Solving Variational Inclusions by a Method Obtained Using a Multipoint Iteration Formula

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Received: March 26, 2007
Accepted: February 18, 2008

ABSTRACT

This paper deals with variational inclusions of the form: $0 \in f(x) + F(x)$ where f is a single function admitting a second order Fréchet derivative and F is a set-valued map acting in Banach spaces. We prove the existence of a sequence (x_k) satisfying $0 \in f(x_k) + \sum_{i=1}^{M} a_i \nabla f\big(x_k + \beta_i(x_{k+1} - x_k)\big)(x_{k+1} - x_k) + F(x_{k+1})$ where the single-valued function involved in this relation is an approximation of the function f based on a multipoint iteration formula and we show that this method is locally cubically convergent.

Key words: set-valued mapping, generalized equations, pseudo-Lipschitz maps, multipoint iteration formula.

2000 Mathematics Subject Classification: 49J53, 47H04, 65K10.

Introduction

This paper deals with the problem of approximating a solution of the "abstract" generalized equation

$$0 \in f(x) + F(x), \tag{1}$$

where f is a function from X into Y, which admits a second order Fréchet derivative, F is a set-valued map from X to the subsets of Y with closed graph, and X, Y are two Banach spaces.

Let us recall that equation (1) is an abstract model for various problems.

ISSN: 1139-1138
http://dx.doi.org/10.5209/rev_REMA.2009.v22.n1.16314

- When $F = \{0\}$, (1) is an equation,
- when F is the positive orthant in $\mathbb{R}^m$, (1) is a system of inequalities,
- when F is the normal cone to a convex and closed set in X, (1) may represent variational inequalities.

When the Fréchet derivative ∇f of f is locally Lipschitz, Dontchev [3, 4] associates to (1) a Newton-type method based on a partial linearization which provides a local quadratic convergence. Following his work, Piétrus [16] obtains a Newton-type sequence which converges whenever ∇f satisfies a Hölder-type condition and in [15] he proves the stability.

Using a second-degree Taylor polynomial expansion of f at x_k, Geoffroy, Hilout, and Piétrus [7] introduced a method involving the second order Fréchet derivative and, when $\nabla^2 f$ is lipschitz, they obtained a cubic convergence. In [8] they proved the stability of the method and in [10] they showed that the previous method is superquadratic when $\nabla^2 f$ satisfies an Hölder condition. Lately, Jean-Alexis presented in [12] a method without second order derivative, which is also cubically convergent and Geoffroy, Jean-Alexis and Piétrus showed the stability of this method in [9]. Our method generalizes this idea by taking more iterates.

For solving (1), we fix an integer $M > 1$ and we consider the sequence

$$0 \in f(x_k) + \sum_{i=1}^{M} a_i \nabla f\big(x_k + \beta_i(x_{k+1} - x_k)\big)(x_{k+1} - x_k) + F(x_{k+1}) \tag{2}$$

where $(a_i)_{1 \leq i \leq M}$ and $(\beta_i)_{1 \leq i \leq M}$ are two sequences of real numbers satisfying

$$\sum_{i=1}^{M} a_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{M} a_i \beta_i = \frac{1}{2}. \tag{3}$$

The inspiration for considering such a method comes from a multipoint iteration formula given in [19] for approximating f.

Let us remark that there is no second order Fréchet derivative of f in this approximation of f but only M values of ∇f computed at $\big(x_k + \beta_i(x_{k+1} - x_k)\big)$, $1 \leq i \leq M$.

The paper is organized as follows: in section 2, we give some definitions and recall a few preliminary results as a fixed-point theorem (lemma 1.3) which is the main tool for proving the existence and the convergence of the sequence defined by (2) to a solution of the equation (1); we also make some fundamental assumptions on f.

The existence and the convergence of the previous sequence defined by (2) are developed in section 3. Moreover, we prove that the convergence of this method is of order 3.

Let us recall some notation:

- The distance from a point x to a set A in the metric space (Z,ρ) is defined by

$$\operatorname{dist}(x,A)=\inf\{\,\rho(x,y),\ y\in A\,\};$$

- The excess e from the set A to the set C is given by

$$e(A,C)=\sup\{\,\operatorname{dist}(x,A),\ x\in C\,\};$$

- Let $\Lambda\colon X\rightrightarrows Y$ be a set-valued map, we write

$$\operatorname{graph}\Lambda=\{\,(x,y)\in X\times Y,\ y\in\Lambda(x)\,\}\quad\text{and}\quad \Lambda^{-1}(y)=\{\,x\in X,\ y\in\Lambda(x)\,\};$$

- $B_r(x)$ is the closed ball centered at x with radius r;
- The norms in the Banach spaces X and Y are both denoted by $\|\cdot\|$.

1. Definitions and preliminary results

In this section, we collect some results that we will need to prove our main theorem.

Definition 1.1. A set-valued Λ is *pseudo-Lipschitz* around $(x_0,y_0)\in\operatorname{graph}\Lambda$ with modulus L if there exist constants a and b such that

$$\sup_{y\in\Lambda(x')\cap B_a(y_0)}\operatorname{dist}(y,\Lambda(x''))\le L\|x'-x''\|,\quad\text{for all } x' \text{ and } x'' \text{ in } B_b(x_0).\tag{4}$$

Using the excess, we have an equivalent definition replacing the inequality (4) by

$$e\big(\Lambda(x')\cap B_a(y_0),\Lambda(x'')\big)\le L\|x'-x''\|,\quad\text{for all } x' \text{ and } x'' \text{ in } B_b(x_0).$$

The pseudo-Lipschitz property has been introduced J.-P. Aubin and he was the first to define this concept as a continuity property. Let us note that sometimes this property is also called “Aubin continuity.” Characterizations of the pseudo-Lipschitz property are also obtained by Rockafellar in [17,18] using the Lipschitz continuity of the distance function $\operatorname{dist}(y,\Lambda(x))$ around (x_0,y_0) and by Mordukhovich in [13,14] using the concept of coderivative of multifunctions. The Aubin-continuity of F is equivalent to the metric regularity of F^{-1}. Lately, Dontchev, Quincampoix, and Zlateva gave in [6] a derivative criterion for the metric regularity of set-valued mappings based on works of Aubin and co-authors. Fore more details and applications of this property, the reader could also refer to [1,2,5].

Definition 1.2. We say that a function f from a metric space (X,ρ) into a metric space (Y,d) is strictly stationary at $x_0\in X$ if, for every $\varepsilon>0$, there exists $\delta>0$ such that

$$d(f(x_1),f(x_2))\le\epsilon\,\rho(x_1,x_2)\quad\text{whenever}\quad\rho(x_i,x_0)<\delta,\quad i=1,2.$$

Lemma 1.3. *Let (Z, ρ) be a complete metric space, let ϕ be a set–valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \leq \lambda < 1$ and*

(i) $\operatorname{dist}\big(\eta_0, \phi(\eta_0)\big) \leq r(1-\lambda)$,

(ii) $e\big(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)\big) \leq \lambda\rho(x_1, x_2), \ \ \forall x_1, x_2 \in B_r(\eta_0)$,

then ϕ has a fixed–point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single-valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

The proof of lemma 1.3 is given in [5] employing the standard iterative concept for nonexpansive mappings. This fixed-point lemma is a generalization of the fixed-point theorem in Ioffe-Tikhomirov [11] where in (ii) the excess e is replaced by the Haussdorf distance.

We make the following assumptions on a neighborhood Ω of x^*:

($\mathcal{H}1$) $\nabla^2 f$ is K_2-Lipschtiz in Ω

($\mathcal{H}2$) F is a set-valued map with closed graph.

($\mathcal{H}3$) $(f+F)^{-1}$ is pseudo-Lipschitz around $(0, x^*)$ with constants a, b and modulus L.

We also define the following functions:

$$\begin{aligned}
\Lambda_k(x) &= f(x_k) + \sum_{i=1}^{M} a_i \nabla f\big(x_k + \beta_i(x - x_k)\big)(x - x_k)\\
\Lambda_{x^*}(x) &= f(x^*) + \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x - x^*)\big)(x - x^*)\\
Q(x) &= \Lambda_{x^*}(x) + F(x)
\end{aligned}$$

and

$$\Psi_k(x) = Q^{-1}\big(\Lambda_{x^*}(x) - \Lambda_k(x)\big). \tag{5}$$

2. Description of the method and convergence results

- From a starting point x_0 in a neighborhood of a solution x^* of (1), applying lemma 1.3, we show that Ψ_0 possesses a fixed point x_1.
- From a current iterate x_k generated by (2) and a function Ψ_k defined on X by (5), applying lemma 1.3, we obtain the existence of the next iterate x_{k+1} which is a fixed point of Ψ_k.

The main result of this study reads as follows:

Theorem 2.1. *Let x^* be a solution of* (1), *and suppose that* $(\mathcal{H}1)$–$(\mathcal{H}3)$ *are satisfied. Then for all* $c > \frac{LK_2}{6}\big(1 + 3\sum_{i=1}^{M}|a_i|\beta_i^2\big)$, *one can find* $\delta > 0$ *such that for every starting point* $x_0 \in B_\delta(x^*)$, *there exists a sequence* $(x_k)_{k\geq 0}$ *defined by* (2) *which satisfies*

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^3. \tag{6}$$

Before proving the theorem, we show that

Proposition 2.2. *The following are equivalent:*

(i) $\big[f(x^*) + \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(\cdot - x^*)\big)(\cdot - x^*) + F(\cdot)\big]^{-1}$ *is pseudo-Lipschitz around* (y^*, x^*),

(ii) $(f + F)^{-1}$ *is pseudo-Lipschitz around* (y^*, x^*).

Proof. According to [5, corollary 2], it suffices to prove, under the assumptions of theorem 2.1, that the function h defined by

$$h(x) = f(x) - \Lambda_{x^*}(x)$$

is strictly stationary at x^*. Without lose of generality, we can suppose that the neighborhood Ω is bounded. This implies, thanks to $(\mathcal{H}1)$, that $\nabla^2 f$ is bounded on Ω by K_1 which implies that ∇f is K_1-Lipschitz on Ω.

Let $\epsilon > 0$ and let us set

$$\alpha_1 = \frac{\epsilon}{2K_1\big[\sum_{i=1}^{M}|a_i|(1 + |\beta_i|)\big]}.$$

Fix $\delta > 0$ such that $\delta \leq \min\{a; \alpha_1\}$ where a is given by the assumption $(\mathcal{H}3)$ and let $x_1, x_2 \in B_\delta(x^*)$.

$$\begin{aligned}
\|h(x_1) - h(x_2)\| &= \|f(x_1) - \Lambda_{x^*}(x_1) - f(x_2) + \Lambda_{x^*}(x_2)\| \\
&\leq \Big\|f(x_1) - f(x_2) - \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x_1 - x^*)\big)(x_1 - x_2)\Big\| \\
&\quad + \Big\|\Big(\sum_{i=1}^{M} a_i\big(\nabla f(x^* + \beta_i(x_2 - x^*)\big) \\
&\quad - \nabla f\big(x^* + \beta_i(x_1 - x^*)\big)\Big)(x_2 - x^*)\Big\|.
\end{aligned} \tag{7}$$

Let us set

$$A = \Big\|f(x_1) - f(x_2) - \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x_1 - x^*)\big)(x_1 - x_2)\Big\|,$$

$$B = \Big\|\Big(\sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x_2 - x^*)\big) - \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x_1 - x^*)\big)\Big)(x_2 - x^*)\Big\|.$$

Using (3), we obtain

$$A = \Big\|\sum_{i=1}^{M} a_i(f(x_1) - f(x_2) - \nabla f\big(x^* + \beta_i(x_1 - x^*)\big)(x_1 - x_2))\Big\|,$$

then,

$$A \leq \sum_{i=1}^{M}|a_i|\|x_1 - x_2\| \int_0^1 \big\|\nabla f\big(x_2 + t(x_1 - x_2)\big) - \nabla f\big(x^* + \beta_i(x_1 - x^*)\big)\big\|\, dt,$$

which implies

$$A \leq K_1 \sum_{i=1}^{M}|a_i|\|x_1 - x_2\| \int_0^1 \|x_2 + t(x_1 - x_2) - x^* - \beta_i(x_1 - x^*)\|\, dt.$$

Since $x_1, x_2 \in B_\delta(x^*)$, we find

$$A \leq K_1\delta\Big[\sum_{i=1}^{M}|a_i|(1 + |\beta_i|)\Big]\|x_1 - x_2\|. \tag{8}$$

In a similar way, one also obtains

$$B \leq 2K_1\delta\Big[\sum_{i=1}^{M}|a_i||\beta_i|\Big]\|x_1 - x_2\|. \tag{9}$$

By (7), (8), and (9), we deduce that

$$\forall x_1, x_2 \in B_\delta(x^*), \quad \|h(x_1) - h(x_2)\| \leq 2K_1\delta\Big[\sum_{i=1}^{M}|a_i|(1 + |\beta_i|)\Big]\|x_1 - x_2\|. \tag{10}$$

According to (10), and the fact that $\delta \leq \alpha_1$, we have

$$\forall x_1, x_2 \in B_\delta(x^*), \quad \|h(x_1) - h(x_2)\| \leq \epsilon\|x_1 - x_2\|,$$

which achieves the proof of the proposition. □

Proposition 2.3. *Under the assumptions of theorem 2.1, there exists $\delta > 0$ such that, for all $x_0 \in B_\delta(x^*)$ and $x_0 \neq x^*$, the map Ψ_0 admits a fixed point $x_1 \in B_\delta(x^*)$.*

Proof. For the proof of this proposition, we show both assertions (i) and (ii) of lemma 1.3.

The assumption $(\mathcal{H}3)$ gives the constants a and b; moreover we set

$$\alpha_2 = \sqrt{\frac{2b}{3K_1\Big[\sum_{i=1}^{M}|a_i|(1+2|\beta_i|)\Big]}}$$
$$\alpha_3 = \sqrt{\frac{1}{c}}$$
$$\alpha_4 = \sqrt[3]{\frac{6b}{K_2\Big[1+3\sum_{i=1}^{M}|a_i|\beta_i^2\Big]}}$$

Let us consider $\delta > 0$ such that

$$\delta < \min\{\, a,\, \alpha_2,\, \alpha_3,\, \alpha_4\,\}. \tag{11}$$

From the definition of the excess e, we have

$$\operatorname{dist}(x^*, \Psi_0(x^*)) \le e\big(Q^{-1}(0)\cap B_\delta(x^*), Q^{-1}(\Lambda_{x^*}(x^*)-\Lambda_0(x^*))\big).$$

We have

$$\|\Lambda_{x^*}(x^*)-\Lambda_0(x^*)\| = \Big\|f(x^*)-f(x_0)-\sum_{i=1}^{M}a_i\nabla f\big(x_0+\beta_i(x^*-x_0)\big)(x^*-x_0)\Big\|,$$

which can be rewritten

$$\begin{aligned}\|\Lambda_{x^*}(x^*)-\Lambda_0(x^*)\| = \Big\|&f(x^*)-f(x_0)-\nabla f(x_0)(x^*-x_0)-\frac{1}{2}\nabla^2 f(x_0)(x^*-x_0)^2\\ &+\nabla f(x_0)(x^*-x_0)+\frac{1}{2}\nabla^2 f(x_0)(x^*-x_0)^2\\ &-\sum_{i=1}^{M}a_i\nabla f\big(x_0+\beta_i(x^*-x_0)\big)(x^*-x_0)\Big\|.\end{aligned} \tag{12}$$

Let us also set

$$D = \Big\|f(x^*)-f(x_0)-\nabla f(x_0)(x^*-x_0)-\frac{1}{2}\nabla^2 f(x_0)(x^*-x_0)^2\Big\|$$

and

$$\begin{aligned}E = \Big\|&\nabla f(x_0)(x^*-x_0)+\frac{1}{2}\nabla^2 f(x_0)(x^*-x_0)^2\\ &-\sum_{i=1}^{M}a_i\nabla f\big(x_0+\beta_i(x^*-x_0)\big)(x^*-x_0)\Big\|.\end{aligned}$$

Using $(\mathcal{H}1)$, we obtain

$$D \leq \frac{K_2}{6} \|x^* - x_0\|^3. \tag{13}$$

Since

$$\nabla f\big(x_0 + \beta_i(x^* - x_0)\big) - \nabla f(x_0) = \beta_i \int_0^1 \nabla^2 f\big(x_0 + \beta_i t(x^* - x_0)\big)(x^* - x_0)\, dt$$

we have

$$E \leq \sum_{i=1}^{M} |a_i \beta_i| \|x^* - x_0\|^2 \int_0^1 \big\|\nabla^2 f\big(x_0 + \beta_i t(x^* - x_0)\big) - \nabla^2 f(x_0)\big\|\, dt.$$

The use of $(\mathcal{H}1)$ gives also

$$E \leq \frac{K_2}{2} \Big[\sum_{i=1}^{M} |a_i| \beta_i^2\Big] \|x^* - x_0\|^3. \tag{14}$$

According to (11), (12), (13), (14), and using the pseudo-Lipschitzness of Q^{-1}, we obtain

$$\|\Lambda_{x^*}(x^*) - \Lambda_0(x^*)\| \leq b$$

and

$$\operatorname{dist}\big(x^*, \Psi_0(x^*)\big) \leq L\|\Lambda_{x^*}(x^*) - \Lambda_0(x^*)\|.$$

Using (12), (13), and (14) one has

$$\operatorname{dist}\big(x^*, \Psi_0(x^*)\big) \leq \frac{LK_2}{6}\Big(1 + 3\sum_{i=1}^{M} |a_i| \beta_i^2\Big) \|x^* - x_0\|^3.$$

By setting

$$r = r_0 = c\|x^* - x_0\|^3, \qquad \lambda = LK_1\delta\Big[\sum_{i=1}^{M} |a_i|\big(1 + 4|\beta_i|\big)\Big],$$

since

$$c > \frac{LK_2}{6}\Big(1 + 3\sum_{i=1}^{M} |a_i| \beta_i^2\Big),$$

one can decrease δ if it is necessary so that $\lambda \in]0, 1[$ and

$$c(1 - \lambda) > \frac{LK_2}{6}\Big(1 + 3\sum_{i=1}^{M} |a_i| \beta_i^2\Big)$$

and assertion (i) in lemma 1.3 is satisfied.

Let us remark that the decreasing of δ implies that the number $r_0 = r_0(\delta)$ also decreases and the above choice of r_0 and (11) imply $r_0 \leq \delta < a$.

Now, let us show that condition (ii) is also satisfied. Let $x \in B_\delta(x^*)$ and let us set $y = \Lambda_{x^*}(x) - \Lambda_0(x)$. We have

$$\begin{aligned}\|y\| \leq{} & \Big\| f(x^*) - f(x_0) - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x - x_0)\big)(x^* - x_0) \Big\| \\ & + \Big\| \Big(\sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x - x^*)\big) \\ & - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x - x_0)\big) \Big)(x - x^*) \Big\|. \end{aligned} \tag{15}$$

Let us also set

$$F = \Big\| f(x^*) - f(x_0) - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x - x_0)\big)(x^* - x_0) \Big\|$$

and

$$G = \Big\| \Big(\sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x - x^*)\big) - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x - x_0)\big) \Big)(x - x^*) \Big\|.$$

Using (3), we have

$$\begin{aligned} F &= \Big\| \sum_{i=1}^{M} a_i \big(f(x^*) - f(x_0) - \nabla f(x_0 + \beta_i(x - x_0))(x^* - x_0) \big) \Big\| \\ &= \Big\| \sum_{i=1}^{M} a_i \int_0^1 \big(\nabla f(x_0 + t(x^* - x_0)) - \nabla f(x_0 + \beta_i(x - x_0)) \big)(x^* - x_0)\, dt \Big\|. \end{aligned}$$

Thus,

$$F \leq \sum_{i=1}^{M} |a_i| \|x^* - x_0\| \int_0^1 \|\nabla f(x_0 + t(x^* - x_0)) - \nabla f(x_0 + \beta_i(x - x_0))\|\, dt.$$

Since x and x_0 belong to $B_\delta(x^*)$, we obtain

$$F \leq \frac{K_1 \delta^2}{2} \sum_{i=1}^{M} |a_i|(1 + 4|\beta_i|). \tag{16}$$

In a similar way we obtain

$$G \leq \delta^2 K_1 \sum_{i=1}^{M} |a_i|(1+|\beta_i|). \tag{17}$$

Thanks to (15), (16), and (17) one has

$$\|y\| \leq \frac{3K_1\delta^2}{2} \sum_{i=1}^{M} |a_i|(1+2|\beta_i|),$$

and the inequality (11) implies $\|y\| \leq b$.

Let us set $H = e\big(\Psi_0(x) \cap B_{r_0}(x^*), \Psi_0(x')\big)$. It follows that , for all $x, x' \in B_{r_0}(x^*)$, we have

$$\begin{aligned} H &\leq e\big(\Psi_0(x) \cap B_\delta(x^*), \Psi_0(x')\big) \\ &\leq L\Big\| \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x-x^*)\big)(x-x^*) - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x-x_0)\big)(x-x_0) \\ &\quad - \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x'-x^*)\big)(x'-x^*) + \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x'-x_0)\big)(x'-x_0) \Big\|. \end{aligned}$$

Thus,

$$\begin{aligned} H \leq \Big\| &\sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x-x^*)\big)(x-x') + \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x-x^*)\big)(x'-x^*) \\ &- \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x'-x^*)\big)(x'-x^*) - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x-x_0)\big)(x-x') \\ &- \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x-x_0)\big)(x'-x_0) + \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x'-x_0)\big)(x'-x_0) \Big\|, \end{aligned}$$

which yields

$$\begin{aligned} H &\leq L\Big\| \Big(\sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x-x^*)\big) - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x-x_0)\big) \Big)(x-x') \Big\| \\ &\quad + L\Big\| \Big(\sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x-x^*)\big) - \sum_{i=1}^{M} a_i \nabla f\big(x^* + \beta_i(x'-x^*)\big) \Big)(x'-x^*) \Big\| \\ &\quad + \Big\| \Big(\sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x'-x_0)\big) - \sum_{i=1}^{M} a_i \nabla f\big(x_0 + \beta_i(x-x_0)\big) \Big)(x'-x_0) \Big\|. \end{aligned}$$

The last inequality implies

$$H \leq LK_1\delta\Big[\sum_{i=1}^{M}|a_i|(1+4|\beta_i|)\Big]\|x-x'\|,$$

then

$$e\big(\Psi_0(x)\cap B_{r_0}(x^*),\Psi_0(x')\big) \leq LK_1\delta\Big[\sum_{i=1}^{M}|a_i|(1+4|\beta_i|)\Big]\|x-x'\|,$$

and using (11), the condition (ii) in lemma 1.3 is satisfied.

Applying this lemma we get the existence of a fixed point $x_1 \in B_{r_0}(x^*)$ of Ψ_0. $\square$

Moreover, x_1 in the preceding proposition satisfies the inequality (6). Proceeding by induction, assuming that $x_k \in B_{r_{k-1}}(x^*)$, keeping $\eta_k = x^*$, and setting $r_k = c\|x_k - x^*\|^3$, we obtain the existence of a fixed point $x_{k+1} \in B_{r_k}(x^*)$ for Ψ_k. This implies

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^3.$$

In other words, $(x_k)_{k\geq 0}$ is cubically convergent to x^*, which completes the proof of theorem 2.1.

Concluding remarks.

- If $M=1$, $a_1=1$, and $\beta_1=0$, (2) is the Newton-type sequence for solving (1).
- If $M=2$, $a_1=a_2=\frac{1}{2}$, $\beta_1=0$, and $\beta_2=1$, (2) is the sequence studied in [12].

Acknowledgement. The authors thank the anonymous referee for his valuable remarks and comments, which improved the presentation of this manuscript.

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