Existence of Renormalized Solution of Some Elliptic Problems in Orlicz Spaces

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ABSTRACT

In this paper, we study the problem

$$-\operatorname{div} a(x,u,\nabla u) - \operatorname{div} \phi(u) + g(x,u) = f$$

in the framework of Orlicz spaces. The main contribution of our work is to prove the existence of a renormalized solution without any restriction on the N-function of the Orlicz space.

Key words: Orlicz Sobolev spaces, boundary value problems, truncations, renormalized solutions.

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Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $Au = -\operatorname{div} a(x,u,\nabla u)$ be a Leray-Lions operator defined in $W_0^{1,p}(\Omega), 1 < p < \infty$.

We consider the following nonlinear elliptic problem:

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) - \operatorname{div} \phi(u) + g(x,u) = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

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where $f \in W^{-1,p'}(\Omega)$, $\phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$, and g is a Carathéodory function satisfying

$$\sup_{|t|\leq n} |g(\cdot, s)| = h_n(\cdot) \in L^1(\Omega) \quad \forall n.$$

Note that no growth hypothesis is assumed on the function ϕ which implies that the term $\operatorname{div} \phi(u)$ may be meaningless, even as a distribution. The notion of renormalized solution (see definition 2.1) gives a meaning to a possible solution of (1).

In the case where $\phi = 0$, existence of a weak solution in the usual sense to (1) is proved by Rakotoson and Temam [16].

The notion of renormalized solutions in the usual case was introduced by R. J. DiPerna and P.-L. Lions [10] for the study of the Boltzmann equations. This notion was then adapted to the study of the problem (1) by L. Boccardo et al. [8] when the right hand side is in $W^{-1,p'}(\Omega)$, by J. M. Rakotoson [15] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso et al. [9] for the case in which the right hand side is general measure data.

The functional setting in these works is the usual Sobolev space $W^{1,p}(\Omega)$. Accordingly the function $a(\cdot)$ is supposed to satisfy polynomial growth conditions with respect to u and its derivatives ∇u.

When trying to perform an analysis for the function $a(\cdot)$ with more general growth conditions, one is led to replace $W^{1,p}$ by a Sobolev-space $W^1 L_M$ built from an Orlicz space L_M instead of L^p. Here the N-function M which defines L_M is related to the actual growth of the function a

Recently Benkirane and Bennouna [5] have generalized the last result of Boccardo et al. [8] to the Orlicz-Sobolev space with some restrictions on the N-function (i.e., the Δ_2-condition).

It is our purpose, in this paper, to prove the existence of renormalized solution for the problem (1) in the setting of the Orlicz Sobolev space $W_0^1 L_M(\Omega)$ without any restriction on the N-function M. (i.e., without the Δ_2-condition). See theorem 2.3. This paper is organized as follows: Section 1 contains some preliminaries and some technical lemmas concerning convergence in Orlicz Sobolev space. In section 2, we state our main result which will be proved in section 3. The proof uses techniques different from that given in [5, 8].

For some existence results for strongly non-linear elliptic equation in Orlicz space see [2–4, 6]

1. Preliminaries

1.1. N-function

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation $M(t) = \int_0^t a(s)\, ds$, where $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing, right continuous function, with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$, conjugate to M, is defined by $\overline{M}(t) = \int_0^t \bar{a}(s)\,ds$, where $\bar{a}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{\, s : a(s) \le t\,\}$.

The N-function M is said to satisfy the Δ_2-condition if for some k

$$M(2t) \le kM(t) \quad \forall t \ge 0. \tag{2}$$

It is readily seen that this will be the case if and only if for every $r > 0$ there exists a positive constant $k = k(r)$ such that for all $t > 0$

$$M(rt) \le kM(t) \quad \forall t \ge 0. \tag{3}$$

When (2) and (3) hold only for $t \ge t_0$, for some $t_0 > 0$, then M is said to satisfy the Δ_2-condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$. Moreover, we have the following Young's inequality:

$$\forall s, t \ge 0, \quad st \le M(t) + \overline{M}(s).$$

Let P and Q be two N-functions. We say that P grows essentially less rapidly than Q near infinity, and denote it $P \ll Q$, if for every $\varepsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$ (see [1, chapter 8]).

1.2. Orlicz space $L_M(\Omega)$

Let M be an N-function and $\Omega \subset \mathbb{R}^N$ be an open and bounded set. The Orlicz class $\mathcal{K}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))\,dx < +\infty \quad (\text{resp. } \int_\Omega M(\frac{u(x)}{\lambda})\,dx < +\infty \text{ for some } \lambda > 0).$$

$L_M(\Omega)$ is a Banach space under the norm,

$$\|u\|_{M,\Omega} = \inf\Big\{\, \lambda > 0 : \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\,dx \le 1\,\Big\}$$

and $\mathcal{K}_M(\Omega)$ is a convex subset of $L_M(\Omega)$ but not necessarily a linear space.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The dual space of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

Let X and Y be arbitrary Banach spaces with bilinear bicontinuous pairing $\langle , \rangle_{X,Y}$.

We say that a sequence $\{u_n\} \subset X$ converges to $u \in X$ with respect to the topology $\sigma(X,Y)$, denoted by $u_n \to u$ $(\sigma(X,Y))$, in X, if $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v \in Y$. For example, if $X = L_M(\Omega)$ and $Y = L_{\overline{M}}(\Omega)$, then the pairing is defined by

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx \qquad \forall u \in X, v \in Y.$$

1.3. Orlicz-Sobolev space

We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lies in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\} \subset L_M(\Omega)$ converges to $u \in L_M(\Omega)$ in the modular sense, denoted $u_n \to u$ (mod) in $L_M(\Omega)$ if for some $\lambda > 0$

$$\int_\Omega M\left(\frac{|u_n(x) - u(x)|}{\lambda}\right) dx \longrightarrow 0 \quad \text{when} \quad n \to +\infty.$$

We say that a sequence $\{u_n\} \subset W^1L_M(\Omega)$ converges to $u \in W^1L_M(\Omega)$ in the modular sense, denoted $u_n \to u$ (mod) in $W^1L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega M\left(\frac{|D^\alpha u_n(x) - D^\alpha u(x)|}{\lambda}\right) dx \longrightarrow 0 \quad \text{when} \quad n \to +\infty \quad \text{for all} \quad |\alpha| \leq 1.$$

If M satisfies the Δ_2-condition (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

1.4. Some lemmas

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

We recall some lemmas introduced in [7] which will be used later.

Lemma 1.1. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$(resp. $W^1E_M(\Omega)$). Moreover, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial}{\partial x_i} u & \text{a.e. in} \quad \{\, x \in \Omega : u(x) \notin D \,\}, \\ 0 & \text{a.e. in} \quad \{\, x \in \Omega : u(x) \in D \,\}, \end{cases}$$

where D is the set of discontinuity points of F'.

Lemma 1.2. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function, then the mapping $T_F\colon W^1L_M(\Omega) \to W^1L_M(\Omega)$ defined by $T_F(u) = F(u)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [7]).

Lemma 1.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P, and Q be N-functions such that $Q \ll P$, and let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$,*

$$|f(x,s)| \leq c(x) + k_1 P^{-1}M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

Below, we will use the following technical Lemmas.

Lemma 1.4 ([7])**.** *Let $(f_n), f, \gamma \in L^1(\Omega)$ such that*

(i) *$f_n \geq \gamma$ a.e. in Ω,*

(ii) *$f_n \to f$ a.e. in Ω,*

(iii) *$\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$.*

Then $f_n \to f$ strongly in $L^1(\Omega)$.

We now turn to the approximation by functions which are smooth up to the boundary, assuming some regularity on Ω. Recall that Ω is said to have the (interior) *segment property* if there exist an open covering $\{U_i\}$ of $\overline{\Omega}$ and corresponding vectors $\{y_i \in \mathbb{R}^N\}$ such that, for $x \in \overline{\Omega} \cap U_i$ and $0 < t < 1$, it is $x + ty_i \in \Omega$.

Lemma 1.5 ([12])**.** *Let Ω have the segment property. Then for each $\nu \in W_0^1L_M(\Omega)$, there exists a sequence $\nu_n \in \mathcal{D}(\Omega)$ such that ν_n converges to ν for the modular convergence in $W_0^1L_M(\Omega)$. Furthermore, if $\nu \in W_0^1L_M(\Omega) \cap L^\infty(\Omega)$ then*

$$\|\nu_n\| \leq (N+1)\|\nu\|_{L^\infty(\Omega)}.$$

2. Main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the segment property. Let $A\colon \mathcal{D}(A) \subset W_0^1L_M(\Omega) \to W^{-1}L_{\overline{M}}(\Omega)$ be a mapping given by $A(u) = -\operatorname{div} a(x,u,\nabla u)$, where a is a function satisfying the following conditions:

(A_1) $a(x,s,\xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

(A_2) There exist two N-functions M and P with $P \ll M$, a function $c(x)$ in $E_{\overline{M}}(\Omega)$, and positive constants k_1, k_2, k_3, k_4 such that

$$|a(x,s,\zeta)| \le c(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|),$$

for a.e. x in Ω and for all $s \in \mathbb{R}$, $\zeta \in \mathbb{R}^N$.

(A_3) For a.e. x in Ω, $s \in \mathbb{R}$ and ζ, ζ' in $\mathbb{R}^N$, with $\zeta' \neq \zeta$

$$[a(x,s,\zeta) - a(x,s,\zeta')](\zeta - \zeta') > 0.$$

(A_4) For a.e. x in Ω, $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$,

$$a(x,s,\zeta)\zeta \ge \alpha M\Big(\frac{|\zeta|}{\lambda}\Big)$$

where $\alpha \in \mathbb{R}_+^*$.

Consider the nonlinear elliptic problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) - \operatorname{div} \phi(u) + g(x,u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

where

$$f \in W^{-1}E_{\overline{M}}(\Omega), \tag{5}$$

and $\phi = (\phi_1, \dots, \phi_N)$ satisfy

$$\phi \in (C^0(\mathbb{R}))^N. \tag{6}$$

Let $g(x,t)$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$

$$g(x,s)\, s \ge 0, \tag{7}$$

$$\sup_{|t| \le n} |g(\cdot,s)| = h_n(\cdot) \in L^1(\Omega) \ \ \forall n. \tag{8}$$

Note that no growth hypothesis is assumed on the function ϕ, which implies that for a solution $u \in W_0^1 L_M(\Omega)$ the term $\operatorname{div} \phi(u)$ may be meaningless, even as a distribution. As in [8] we define the following notion of renormalized solution, which gives a meaning to a possible solution of (4).

Definition 2.1. Assume that (A_1)–(A_4), (5)–(8) hold true. A function u is a renormalized solution of the problem (4) if

$$\begin{cases} u \in W_0^1 L_M(\Omega), g(x,u) \in L^1(\Omega), \quad u\, g(x,u) \in L^1(\Omega) \\ -\operatorname{div} a(x,u,\nabla u)\, h(u) - \operatorname{div}(\phi(u)h(u)) + \phi(u)h'(u)\nabla u \\ \qquad +g(x,u)h(u) = fh(u) \text{ in } \mathcal{D}'(\Omega), \quad \forall h \in \mathcal{C}_c^1(\mathbb{R}^N). \end{cases} \tag{9}$$

The weaker problem (9) is obtained by using the test function $h(u)$ where $h \in \mathcal{C}_c^1(\mathbb{R})$ in (4).

Remark 2.2. Let us note that in (9) every term is meaningful in the distributional sense.

It's easy to see that for $\varphi \in \mathcal{D}(\Omega)$ and $u \in W_0^1 L_M(\Omega)$ we have $\varphi h(u) \in W_0^1 L_M(\Omega)$ (one can apply Lemma 1.2) and

$$\langle f h(u), \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle f, \varphi h(u) \rangle_{W^{-1}E_{\overline{M}}(\Omega), W_0^1 L_M(\Omega)}.$$

We have also $[-\operatorname{div} a(x,u,\nabla u)] \in W^{-1}L_{\overline{M}}(\Omega)$ and

$$\begin{aligned}\langle -\operatorname{div} a(x,u,\nabla u) h(u), \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ = \langle -\operatorname{div} a(x,u,\nabla u), \varphi h(u) \rangle_{W^{-1}L_{\overline{M}}(\Omega), W_0^1 L_M(\Omega)}.\end{aligned}$$

Finally since ϕh and $\phi h' \in (\mathcal{C}_c^0(\mathbb{R}))^N$ we have $\phi(u)h(u)$ and $\phi(u)h'(u) \in (L^\infty(\Omega))^N$, for any measurable function u and then

$$\operatorname{div}(\phi(u)h(u)) \in W^{-1,\infty}(\Omega), \qquad \phi(u)h'(u)\nabla u \in L_M(\Omega).$$

Theorem 2.3. *Under assumptions* (A_1)*–*(A_4)*,* (5)*–*(8)*, there exists a renormalized solution u (in the sense of definition* 2.1*) of problem* (4)*.*

3. Proof of the main result

We state and prove the following lemmas that will be used later

3.1. Some lemmas

Lemma 3.1. *Assume that* (A_1)*–*(A_4) *are satisfied, and let* (z_n) *be a sequence in* $W_0^1 L_M(\Omega)$ *such that*

(i) $z_n \rightharpoonup z$ *in* $W_0^1 L_M(\Omega)$ *for* $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$*;*

(ii) $(a(x,z_n,\nabla z_n))_n$ *is bounded in* $(L_{\overline{M}}(\Omega))^N$*;*

(iii) $\int_\Omega [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z \chi_s)][\nabla z_n - \nabla z \chi_s]\,dx \to 0$ *as* $n,s \to +\infty$ *(where* χ_s *is the characteristic function of* $\Omega_s = \{\, x \in \Omega, |\nabla z| \le s \,\}$*).*

Then

$$M\Big(\frac{|\nabla z_n|}{\lambda}\Big) \longrightarrow M\Big(\frac{|\nabla z|}{\lambda}\Big) \quad in \quad L^1(\Omega).$$

Remark 3.2. The condition (ii) is not necessary if the N-function M satisfies the Δ_2-condition.

Proof of Remark 3.2. The condition (i) implies that the sequence $(z_n)_n$ is bounded in $W_0^1 L_M(\Omega)$, hence there exists two positive constants λ, C such that

$$\int_\Omega M(\lambda|\nabla z_n|)\,dx \leq C. \tag{10}$$

On the other hand, let Q be an N-function such that $M \ll Q$ and the continuous embedding $W_0^1 L_M(\Omega) \subset E_Q(\Omega)$ hold (see [11]). Let $\varepsilon > 0$. Then there exists $C_\varepsilon > 0$, as in [7], such that

$$|a(x,s,\zeta)| \leq c(x) + C_\varepsilon + k_1 \overline{M}^{-1} Q(\varepsilon|s|) + k_3 \overline{M}^{-1} M(\varepsilon|\zeta|) \tag{11}$$

for a.e. $x \in \Omega$ and for all $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$. From (10) and (11) we deduce that $(a(x,z_n,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$. □

Proof of Lemma 3.1. Let $s > 0$. Let $\Omega_s = \{\, x \in \Omega, |\nabla u(x)| \leq s \,\}$ and denote by χ_s the characteristic function of Ω_s. Fix $r > 0$ and let $s > r$. We have

$$\begin{aligned}
0 &\leq \int_{\Omega_r} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z)][\nabla z_n - \nabla z]\,dx \\
&\leq \int_{\Omega_s} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z)][\nabla z_n - \nabla z]\,dx \\
&= \int_{\Omega_s} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx \\
&\leq \int_{\Omega} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx,
\end{aligned}$$

which with (iii) implies

$$\lim_{n\to\infty} \int_{\Omega_r} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z)][\nabla z_n - \nabla z]\,dx = 0.$$

So, as in [11]

$$\nabla z_n \longrightarrow \nabla z \qquad \text{a.e. in} \quad \Omega. \tag{12}$$

On the other hand, we have

$$\begin{aligned}
\int_\Omega a(x,z_n,\nabla z_n)\nabla z_n\,dx = &\int_\Omega [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)] \\
&\quad \times [\nabla z_n - \nabla z\chi_s]\,dx \\
&+ \int_\Omega a(x,z_n,\nabla z\chi_s)(\nabla z_n - \nabla z\chi_s)\,dx \\
&+ \int_\Omega a(x,z_n,\nabla z_n)\nabla z\chi_s\,dx.
\end{aligned} \tag{13}$$

Since $(a(x,z_n,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, and using (12), we obtain

$$a(x,z_n,\nabla z_n) \longrightarrow a(x,z,\nabla z) \quad \text{weakly in} \quad (L_{\overline{M}}(\Omega))^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M),$$

which implies that

$$\int_\Omega a(x,z_n,\nabla z_n)\nabla z \chi_s \, dx \longrightarrow \int_\Omega a(x,z,\nabla z)\nabla z \chi_s \, dx \tag{14}$$

as $n \to \infty$. Letting also $s \to \infty$, we obtain

$$\int_\Omega a(x,z,\nabla z)\nabla z \chi_s \, dx \longrightarrow \int_\Omega a(x,z,\nabla z)\nabla z \, dx. \tag{15}$$

On the other hand, it is easy to see that the second term of the right hand side of (13) tends to 0 as $n \to \infty$. Consequently, from (iii), (14), and (15) we have

$$\lim_{n\to\infty} \int_\Omega a(x,z_n,\nabla z_n)\nabla z_n \, dx = \int_\Omega a(x,z,\nabla z)\nabla z \, dx.$$

Using (A_4), we obtain, by lemma 1.4 and Vitali's Theorem,

$$M\Big(\frac{|\nabla z_n|}{\lambda}\Big) \to M\Big(\frac{|\nabla z|}{\lambda}\Big) \quad \text{in} \quad L^1(\Omega). \qquad \square$$

The following lemma will be used in the proof of the propositions 3.4 and 3.5.

Lemma 3.3. *Let Ω be an open bounded subset of $\mathbb{R}^N$ satisfying the segment property. If $u \in W_0^1 L_M(\Omega)$, then*

$$\int_\Omega \operatorname{div} u \, dx = 0.$$

For the proof we refer to [4].

3.2. The approximate problem

Let us define, for each $k > 0$, the truncation

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\frac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and, for each $n \in \mathbb{N}^*$, the approximations

$$\phi_n(s) = \phi(T_n(s)), \qquad g_n(x,t) = T_n(g(x,t)).$$

Consider the nonlinear elliptic problem

$$\begin{cases} u_n \in W_0^1 L_M(\Omega) \\ -\operatorname{div} a(x,u_n,\nabla u_n) - \operatorname{div} \phi_n(u_n) + g_n(x,u_n) = f \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{16}$$

which is equivalent to

$$\begin{cases} u_n \in W_0^1 L_M(\Omega) \\ -\operatorname{div} \tilde{a}(x, u_n, \nabla u_n) + g_n(x, u_n) = f \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{17}$$

where $\tilde{a}(x,t,\xi) = a(x,t,\xi) + \phi_n(t)$.

Since $|T_n(t)| \leq n$ and ϕ is continuous, we have $|\phi_n(t)| = |\phi(T_n(t))| \leq c_n$. From Gossez and Mustonen [13, Proposition 1 and Remark 2], the problem (16), and its equivalent (17), have at least one solution u_n.

3.3. Some intermediate results

Proposition 3.4. *Assume that (A_1)–(A_4), (5)–(8) hold true, and let u_n be a solution of the approximate problem (16). Then we have the following properties:*

(i) *$(u_n)_n$ is bounded in $W_0^1 L_M(\Omega)$, and there exists a function u in $W_0^1 L_M(\Omega)$ such that*

$$\begin{aligned} &u_n \rightharpoonup u \quad \textit{weakly in } W_0^1 L_M(\Omega) \quad \textit{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}), \\ &u_n \longrightarrow u \quad \textit{strongly in } E_M(\Omega) \quad \textit{and} \quad \textit{a.e. in } \Omega. \end{aligned}$$

(ii) *$(a(x, u_n, \nabla u_n))_n$ is bounded in $L_{\overline{M}}(\Omega)$.*

(iii) *$g_n(x, u_n) \to g(x, u)$ strongly in $L^1(\Omega)$.*

Proof. We divide the proof in several steps.

Step 1: Boundedness of $(u_n)_n$ in $W_0^1 L_M(\Omega)$. Taking u_n as test function in (16), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \, dx + \int_\Omega \phi(T_n(u_n))\nabla u_n \, dx + \int_\Omega g_n(x, u_n)u_n \, dx \leq \langle f, u_n\rangle.$$

Define $\tilde{\phi_n}(t) = \int_0^t \phi_n(\tau)\, d\tau$. We have $\tilde{\phi_n}(u_n) \in (W_0^1 L_M(\Omega))^N$. (We can apply Lemma 1.1 since each component of $\tilde{\phi}_n$ is uniformly Lipschitzian and $\tilde{\phi}_n(0) = 0$.) We obtain

$$\int_\Omega \phi_n(u_n)\, \nabla u_n \, dx = \int_\Omega \operatorname{div}(\tilde{\phi_n}(u_n))\, dx = 0.$$

(See Lemma 3.3.) By (7), we get

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \, dx \leq \langle f, u_n\rangle. \tag{18}$$

$$\int_\Omega g_n(x, u_n)u_n \, dx \leq \langle f, u_n\rangle. \tag{19}$$

On the other hand, f can be written as $f = f_0 - \operatorname{div} F$ where $f_0 \in E_{\overline{M}}(\Omega)$, $F \in (E_{\overline{M}}(\Omega))^N$. Using [11, Lemma 5.7] and Young's inequality we deduce

$$\begin{aligned} \int_\Omega f_0 u_n \, dx &\le C_1 + \frac{\alpha}{4} \int_\Omega M(|\nabla u_n|) \, dx, \\ \int_\Omega F \nabla u_n \, dx &\le C_2 + \frac{\alpha}{4} \int_\Omega M(|\nabla u_n|) \, dx. \end{aligned} \tag{20}$$

Combining (18) and (20), we get

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \, dx \le \frac{\alpha}{2} \int_\Omega M(|\nabla u_n|) \, dx + C_3. \tag{21}$$

This implies, by using (A_4), that

$$\int_\Omega M(|\nabla u_n|) \, dx \le C_4, \tag{22}$$

which gives

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^1 L_M(\Omega) \quad \text{for } \sigma(\Pi L_M, \Pi E_{\overline{M}}).$$

Using the compact embedding $W_0^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$, we get

$$u_n \longrightarrow u \quad \text{strongly in } E_M(\Omega) \quad \text{and} \quad \text{a.e. in } \Omega.$$

Step 2: Boundedness of $(a(x, u_n, \nabla u_n))_n$ *in* $(L_{\overline{M}}(\Omega))^N$*.* Let $w \in (E_M(\Omega))^N$ be arbitrary. By (A_3), we have

$$(a(x, u_n, \nabla u_n) - a(x, u_n, w))(\nabla u_n - w) > 0,$$

which implies that

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) w \, dx \\ &\qquad \le \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \, dx + \int_\Omega a(x, u_n, w)(w - \nabla u_n) \, dx. \end{aligned} \tag{23}$$

Combining (21) and (22), we get

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \, dx \le C_5,$$

with C_5 a positive constant.

On the other hand, for λ large and using (A_2), we have

$$\int_\Omega \overline{M}\Big(\frac{a(x, u_n, w)}{\lambda}\Big) \, dx \le \int_\Omega \overline{M}\Big(\frac{c(x)}{\lambda}\Big) + \int_\Omega \frac{k_3}{\lambda} M(k_4|w|) + C_7 \le C_8.$$

Thus, $|a(x, u_n, w)|$ is bounded in $L_{\overline{M}}(\Omega)$. This condition, additionally to (22), implies that the second term of the right hand side of (23) is bounded. Consequently, we obtain

$$\int_\Omega a(x, u_n, \nabla u_n) w \, dx \leq C_9,$$

with C_9 a positive constant. Hence, thanks to the Banach-Steinhaus theorem, the sequence $(a(x, u_n, \nabla u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

Step 3: Strongly convergence of the nonlinearity. Since $g_n(x, u_n) \to g(x, u)$ a.e. in Ω, by the sign condition (7) and Fatou's Lemma we obtain from (19) and (22) that

$$g(x, u) \, u \in L^1(\Omega),$$

and by Vitali's theorem we have

$$g_n(x, u_n) \longrightarrow g(x, u) \quad \text{strongly in } L^1(\Omega),$$

which completes the proof. □

Proposition 3.5. *Assume that (A_1)–(A_4), (5)–(8) hold true, and let u_n be a solution of the approximate problem* (16). *Then, we have (for a subsequence noted again u_n)*

$$\nabla u_n \longrightarrow \nabla u \quad \textit{a.e. in } \Omega.$$

Proof. Again we divide the proof in several steps.

Step 1. $\limsup_{n \to +\infty} \int_{\{|u_n| > h\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq \langle f, u - T_h(u) \rangle$ where $h > 0$.

The idea is to use in (16) the test function $u_n - T_h(u_n)$ (which is in $W_0^1 L_M(\Omega)$). Consider

$$\begin{aligned} \theta(t) &= \phi_n(t) \, \chi_{\{s \in \mathbb{R}, |s| \geq h\}}(t), \\ \tilde{\theta}(t) &= \int_0^t \theta(\tau) \, d\tau, \end{aligned}$$

hence $\tilde{\theta}(u_n) \in (W_0^1 L_M(\Omega))^N$ (by Lemma 1.1). We obtain, by Lemma 3.3,

$$\begin{aligned} \int_\Omega \phi_n(u_n) \nabla(u_n - T_h(u_n)) \, dx &= \int_\Omega \phi_n(u_n) \chi_{\{s \in \mathbb{R}, |s| \geq h\}}(u_n) \nabla u_n \, dx \\ &= \int_\Omega \theta(u_n) \nabla u_n \, dx = \int_\Omega \operatorname{div}(\tilde{\theta}(u_n)) \, dx = 0. \end{aligned}$$

Using the sign condition (7) we have $g_n(x, u_n)(u_n - T_h(u_n)) \geq 0$ a.e. in Ω. Then, for any fixed $h > 0$, we have

$$\int_{\{|u_n| > h\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq \langle f, u_n - T_h(u_n) \rangle.$$

Since $u_n - T_h(u_n) \rightharpoonup u - T_h(u)$ weakly in $W_0^1 L_M(\Omega)$ and $f \in W^{-1}E_{\overline{M}}(\Omega)$, we have

$$\limsup_{n\to+\infty} \int_{\{|u_n|>h\}} a(x, u_n, \nabla u_n)\nabla u_n \, dx \le \langle f, u - T_h(u)\rangle. \tag{24}$$

Step 2. We shall prove that $\nabla u_n \to \nabla u$ a.e. in Ω.

By Lemma 1.5 there exists a sequence $v_j \in \mathcal{D}(\Omega)$ which converges to u for the modular convergence in $W_0^1 L_M(\Omega)$. Let $s, j > 0$. Let $\Omega_s^j = \{x \in \Omega, |\nabla v_j(x)| \le s\}$ and denote by χ_s^j the characteristic function of Ω_s^j. We will note by $\epsilon(n, j, h)$ any quantity such that

$$\lim_{h\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \epsilon(n, j, h) = 0.$$

If the quantity we consider does not depend on one parameter among n, j, and h, we will omit the dependence on the corresponding parameter: as an example, $\epsilon(n, h)$ is any quantity such that

$$\lim_{h\to+\infty} \lim_{n\to+\infty} \epsilon(n, h) = 0.$$

Finally, we will note (for example) by $\epsilon_h(n, j)$ a quantity that depends on n, j, h, and is such that

$$\lim_{j\to+\infty} \lim_{n\to+\infty} \epsilon_h(n, j) = 0$$

for any fixed value of h.

We have

$$\begin{aligned}
\int_\Omega &[a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u\chi_s)][\nabla u_n - \nabla u\chi_s] \, dx \\
&= \int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla v_j\chi_s^j)][\nabla u_n - \nabla v_j\chi_s^j] \, dx \\
&\quad + \int_\Omega a(x, u_n, \nabla v_j\chi_s^j)(\nabla u_n - \nabla v_j\chi_s^j) \, dx \\
&\quad - \int_\Omega a(x, u_n, \nabla u\chi_s)(\nabla u_n - \nabla u\chi_s) \, dx \\
&\quad + \int_\Omega a(x, u_n, \nabla u_n)(\nabla v_j\chi_s^j - \nabla u\chi_s) \, dx.
\end{aligned}$$

We pass to the limit in n and j in the last three terms of the right hand side of the last equality, we get

$$\int_\Omega a(x, u_n, \nabla v_j\chi_s^j)[\nabla u_n - \nabla v_j\chi_s^j] \, dx - \int_\Omega a(x, u_n, \nabla u\chi_s)[\nabla u_n - \nabla u\chi_s] \, dx = \epsilon(n, j),$$

and

$$\int_\Omega a(x, u_n, \nabla u_n)[\nabla v_j\chi_s^j - \nabla u\chi_s] \, dx = \epsilon(n, j).$$

This implies

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi_s)][\nabla u_n - \nabla u\chi_s]\,dx$$
$$= \int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n - \nabla v_j\chi_s^j]\,dx + \epsilon(n,j). \quad (25)$$

The term in the right hand side of the last equality can be estimated as follows:

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n - \nabla v_j\chi_s^j]\,dx$$
$$\le \int_{\{|u_n - T_h(v_j)|\le 2h\}} [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n - \nabla v_j\chi_s^j]\,dx$$
$$+ \int_{\{|u_n|>h\}} [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n - \nabla v_j\chi_s^j]\,dx. \quad (26)$$

The first term of the right hand side of (26) can be written as

$$\int_{\{|u_n - T_h(v_j)|\le 2h\}} [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n - \nabla v_j\chi_s^j]\,dx$$
$$= \int_{\{|u_n - T_h(v_j)|\le 2h\}} a(x,u_n,\nabla u_n)\nabla(u_n - T_h(v_j))\,dx$$
$$+ \int_{\{|u_n - T_h(v_j)|\le 2h\}} a(x,u_n,\nabla u_n)(\nabla T_h(v_j) - \nabla v_j\chi_s^j)\,dx$$
$$- \int_{\{|u_n - T_h(v_j)|\le 2h\}} a(x,u_n,\nabla v_j\chi_s^j)[\nabla u_n - \nabla v_j\chi_s^j]\,dx \quad (27)$$

If we take $T_{2h}(u_n - T_h(v_j))$ as test function in (16), we have for n large enough

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_{2h}(u_n - T_h(v_j))\,dx + \int_\Omega \phi(u_n)\nabla T_{2h}(u_n - T_h(v_j))\,dx$$
$$+ \int_\Omega g_n(x,u_n)T_{2h}(u_n - T_h(v_j))\,dx = \langle f, T_{2h}(u_n - T_h(v_j))\rangle. \quad (28)$$

Using (i) of proposition 3.4 and the modular convergence of v_j, we have

$$\int_\Omega \phi(u_n)\nabla T_{2h}(u_n - T_h(v_j))\;dx = \int_\Omega \phi(u)\nabla T_{2h}(u - T_h(u))\;dx + \epsilon_h(n,j) = \epsilon(n,j,h),$$
$$\int_\Omega g_n(x,u_n)T_{2h}(u_n - T_h(v_j))\;dx = \epsilon(n,j,h),$$
$$\langle f, T_{2h}(u_n - T_h(v_j))\rangle = \epsilon(n,j,h),$$

which, with (28), implies that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_{2h}(u_n - T_h(v_j))\,dx = \epsilon(n,j,h). \tag{29}$$

Now, since $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, we have, for a subsequence

$$a(x,u_n,\nabla u_n) \rightharpoonup \rho \tag{30}$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $(\sigma(L_{\overline{M}}(\Omega),E_M(\Omega)))^N$ as n tends to infinity, that

$$\int_{\{|u_n-T_h(v_j)|\le 2h\}} a(x,u_n,\nabla u_n)(\nabla T_h(v_j) - \nabla v_j\chi_s^j)\,dx \longrightarrow \int_{\{|u-T_h(v_j)|\le 2h\}} \rho(\nabla T_h(v_j) - \nabla v_j\chi_s^j)\,dx$$

as n tends to infinity.

Using now the modular convergence of v_j, we get

$$\int_{\{|u-T_h(v_j)|\le 2h\}} \rho(\nabla T_h(v_j) - \nabla v_j\chi_s^j)\,dx \longrightarrow \int_{\{|u-T_h(u)|\le 2h\}} \rho(\nabla T_h(u) - \nabla u\chi_s)\,dx$$

as j tends to infinity. Letting also h to infinity, we can easy deduce

$$\int_{\{|u-T_h(u)|\le 2h\}} \rho(\nabla T_h(u) - \nabla u\chi_s)\,dx \longrightarrow \int_{\Omega\setminus\Omega_s} \rho\nabla u\,dx.$$

Finally

$$\int_{\{|u_n-T_h(v_j)|\le 2h\}} a(x,u_n,\nabla u_n)(\nabla T_h(v_j) - \nabla v_j\chi_s^j)\,dx = \int_{\Omega\setminus\Omega_s} \rho\nabla u\,dx + \epsilon(n,j,h). \tag{31}$$

For the third term of the right hand side of (27), we have for a subsequence (use Lemma 1.3)

$$a(x,u_n,\nabla v_j\chi_s^j)\chi_{\{|u_n-T_h(v_j)|\le 2h\}} \longrightarrow a(x,u,\nabla v_j\chi_s^j)\chi_{\{|u-T_h(v_j)|\le 2h\}} \quad \text{strongly in } (L_{\overline{M}}(\Omega))^N \quad \text{for } \sigma(L_{\overline{M}}(\Omega),E_M(\Omega))$$

as n tends to infinity, and

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^1L_M(\Omega) \quad \text{for } \sigma(\Pi L_{\overline{M}}(\Omega),\Pi E_M(\Omega))$$

as n tends to infinity. Hence

$$\int_{\{|u_n-T_h(v_j)|\leq 2h\}} a(x,u_n,\nabla v_j\chi_s^j)[\nabla u_n-\nabla v_j\chi_s^j]\,dx \longrightarrow \int_{\{|u-T_h(v_j)|\leq 2h\}} a(x,u,\nabla v_j\chi_s^j)[\nabla u-\nabla v_j\chi_s^j]\,dx.$$

Using now the modular convergence of (v_j), we get

$$\int_{\{|u-T_h(v_j)|\leq 2h\}} a(x,u,\nabla v_j\chi_s^j)[\nabla u-\nabla v_j\chi_s^j]\,dx \longrightarrow \int_{\{|u-T_h(u)|\leq 2h\}} a(x,u,\nabla u\chi_s)[\nabla u-\nabla u\chi_s]\,dx=0$$

as j tends to infinity.

Finally,

$$\int_{\{|u_n-T_h(v_j)|\leq 2h\}} a(x,u_n,\nabla v_j\chi_s^j)[\nabla u_n-\nabla v_j\chi_s^j]\,dx=\epsilon(n,j,h). \tag{32}$$

Combining (27), (29), (31), and (32), we deduce

$$\int_{\{|u_n-T_h(v_j)|\leq 2h\}} [a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n-\nabla v_j\chi_s^j]\,dx = \int_{\Omega\setminus\Omega_s} \rho\nabla u\,dx+\epsilon(n,j,h). \tag{33}$$

The second term of the right hand side of the (26) can be written as

$$\begin{aligned}\int_{\{|u_n|>h\}} [a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n-\nabla v_j\chi_s^j]\,dx &= \int_{\{|u_n|>h\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \\ &\quad -\int_{\{|u_n|>h\}} a(x,u_n,\nabla u_n)\nabla v_j\chi_s^j\,dx \\ &\quad -\int_{\{|u_n|>h\}} a(x,u_n,\nabla v_j\chi_s^j)][\nabla u_n-\nabla v_j\chi_s^j]\,dx.\end{aligned}$$

Letting h to infinity in (24), we get

$$\int_{\{|u_n|>h\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx\leq\epsilon(n,h),$$

and, reasoning as above, it is easy to see that

$$\int_{\{|u_n|>h\}} a(x,u_n,\nabla u_n)\nabla v_j \chi_s^j \, dx = \epsilon(n,j,h),$$

$$\int_{\{|u_n|>h\}} a(x,u_n,\nabla v_j \chi_s^j)][\nabla u_n - \nabla v_j \chi_s^j] \, dx = \epsilon(n,j,h).$$

Finally

$$\int_{\{|u_n|>h\}} [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j \chi_s^j)][\nabla u_n - \nabla v_j \chi_s^j] \, dx = \epsilon(n,j,h). \tag{34}$$

Combining (33) and (34), we deduce from (26) that

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j \chi_s^j)][\nabla u_n - \nabla v_j \chi_s^j] \, dx \leq \int_{\Omega\setminus\Omega_s} \rho \nabla u \, dx + \epsilon(n,j,h). \tag{35}$$

Letting s to infinity, we get by using (25) and (35)

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi_s)][\nabla u_n - \nabla u \chi_s] \, dx \longrightarrow 0 \tag{36}$$

as $n, s \to \infty$. Using Lemma 3.1 we can conclude the result of Proposition 3.5. □

Proof of Theorem 2.3. *Step 1.* We shall prove that

$$a(x,u_n,\nabla u_n)\nabla u_n \longrightarrow a(x,u,\nabla u)\nabla u \quad \text{strongly in} \quad L^1(\Omega). \tag{37}$$

We have

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \, dx &= \int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi_s)][\nabla u_n - \nabla u\chi_s] \, dx \\ &\quad + \int_\Omega a(x,u_n,\nabla u_n)\nabla u \chi_s \, dx \\ &\quad + \int_\Omega a(x,u_n,\nabla u\chi_s)[\nabla u_n - \nabla u \chi_s] \, dx.\end{aligned}$$

By (36) the first term of the last equality tends to 0. By the Proposition 3.5 and (30), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u \chi_s \, dx \longrightarrow \int_\Omega a(x,u,\nabla u)\nabla u \chi_s \, dx$$

as n tends to infinity. Letting also s to infinity, we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u \chi_s \, dx \longrightarrow \int_\Omega a(x,u,\nabla u)\nabla u \, dx.$$

The third term of the last equality tends to 0 as n and $s \to \infty$. We deduce

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n\, dx \longrightarrow \int_\Omega a(x, u, \nabla u)\nabla u\, dx. \tag{38}$$

Using Lemma 1.4 we get the result.

Step 2. Passing to the limit. Using in (16) the test function $h(u_n)\varphi$ with $h \in C_c^1(\mathbb{R})$ and $\varphi \in \mathcal{D}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)\varphi\, dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla\varphi h(u_n)\, dx \\ + \int_\Omega \phi_n(u_n)\nabla(h(u_n)\varphi)\, dx + \int_\Omega g_n(x, u_n)h(u_n)\varphi\, dx = \langle f, h(u_n)\varphi\rangle. \end{aligned}\tag{39}$$

We shall pass to the limit in each term of last equality.

Since h and h' have compact support on $\mathbb{R}$, there exist $\eta > 0$ such that $\operatorname{supp} h$ and $\operatorname{supp} h' \in [-\eta, \eta]$. We have for n large enough

$$\begin{aligned}\phi_n(t)h(t) = \phi(T_n(t))h(t) = \phi(T_\eta(t))h(t),\\ \phi_n(t)h'(t) = \phi(T_n(t))h'(t) = \phi(T_\eta(t))h'(t)\end{aligned}$$

and the functions ϕh and $\phi h'$ belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$.

First we have that $h(u_n)\varphi$ is bounded in $W_0^1L_M(\Omega)$. Indeed, since u_n is bounded in $W_0^1L_M(\Omega)$ there exists two constants positive $c, \lambda > 0$ such that $\int_\Omega M\big(\frac{|\nabla u_n|}{\lambda}\big)\, dx \le c$. Let c_1 be a constant positive such that $\|h(u_n)|\nabla\varphi|\|_\infty \le c_1$ and $\|h'(u_n)\varphi\|_\infty \le c_1$. For μ large enough, we have

$$\begin{aligned}\int_\Omega M\Big(\frac{h(u_n)\nabla\varphi + h'(u_n)\varphi|\nabla u_n|}{\mu}\Big)\, dx \\ &\le \int_\Omega M\Big(\frac{c_1 + c_2|\nabla u_n|/\lambda}{\mu}\Big)\, dx \qquad \text{with} \quad c_2 = c_1\lambda \\ &\le c_3 + \frac{c_2}{\mu}\int_\Omega M\Big(\frac{|\nabla u_n|}{\lambda}\Big)\, dx \le c_4.\end{aligned}$$

This implies that

$$h(u_n)\varphi \rightharpoonup h(u)\varphi \quad \text{weakly in } W_0^1L_M(\Omega) \quad \text{for } \sigma(\textstyle\prod L_M, \prod E_{\overline{M}}). \tag{40}$$

By the convergence of (40), and since

$$\phi(T_\eta(u_n)) \longrightarrow \phi(T_\eta(u)) \quad \text{strongly in } (E_{\overline{M}})^N,$$

the third term of (39) tends to $\int_\Omega \phi(T_\eta(u))\nabla(h(u)\varphi)\, dx$, and the right hand side of (39) tends to $\langle f, h(u)\varphi\rangle$. For the first term of (39), we remark that

$$|a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)\varphi| \le c_1 a(x, u_n, \nabla u_n)\nabla u_n;$$

consequently, Vitali's theorem and (37) give that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'(u_n)\varphi\,dx \longrightarrow \int_\Omega a(x,u,\nabla u)\nabla u h'(u)\varphi\,dx.$$

For the second term of (39), we have

$$h(u_n)\nabla\varphi \longrightarrow h(u)\nabla\varphi \quad \text{strongly in} \quad (E_M(\Omega))^N,$$

and

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N \quad \text{for } \sigma(\prod L_{\overline{M}},\prod E_M),$$

then

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi h(u_n)\,dx \longrightarrow \int_\Omega a(x,u,\nabla u)\nabla\varphi h(u)\,dx.$$

The fourth term of (39) tends to $\int_\Omega g(x,u)h(u)\varphi\,dx$.

Using the limite proved above we can easily pass to the limit in each term of (39) and obtain

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)[h'(u)\varphi\nabla u + h(u)\nabla\varphi]\,dx \\ + \int_\Omega \phi(u)h'(u)\varphi\nabla u\,dx + \int_\Omega \phi(u)h(u)\nabla\varphi\,dx + \int_\Omega g(x,u)h(u)\varphi\,dx \\ = \langle f, h(u)\varphi\rangle \quad \forall h\in\mathcal{C}_c^1(\mathbb{R}),\ \forall\varphi\in\mathcal{D}(\Omega),\end{aligned}$$

which proves Theorem 2.3. □

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