrac{1}{t+b} > 0.$$

The assumption $F(t_0) \le 1$ implies that $a + \log[t_0 + b] \le 1 + \log t_0$. Hence $a < 1$. Let $s > 0$ denote the unique zero of the increasing function $N'(t)$. Then $N(t)$ is increasing on $[s, \infty)$. In view of $F(t_0) \le 1$, we obtain

$$\log[1+\tfrac{b}{t_0}] \le 1 - a = \log[1+\tfrac{b}{s}].$$

Thus $s \le t_0$, and $N(t)$ is increasing on $[t_0, \infty)$. If $N(t_0) \ge 0$, it follows that $F(t)$ is increasing on $[t_0, \infty)$. In the case that $N(t_0) < 0$, there exists a point $t_{\min} > t_0$ at which $N(t)$ changes from $-$ to $+$, which means that $F(t)$ decreases on $[t_0, t_{\min}]$ and increases on $[t_{\min}, \infty)$. $\square$

A.3 Let

$$G(t) := \frac{c+t}{1+\log t} \quad \text{for} \quad t \ge 1.$$

If $c \ge 0$, then the function $F(t)$ decreases on $[1, t_{\min}]$ and increases on $[t_{\min}, \infty)$, where $t_{\min} \ge 1$ is the unique solution of the equation $t \log t = c$.

A.4 If a function H decreases on $[t_0, t_{\min}]$ and increases on $[t_{\min}, \infty)$, then

$$H(t) \le \max\{H(t_1), H(t_2)\} \quad \text{if} \quad t_0 \le t_1 < t < t_2 < \infty.$$

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