Cofibrations and Bicofibrations for C^*-Algebras

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ABSTRACT

The paper deals with the correlated concepts of cofibration and bicofibration in C^*-algebra theory. We study cofibrations of C^*-algebras introduced by Claude Schochet in [9] (see also [7]). Cofibrations are characterized by means of the mapping cylinder C^*-algebras. We also define and analyse the notion of bicofibration for C^*-algebras based on the topological model from [8] (see also [5]). As an application, an exact sequence of Čerin's homotopy groups [1] is obtained.

Key words: C^*-algebra, homotopic $*$-homomorphisms, cofibration (bicofibration) of C^*-algebras, mapping cylinder (cone), double mapping cylinder, Čerin's homotopy groups for C^*-algebras.

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Introduction

We recall that a continuous map $f : X' \to X$ is called a *cofibration* if, whenever we are given a space Y, a map $g : X \to Y$ and a homotopy $H : X' \times I \to Y$, starting with $g \circ f$, there is a homotopy $G : X \times I \to Y$ that starts with g, and satisfies $H = G \circ (f \times 1_I)$. A well- known example is that one of the inclusion map $i : L \hookrightarrow K$ for a CW-pair (K, L) (see [6, p. 285]). Secondly every continuous map $f : X \to Y$ can be written as a composition $f = r \circ i$ between a cofibration $i : X \to Z_f$ and a strong deformation retract $r : Z_f \to Y$ (see [10, ch. I, §4]). The notion of cofibration and respectivly the homotopy extension property play an important role in the general homotopy theory (see for example [2, ch. I; 4, ch. 6; 6, ch. 6, §5; 10, ch. 2, §8; 11, ch. I]).

The notion of *bicofibration* was introduced by the first author in [8] and then it was also studied by R. W. Kieboom in [5]. This is a generalization of the topological sum of two spaces and of the joining of complexes. A bicofibration is a pair of cofibration $X_1 \xrightarrow{f_1} X \xleftarrow{f_2} X_2$, either having two retract functions mutually stationary [8], or being strictly separated, which means that there exists a map $u : X \to I$ such that $f_1(X_1) \subset u^{-1}(0)$ and $f_2(X_2) \subset u^{-1}(1)$, see [5].

The idea to consider these notions in noncommutative context came to us in connection with the study of the existence of some homotopy commutative diagrams of $*$-homomorphisms [7]. In [9] the cofibrations were used to define the so-called cofibre homology and cohomology theories.

The aim of the paper is the translation of the usual properties of these structures from the usual case in the language of noncommutative homotopy theory of C^*-algebras. Most of the properties of the usual cofibrations and bicofibrations have interesting statements and require nontrivial proofs in the noncommutative approach. But a series of new results also appears, for example the ones in section 5, connected to the Čerin's homotopy groups [1]. In section 1 we give the definition of cofibrations of C^*-algebras and we establish some general results (Theorem 1.4, Theorem 1.7, Corollary 1.11) which produce a lot of examples. These examples start either from a C^*-algebra and its cylinder, cone and suspension, or from a $*$-homomorphism and its mapping cylinder and mapping cone. In section 2 we prove that a $*$-homomorphism $\phi : A \to B$ is a cofibration if and only if its mapping cylinder M_ϕ is a canonical retract of the cylinder AI (Corollary 2.3). In section 3 a series of properties of cofibrations of C^* -algebras is proved inspired from some results on the topological cofibrations given in the book of I. M. James [4, ch. 6]. Section 4 is devoted to the introduction and study of the notion of bicofibration of C^*-algebras. A series of examples of bicofibrations is given. It is illustrated that not each pair of cofibrations is a bicofibration. It is emphasized that every cofibration $\phi : A \to B$ can be considered as a trivial bicofibration $0 \leftarrow A \xrightarrow{\phi} B$. A characterization of bicofibrations is established on the model of cofibrations by means of a canonical pair retracts (Corollary 4.11). Using this characterization other examples are obtained and, among these, that one for a fixed nuclear C^*-algebra F, the functor $A \to A \otimes_{\min} F$ preserves bicofibrations. In section 5 we establish some properties (Theorem 5.1, Theorem 5.2, Theorem 5.7) in connection with the Čerin's homotopy groups of C^*-algebras [1]. The main result in this section is the construction, for a cofibration $\phi : A \to B$, an arbitrary C^*-algebra K, and an integer $n \geq 0$, of an exact sequence

$$\pi_{n+1}(K; B) \xrightarrow{\partial_*} \pi_n(K; C_\phi) \xrightarrow{\pi(\phi)_*} \pi_n(K; A) \xrightarrow{\phi_*} \pi_n(K; B)$$

of Čerin's homotopy groups. Then this applied to obtain an exact sequence

$$\pi_{n+1}(K; B) \xrightarrow{\partial_*} \pi_n(K; C_\phi) \xrightarrow{i'_*} \pi_n(K; M_\phi) \xrightarrow{\iota_*} \pi_n(K; B)$$

for an arbitrary $*$-homomorphism $\phi : A \to B$.

Notation (cf. [3, ch. I]). By a morphism or a morphism of C^*-algebras we mean a $*$-homomorphism.

Given a C^*-algebra A and a (locally) compact space Y, denote by AY the C^*-algebra of (vanishing at infinity) continuous functions of Y into A. If $\phi : A \to B$ is a $*$-homomorphism and Y is a (locally) compact space, then ϕ induces a $*$-homorphism $\phi Y : AY \to BY$ by $(\phi Y)(u) = \phi \circ u$, $\forall u \in AY$. If $Y = I = [0,1]$, then for every $t \in I$, denote by $\rho_t : AI \to A$ the $*$-homomorphism defined by $\rho_t(u) = u(t)$, $\forall u \in AI$.

Two morphisms of C^*-algebras $\eta : A \to B$ and $\phi : A \to B$ are said to homotopic, written $\eta \overset{h}{\sim} \phi$, if there is a morphism $\Psi : A \to BI$ such that $\rho_0 \circ \Psi = \eta$ and $\rho_1 \circ \Psi = \phi$. The morphism Ψ is called a homotopy (morphism).

A morphism $\eta : A \to B$ is called a homotopy equivalence when there is a morphism $\xi : B \to A$ such that $\xi \circ \eta$ and $\eta \circ \xi$ are homotopic to the respective identity maps of A and B.

If $\eta : A \to B$ and $\xi : B \to A$ are two morphisms such that $\xi \circ \eta = \mathrm{id}_A$ and $\eta \circ \xi \overset{h}{\sim} \mathrm{id}_B$, by a homotopy morphism $\Phi : B \to BI$, such that $\rho_t \circ \Phi \circ \eta = \eta$, $\forall t \in I$, and $\rho_1 \Phi(\ker \xi) = 0$, the C^*-algebra A is called a deformation retract of the C^*-algebra B ([7]; see also [9]).

Given a commutative diagram of $*$-homomorphisms

$$\begin{tikzcd} A_1 \arrow[dd, "\chi"'] \arrow[dr, "\phi_1"] & \\ & B \\ A_2 \arrow[ur, "\phi_2"] & \end{tikzcd}$$

χ is called a morphism over B. If $\chi, \theta : A_1 \to A_2$ are morphisms over B, then a homotopy over B of χ into θ is a homotopy in the ordinary sense which is a morphism over B at each stage of "deformation."

1. Cofibrations: definition and examples

Definition 1.1 ([9], see also [7]). A $*$-homomorphism $\phi : A \to B$ is said to be a cofibration if it satisfies the following ("homotopy lifting") property: for a C^*-algebra D, a $*$-homomorphism $\psi : D \to A$, and a homotopy $*$-homomorphism $\Phi : D \to BI$ of $\phi \circ \psi$, there exists a homotopy $*$-homomorphism $\Psi : D \to AI$ of ψ, such that $\phi I \circ \Psi = \Phi$.

$$\begin{tikzcd} A \arrow[rr, "\phi"] & & B \\ & D \arrow[ul, "\psi"'] \arrow[dl, dotted, "\Psi"] \arrow[dr, "\Phi"] & \\ AI \arrow[uu, "\rho_0"] \arrow[rr, "\phi I"'] & & BI \arrow[uu, "\rho_0"'] \end{tikzcd}$$

Example 1.2. For A, B arbitrary C^*-algebras, the projections $p_A : A \oplus B \to A$ and $p_B : A \oplus B \to B$ are cofibrations.

Consider the projection p_B. First we observe that $(A \oplus B)I \cong AI \oplus BI$ and that the $*$-homomorphism $p_B I$ can be identified with p_{BI}. Then if $\psi : D \to A \oplus B$ is a morphism and $\Phi : D \to BI$, a homotopy of ψ, i.e., $\rho_0 \circ \Phi = p_B \circ \psi$, we can define a homotopy $*$-homomorphism $\Psi : D \to (A \oplus B)I \cong AI \oplus BI$ by $\Psi(d)(t) = (p_A(\psi(d)), \Phi(d)(t))$. For this homotopy we have $\Psi(d)(0) = (p_A(\psi(d)), \Phi(d)(0)) = (p_A(\psi(d)), p_B(\psi(d)) = \psi(d)$, i.e., $\rho_0 \circ \Psi = \psi$, and $(p_B \circ \Psi)(d)(t) = p_B((p_A(\psi(d)), \Phi(d)(t))) = \Phi(d)(t)$, i.e., $p_B \circ \Psi = \Phi$.

Remark 1.3. The example of the above proposition corresponds to the topological cofibrations $i_X : X \to X \vee Y$ and $i_Y : Y \to X \vee Y$, where $X \vee Y$ is the disjoint union of the spaces X and Y.

Afterwards we give two theorems which offer a series of interesting examples of cofibrations.

Theorem 1.4 ([7, 9])**.** *Let $\phi : A \to B$ be an arbitrary $*$-homomorphism with the mapping cylinder C^*-algebra $M_\phi = \{(a, \beta) \in A \oplus BI : \phi(a) = \beta(1)\}$ (*[3, p. 23]*). The map $\iota : M_\phi \to B$, defined by $\iota((a, \beta)) = \beta(0)$, is a cofibration.*

Proof. Suppose that the following diagram is given

$$\begin{tikzcd} M_\phi \arrow[rr, "\iota"] & & B \\ & D \arrow[ul, "\psi"'] \arrow[dr, "\Phi"] & \\ M_\phi I \arrow[uu, "\rho_0"] \arrow[rr, "\iota I"'] & & BI \arrow[uu, "\rho_0"'] \end{tikzcd}$$

and we need to define a homotopy morphism $\Psi : D \to M_\phi I$. for ψ.

If for $d \in D$, $\psi(d) = (a, u), u \in BI$ with

$$u(1) = \phi(a), \tag{1}$$

then $(\iota \circ \psi)(d) = u(0)$. On the other hand, $(\rho_0 \circ \Phi)(d) = \Phi(d)(0)$, hence we have

$$u(0) = \Phi(d)(0). \tag{2}$$

We shall define Ψ as $\Psi(d)(t) = (a, u_t)$, with $u_t \in BI$, satisfying

$$u_t(1) = \phi(a), \tag{3}$$

in order to fulfill $(a, u_t) \in M_\phi$. Moreover the condition $\rho_0 \circ \Psi = \psi$ implies $\Psi(d)(0) = (a, u_0)$, so the equality

$$u_0 = u \tag{4}$$

is necessary. And, finally, since $\iota I \circ \Psi = \Phi$ we have

$$\iota I(\Psi(d))(t) = \Phi(d)(t) \Longrightarrow \iota(\Psi(d))(t) = \Phi(d)(t)$$

so that it is also necessary that the condition

$$u_t(0) = \Phi(d)(t) \tag{5}$$

is fulfilled.

These conditions (1)–(5) are satisfied by the path

$$u_t(\tau) = \begin{cases} \Phi(d)((t - 2\tau)), & 0 \le \tau \le \dfrac{t}{2}, \\ u\left(\dfrac{2\tau - t}{2 - t}\right), & \dfrac{t}{2} \le \tau \le 1. \end{cases}$$

Thus $\iota : M_\phi \to B$ is a cofibration and this finishes the proof. □

Remark 1.5. The above example is inspired from the topological cofibration $i : X \to M_f$, $i(x) = [x, 0]$, for a continuous map $f : X \to Y$ (see [10, ch. I , §4, Th. 12]).

In section 2 the mapping cylinder will be used for a characterization of an arbitrary cofibration.

Remark 1.6. In [7] (see also [9]) there was proved that there exists a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\varsigma"'] & & B \\ & M_\phi \arrow[ur, "\iota"'] & \end{tikzcd}$$

with ς a deformation retract $*$-homomorphism and ι the cofibration from Theorem 1.4.

The following theorem is a slight generalization of [9, Prop. 1.5].

Theorem 1.7. *Consider a commutative diagram of C^*-algebras*

$$
\begin{tikzcd}
P \arrow[r, "\bar{\phi}"] \arrow[d, "\bar{q}"'] & C \arrow[d, "q"] \\
A \arrow[r, "\phi"'] & B
\end{tikzcd}
$$

with the property that the pullback product $*$-morphism $\bar{q} \times_B \bar{\phi} : P \to A \times_B C$ admits a left inverse $\tau : A \times_B C \to P$. In these conditions if ϕ is a cofibration then $\bar{\phi}$ is also a cofibration.

Particularly the pullback of a cofibration ϕ by an arbitrary $*$-morphism q is a cofibration $\bar{\phi}$.

Proof. Suppose that we have a commutative diagram

$$
\begin{tikzcd}
P \arrow[rr, "\bar{\phi}"] & & C \\
& D \arrow[ul, "\bar{\psi}"'] \arrow[dr, "\bar{\Phi}"] & \\
PI \arrow[uu, "\rho_0"] \arrow[rr, "\bar{\phi}I"'] & & CI. \arrow[uu, "\rho_0"']
\end{tikzcd}
$$

Then the following commutative diagram exists:

$$
\begin{tikzcd}
A \arrow[rr, "\phi"] & & B \\
& D \arrow[ul, "\bar{q}\circ\bar{\psi}"'] \arrow[dr, "qI\circ\bar{\Phi}"] & \\
AI \arrow[uu, "\rho_0"] \arrow[rr, "\phi I"'] & & BI. \arrow[uu, "\rho_0"']
\end{tikzcd}
$$

By hypothesis there is a homotopy $\Psi : D \to AI$, with $\rho_0 \circ \Psi = \bar{q} \circ \bar{\psi}$ and $\phi I \circ \Psi = qI \circ \bar{\Phi}$. We need to define an extension homotopy $\bar{\Psi} : D \to PI$ of $\bar{\Phi}$. For this we observe that for each $d \in D$ and $t \in I$ the pair $(\Psi(d)(t), \bar{\Phi}(d)(t)) \in A \times_B C$. Then for the $*$- morphism $\tau : A \times_B C \to P$ we have $\tau((\bar{q}(x), \bar{\phi}(x))) = x$, for any $x \in P$, and $(\bar{\phi} \circ \tau)((a, b)) = b$. Define $\bar{\Psi}(d)(t) = \tau((\Psi(d)(t), \bar{\Phi}(d)(t)))$. This satisfies

$$(\rho_0 \circ \bar{\Psi})(d) = \bar{\Psi}(d)(0) = \tau((\Psi(d)(0), \bar{\Phi}(d)(0))) = \tau(\bar{q}(\bar{\psi}(d)), \bar{\phi}(\bar{\psi}(d))) = \bar{\psi}(d),$$

i.e., $\rho_0 \circ \bar{\Psi} = \bar{\psi}$, and

$$(\bar{\phi} I \circ \bar{\Psi})(d)(t) = \bar{\phi}(\tau((\Psi(d)(t), \bar{\Phi}(d)(t))) = \bar{\Phi}(d)(t)),$$

i.e., $\bar{\phi} I \circ \bar{\Psi} = \bar{\Phi}$. □

We shall also use the following lemma of which proof is immediate.

Lemma 1.8. *Let $\phi : A \to B$ and $\phi' : A' \to B$ be two $*$-homomorphisms such that A and A' are isomorphic over B. Then, if ϕ is a cofibration, ϕ' is also a cofibration.*

Example 1.9. The $*$-homomorphism $\rho_0 : BI \to B$ is a cofibration.

We obtain this by using Theorem 1.4 by taking $\phi = \mathrm{id}_B$, for which $M_\phi \cong BI$, and then the morphism ι can be identified with ρ_0.

Example 1.10. The $*$-homomorphism $\rho_t : BI \to B$ is a cofibration for each $t \in [0, 1]$ (see also [9, Lemma 1. 3]).

To verify this, consider the map $\zeta : BI \to BI$ given by $\zeta(\beta) = \beta'$ with

$$\beta'(\tau) = \begin{cases} \beta(t - \tau), & \text{if} \quad \tau \leq t, \\ \beta(\tau - t), & \text{if} \quad \tau \geq t. \end{cases}$$

This is a $*$-isomorphism over B along the pair (ρ_0, ρ_t). Then we can apply Lemma 1.8 and Example 1.9.

To give other examples of cofibrations, consider two $*$-homomorphisms $B_1 \xrightarrow{\varphi_1} C \xleftarrow{\varphi_2} B_2$ and the double mapping cylinder

$$M_{(\varphi_1,\varphi_2)} = \{\, (b_1, b_2, \gamma) \in B_1 \oplus B_2 \oplus CI \;:\; \gamma(0) = \varphi_1(b_1),\;\; \gamma(1) = \varphi_2(b_2) \,\},$$

see [7].

Corollary 1.11. *The projections $p_i : M_{(\varphi_1,\varphi_2)} \to B_i, p_i((b_1, b_2, \gamma)) = b_i$, $i = 1, 2$, are cofibrations.*

Proof. At first we observe that $M_{(\varphi_1,\varphi_2)}$ is in fact the pullback along the pair of morphisms $\iota : M_{\varphi_2} \to C$, $\varphi_1 : B_1 \to C$ and that p_1 is the pullback projection opposite to ι. Then by applying Theorem 1.4 and Theorem 1.7 we deduce that p_1 is a cofibration. By analogy, the morphism $p'_1 : M_{(\varphi_2,\varphi_1)} \to B_2, p'_1((b_2, b_1, \gamma)) = b_2$ is a cofibration. Then we apply Lemma 1.8 for the morphisms $p_2 : M_{(\varphi_1,\varphi_2)} \to B_2$ and $p'_1 : M_{(\varphi_2,\varphi_1)} \to B_2$. □

Example 1.12 ([9, p. 409]). For any $*$-homomorphism $\phi : A \to B$, the projection $p_A : M_\phi \to A$ is a cofibration.

We apply Corollary 1.11 for the morphisms $B \xrightarrow{\mathrm{id}_B} B \xleftarrow{\phi} A$. Then $M_{(id_B,\phi)} \cong M_\phi$ and the projection $M_{(\mathrm{id}_B,\phi)} \to A$ can be identified with the projection $p_A : M_\phi \to A$.

Example 1.13. If for a morphism $\phi : A \to B$, denote by C_ϕ the mapping cone C^*-algebra of ϕ, i.e.,

$$C_\phi \coloneqq \{(a, \beta) \in A \oplus BI : \beta(1) = \phi(a), \beta(0) = 0\} = \{(a, \beta) \in M_\phi : \beta(0) = 0\},$$

then the projection $\pi(\phi) : C_\phi \to A$, $\pi(\phi)((a, \beta)) = a$, is a cofibration. This results from Corollary 1.11 by taking the pair of morphisms $0 \to B \xleftarrow{\phi} A$. For this we have $M_{(0,\phi)} = \{(0, a, \beta) : \beta(0) = 0, \beta(1) = \phi(a)\} = C_\phi$ and $\pi(\phi)$ is the projection p_2.

Particularly, if CB is the cone algebra over B, i.e.,

$$CB = C_{\mathrm{id}_B} = \{\beta \in BI : \beta(0) = 0\},$$

and then $\rho_1' \coloneqq \rho_1/CB : CB \to B$ is a cofibration.

Example 1.14. If $\phi : A \to B$ is a cofibration then the projection $p_{CB} : C_\phi \to CB$, $p_{CB}((a, \beta)) = \beta$ is also a cofibration. This results from Theorem 1.7 since C_ϕ is the pullback along the morphisms ϕ and ρ_1' and p_{CB} is opposite to ϕ.

Proposition 1.15. *Let $\phi_i : A \to B_i, i = 1, 2$, be $*$-homomorphisms with ϕ_1 a cofibration. Suppose that there exist $f : B_1 \to B_2$ and $g : B_2 \to B_1$ such that $f \circ \phi_1 = \phi_2$, $g \circ \phi_2 = \phi_1$, and $f \circ g = 1_{B_2}$.*

Then ϕ_2 is also a cofibration.

Proof. Let a diagram

$$\begin{tikzcd}
A \arrow[rr, "\phi_2"] & & B_2 \\
& D \arrow[ul, "\psi"'] \arrow[dr, "\Phi"] & \\
AI \arrow[uu, "\rho_0"] \arrow[rr, "\phi_2 I"'] & & B_2 I \arrow[uu, "\rho_0"']
\end{tikzcd}$$

with $\rho_0 \circ \Phi = \phi_2 \circ \psi$ be given. Then there exists the commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\phi_1"] & & B_1 \\
& D \arrow[ul, "\psi"'] \arrow[dl, "\Psi"] \arrow[dr, "(gI)\circ\Phi"] & \\
AI \arrow[uu, "\rho_0"] \arrow[rr, "\phi_1 I"'] & & B_1 I \arrow[uu, "\rho_0"']
\end{tikzcd}$$

with

$$\rho_0 \circ \Psi = \psi \tag{6}$$

and $(\phi_1 I) \circ \Psi = (gI) \circ \Phi$. By this we deduce that

$$(fI) \circ ((\phi_1 I) \circ \Psi) = (fI) \circ ((gI) \circ \Phi) \Longleftrightarrow ((f \circ \phi_1)I) \circ \Psi = ((f \circ g)I) \circ \Phi,$$

i.e.,

$$(\phi_2 I) \circ \Psi = \Phi. \tag{7}$$

Thus, the relations (6) and (7) show that ϕ_2 is a cofibration. □

2. The role of the mapping cylinder in the general case

Theorem 2.1. *A $*$-homomorphism $\phi : A \to B$ is a cofibration if and only if there exists a $*$-homomorphism $r : M_\phi \to AI$ satisfying the following conditions:*

(i) $r((a, \beta))(0) = a$,

(ii) $(\phi I \circ r)((a, \beta)) = \hat{\beta}$, $\forall (a, \beta) \in M_\phi$.

($\hat{\beta}$ denotes the inverse path of β, i.e., $\hat{\beta}(t) = \beta(1 - t)$, $\forall t \in I$).

Proof. Suppose that there exists a $*$-homomorphism $r : M_\phi \to AI$ with the properties (i), (ii).

Let $\psi : D \to A, \Phi : D \to BI$ be $*$-homomorphisms such that $\rho_0 \circ \Phi = \phi \circ \psi$. Thus we have $\Phi(d)(0) = \phi(\psi(d))$ and we can define $\Psi : D \to AI$, by $\Psi(d) = r((\psi(d), \widehat{\Phi(d)}))$.

For this morphism we have

$$(\rho_0 \circ \Psi)(d) = \Psi(d)(0) = r((\psi(d), \widehat{\Phi(d)}))(0) = \psi(d)$$

and

$$(\phi I \circ \Psi)(d) = (\phi I \circ r)((\psi(d), \widehat{\Phi(d)})) = \Phi(d),$$

i.e., $(\phi I) \circ \Psi = \Phi$. Thus ϕ is a cofibration.

Conversely, suppose that ϕ is a cofibration . Consider $D = M_\phi$ and $\psi : D \to A$, $\Phi : D \to BI$ defined by $\psi((a, \beta)) = a$, and $\Phi((a, \beta)) = \hat{\beta}$, $\forall (a, \beta) \in M_\phi$. Then

$$(\rho_0 \circ \Phi)((a, \beta)) = \Phi((a, \beta))(0) = \hat{\beta}(0) = \beta(1) = \phi(a) = (\phi \circ \psi)((a, \beta)),$$

i.e., $\rho_0 \circ \Phi = \psi$ and this implies that there exists $\Psi : M_\phi \to AI$, with $\Psi((a, \beta))(0) = \psi((a, \beta)) = a$ and $(\phi I \circ \Psi)((a, \beta)) = \Phi((a, \beta)) = \hat{\beta}$. Thus $r = \Psi$ verifies the conditions (i), (ii). □

We can formulate this characterization of cofibrations also in terms of retracts, as follows.

Definition 2.2. For a $*$-homomorphism $\phi : A \to B$ we can define a morphism $\varkappa : AI \to M_\phi$ by $\varkappa(\alpha) = (\alpha(0), \phi \circ \hat{\alpha})$. We say that M_ϕ is a "canonical retract" of AI if there exists a $*$-homomorphism $\gamma : M_\phi \to AI$ such that $\varkappa \circ \gamma = 1_{M_\phi}$.

Corollary 2.3. *A $*$-homomorphism $\phi : A \to B$ is a cofibration if and only if M_ϕ is a "canonical retract" of AI.*

Proof. Suppose that ϕ is a cofibration and $r : M_\phi \to AI$ is the $*$-homomorphism from Theorem 2.1. Then if we put $\gamma = r$, we have

$$(\varkappa \circ \gamma)((a,\beta)) = (r((a,\beta))(0), \phi \circ \widehat{r((a,\beta))}) = (a, \widehat{\phi \circ r((a,\beta))}) = (a,\beta) \\ \Longrightarrow \varkappa \circ \gamma = 1_{M_\phi}.$$

Conversely, suppose that there is a retraction γ, as above. Then if $(a,\beta) \in M_\phi$,

$$(a,\beta) = (\varkappa \circ \gamma)((a,\beta)) = (\gamma((a,\beta))(0), \widehat{\phi \circ \gamma((a,\beta))}) \Longrightarrow \gamma((a,\beta))(0) = a,$$

and $\widehat{\phi \circ \gamma((a,\beta))}) = \beta$. Therefore, if we put $r = \gamma$, the conditions of Theorem 2.1 are verified and thus ϕ is a cofibration. $\square$

Remark 2.4. In [9, Prop. 1.10] a variant of Corollary 2.3 also exists.

Corollary 2.5. *A composition of two cofibrations is also a cofibration.*

Proof. Let $\phi_1 : A \to B$, $\phi_2 : B \to C$ be cofibrations with canonical retracts $r_1 : M_{\phi_1} \to AI$ and, respectively, $r_2 : M_{\phi_2} \to BI$. Then we can define $r : M_{\phi_2 \circ \phi_1} \to AI$ by $r((a,\gamma)) = r_1((a, \widehat{r_2((\phi_1(a),\gamma)}))$, which is a canonical retract. $\square$

Corollary 2.6. *If $\phi : A \to B$ is a cofibration, then $\phi I : AI \to BI$ is also a cofibration.*

Proof. $M_{\phi I} = \{(\alpha, F) \in AI \oplus (BI)I : F(1) = \phi \circ \alpha\}$ and $\varkappa_{\phi I} : (AI)I \to M_{\phi I}$, $\varkappa_{\phi I}(G) = (G(0), \phi I \circ \hat{G})$.

If $r : M_\phi \to AI$ is a canonical retract for ϕ, we can obtain a morphism $R : M_{\phi I} \to (AI)I$. If $(\alpha, F) \in M_{\phi I}$, and $t \in I$, considering $\beta_t \in BI$ with $\beta_t(t') = F(t')(t)$. Then $\beta_t(1) = F(1)(t) = \phi(\alpha(t))$, which implies that $(\alpha(t), \beta_t) \in M_\phi$.

We define $R((\alpha,F))(t')(t) = r((\alpha(t),\beta_t))(t')$. This morphism satisfies $R((\alpha,F))(0)(t) = r((\alpha(t),\beta_t))(0) = \alpha(t)$ and

$$(\phi I \circ \widehat{R((\alpha,F))})(t')(t) = (\phi \circ \widehat{r((\alpha(t),\beta_t))})(t') = \beta_t(t') = F(t')(t) \\ \Longrightarrow \phi I \circ \widehat{R((\alpha,F))} = F.$$

These relations show that R is a canonical retract. $\square$

The proof of Corollary 2.6 can be adapted to obtain the following corollary.

Corollary 2.7. *If $\phi : A \to B$ is a cofibration, then $C(\phi) : CA \to CB$ is also a cofibration.*

3. Other properties of the cofibrations [4]

The following theorem is inspired from some results on the topological cofibrations given in the book of I. M. James [4, ch. 6].

Theorem 3.1.

(i) *A cofibration of C^*-algebras is a surjective $*$-homomorphism.*

(ii) *Let $\phi_1 : A_1 \to B$ be a cofibration and $\phi_2 : A_2 \to B$ an arbitrary morphism. Let $\chi : A_2 \to A_1$ be a morphism such that $\phi_1 \circ \chi \overset{h}{\sim} \phi_2$. Then $\chi \overset{h}{\sim} \chi'$ for $\chi' : A_2 \to A_1$ a morphism over B.*

(iii) *If a cofibration $\phi : A \to B$ admits a right inverse up to homotopy then ϕ admits a right inverse.*

(iv) *Let $\phi : A \to B$ be a cofibration. Let $\theta : A \to A$ a morphism over B, and suppose that $\theta \overset{h}{\sim} 1_A$. Then there exists a morphism $\theta' : A \to A$ over B such that $\theta \circ \theta' \overset{h}{\sim} 1_A$ over B.*

(v) *Let $\phi_i : A_i \to B$, $i = 1, 2$, be cofibrations. Let $\gamma : A_2 \to A_1$ a morphism over B. Suppose that γ, as an ordinary morphism, is a homotopy equivalence. Then γ is a homotopy equivalence over B.*

(vi) *If a cofibration $\phi : A \to B$ admits a right inverse $\phi' : B \to A$ and it is a homotopy equivalence then ϕ is a homotopy equivalence over B.*

Proof. (i) Consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\phi"] & & B \\
& M_\phi \arrow[ul, "p_A"'] \arrow[dl, "\Psi"] \arrow[dr, "\Phi"] & \\
AI \arrow[uu, "\rho_0"] \arrow[rr, "\phi I"'] & & BI \arrow[uu, "\rho_0"']
\end{tikzcd}$$

with $p_A((a,\beta)) = a, \Phi((a,\beta)) = \hat{\beta}$, satisfying $\phi \circ p_A = \rho_0 \circ \Phi$, and $\rho_0 \circ \Psi = p_A$, $\phi I \circ \Psi = \Phi$. The last relation implies $\phi(\Psi((a,\beta))(1)) = \beta(0)$ for each pair $(a,\beta) \in M_\phi$. If $b \in B$ is an arbitrary element, consider the path $\beta_b \in BI$, defined by $\beta_b(t) = (1-t)b$, for any $t \in I$. Then $(0_A, \beta_b) \in M_\phi$ since $\phi(0_A) = 0_B = \beta_b(1)$. Thus we can write $b = \beta_b(0) = \phi(\Psi((0_A, \beta_b))(1))$, i.e., $b \in \operatorname{Im}\phi$.

(ii) Let $\Phi : A_2 \to BI$ be a homotopy of $\phi_1 \circ \chi$ into ϕ_2. Since $\rho_0 \circ \Phi = \phi_1 \circ \chi$ and ϕ_1 is a cofibration there exists a homotopy $\Psi : A_2 \to A_1 I$ with $\rho_0 \circ \Psi = \chi$ and

$(\phi_1 I) \circ \Psi = \Phi$. Taking χ' to be $\rho_1 \circ \Psi$, we have $\chi' \overset{h}{\sim} \chi$ and

$$\phi_1 \circ \chi' = \phi_1 \circ \rho_1 \circ \Psi = \rho_1 \circ \Phi = \phi_2.$$

(iii) This assertion is a special case of (ii) for $\phi_1 = \phi : A \to B, \phi_2 = 1_B$ and χ a homotopic right inverse of ϕ. Then $\chi \overset{h}{\sim} \chi'$ for a morphism $\chi' : B \to A$ over B. This means that $\phi \circ \chi' = 1_B$.

(iv) Let $\Phi : A \to AI$ be a homotopy of θ with 1_A, i.e., $\rho_0 \circ \Phi = \theta$ and $\rho_1 \circ \Phi = 1_A$. The property of the $*$-morphism θ to be over B is expressed by the relation $\phi \circ \theta = \phi$. Then the $*$-homotopy $\phi I \circ \Phi : A \to BI$ satisfies the relation

$$\rho_0 \circ (\phi I \circ \Phi) = \phi \circ (\rho_0 \Phi) = \phi \circ \theta = \phi.$$

Since ϕ is a cofibration, there exists a $*$-homotopy $\Psi : A \to AI$ such that $\rho_0 \circ \Psi = 1_A$ and $\phi I \circ \Psi = \phi I \circ \Phi$. Define $\theta' = \rho_1 \circ \Psi$. For this we have

$$\phi \circ \theta' = \phi \circ \rho_1 \circ \Psi = \phi \circ \rho_0 \circ \Psi = \phi \circ \theta = \phi$$

and $\theta' \overset{h}{\sim} 1_A$. We shall prove that $\theta \circ \theta' \overset{h}{\sim} 1_A$ over B. A simple homotopy of these morphisms is $\Gamma : A \to AI$, being defined by

$$\Gamma(a)(t) = \begin{cases} \theta((\Psi(a)(1-2t)), & 0 \le t \le 1/2, \\ \Phi(a)(2t-1), & 1/2 \le t \le 1, \end{cases} \quad \rho_0 \circ \Gamma = \theta \circ \theta', \quad \rho_1 \circ \Gamma = 1_A.$$

But this is not a $*$-homotopy over B since

$$(\phi \circ \Gamma)(a)(t) = \begin{cases} \phi((\Phi(a)(1-2t)), & 0 \le t \le 1/2, \\ \phi(\Phi(a)(2t-1)), & 1/2 \le t \le 1, \end{cases} \quad \phi \circ \Gamma_t \ne \phi.$$

We shall replace this $*$- homotopy Γ by a $*$-homotopy of $\theta \circ \theta'$ with 1_A over B. For this we consider first a homotopy $\Lambda : A \to (BI)I$ defined by

$$\Lambda(a)(t)(t') = \begin{cases} \phi((\Phi(a)(1-2t'(1-t)), & 0 \le t' \le \frac{1}{2}, \quad t \in I \\ \phi(\Phi(a)(1-2(1-t')(1-t))), & \frac{1}{2} \le t' \le 1, \quad t \in I \end{cases}$$

Then $\rho_0 \circ \Lambda = (\phi I) \circ \Gamma$ and since ϕI is a cofibration (Corollary 2.6) there exists a homotopy $\Lambda' : A \to (AI)I$ with $\rho_0 \circ \Lambda' = \Gamma$ and $((\phi I)I) \circ \Lambda' = \Lambda$. Then

$$\theta \circ \theta' = \rho_0 \circ \Gamma = \rho_0 \circ \rho_0 \circ \Lambda' \overset{h}{\sim} \rho_1 \circ \rho_0 \circ \Lambda' \overset{h}{\sim} \rho_1 \circ \rho_0 \circ \Lambda' = \rho_1 \circ \Gamma = 1_A,$$

all homotopies being over B.

(v) Let $\gamma' : A_1 \to A_2$ be a homotopy inverse of γ, as an ordinary morphism. Then $\phi_2 \circ \gamma' = \phi_1 \circ \gamma \circ \gamma' \overset{h}{\sim} \phi_1$. By (i), $\gamma' \overset{h}{\sim} \gamma''$ for some morphism $\gamma'' : A_1 \to A_2$

over B. Since $\gamma \circ \gamma'' \overset{h}{\sim} 1_{A_1}$ and, since $\gamma \circ \gamma''$ is over B, by (iii) there exists a morphism $\delta : A_1 \to A_1$ over B such that $\gamma \circ \gamma'' \circ \delta \overset{h}{\sim} 1_{A_1}$ over B. Thus γ admits a homotopy right inverse $\tilde{\gamma} = \gamma'' \circ \delta$ over B.

Now $\tilde{\gamma}$ is a homotopy equivalence, since γ is a homotopy equivalence, and so the same argument, applied to $\tilde{\gamma}$ instead of γ, shows that $\tilde{\gamma}$ admits a homotopy right inverse $\tilde{\tilde{\gamma}}$ over B. Thus $\tilde{\gamma}$ admits both a homotopy left inverse γ over B and a homotopy right inverse $\tilde{\tilde{\gamma}}$ over B. Hence $\tilde{\gamma}$ is a homotopy equivalence over B, and so γ itself is a homotopy equivalence over B, as asserted.

(vi) If $\phi \circ \phi' = 1_B$ we have that ϕ' is a morphism over B. And if ϕ is a homotopy equivalence we can suppose that ϕ'.is a homotopy equivalence. Then we apply (v) for $\phi_1 = \phi$, $\phi_2 = 1_B$, and $\gamma = \phi'$. Therefore ϕ' is a homotopy equivalence over B, and so ϕ itself is a homotopy equivalence over B. □

4. Bicofibrations

In this part of the paper our notion of bicofibration and also some properties of this structure are a noncommutative version of the notion of (topological) bicofibration [8] and of some properties of this given in [5].

Definition 4.1. A pair of $*$-homomorphisms $\phi_i : A \to B_i$, $i = 1, 2$, is a bicofibration of C^*-algebras if given a $*$-homomorphism $\psi : D \to A$ and homotopy $*$-homomorphisms $\Phi_i : D \to B_i I, i = 1, 2$, satisfying $\rho_0 \circ \Phi_i = \phi_i \circ \psi$, $i = 1, 2$, there exist homotopy $*$-homomorphisms $\Psi_i : D \to AI$, $i = 1, 2$, such that:

(i) $\rho_0 \circ \Psi_i = \psi$, $i = 1, 2$,

(ii) $\phi_i I \circ \Psi_i = \Phi_i$, $i = 1, 2$, and

(iii) $(D \xrightarrow{\Psi_1} AI \xrightarrow{\rho_t} A \xrightarrow{\phi_2} B_2) = (D \xrightarrow{\psi} A \xrightarrow{\phi_2} B_2)$, $\forall t \in I$,

(iv) $(D \xrightarrow{\Psi_2} AI \xrightarrow{\rho_t} A \xrightarrow{\phi_1} B_1) = (D \xrightarrow{\psi} A \xrightarrow{\phi_1} B_1)$, $\forall t \in I$.

$$
\begin{tikzcd}
B_1 & A \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] & B_2 \\
 & D \arrow[u, "\psi"'] \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] \arrow[d, shift right, "\Psi_1"'] \arrow[d, shift left, "\Psi_2"] & \\
B_1 I \arrow[uu, "\rho_0"] & AI \arrow[l, "\phi_1 I"] \arrow[r, "\phi_2 I"'] \arrow[d, "\rho_t"] & B_2 I \arrow[uu, "\rho_0"'] \\
B_1 & A \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] & B_2
\end{tikzcd}
$$

Example 4.2. Let $\phi_i : A_i \to B_i$, $i = 1, 2$, be cofibrations. Define $\phi_i' : A_1 \oplus A_2 \to B_i$ by $\phi_i' = \phi_i \circ p_i, i = 1, 2$, where $p_i : A_1 \oplus A_2 \to A_i$ are the sum projections. Then the pair of $*$-homomorphisms $B_1 \xleftarrow{\phi_1'} A_1 \oplus A_2 \xrightarrow{\phi_2'} B_2$ constitutes a bicofibration.

Particularly, for two arbitrary C^*-algebras A_i, $i = 1, 2$, the pair of the projections $A_1 \xleftarrow{p_1} A_1 \oplus A_2 \xrightarrow{p_2} A_2$ is a bicofibration.

To see this, let $\psi : D \to A_1 \oplus A_2$ be a $*$-homomorphism and homotopy morphisms $\Phi_i : D \to B_i I$,with $\rho_0 \circ \Phi_i = \phi_i' \circ \psi, i = 1, 2$. Consider $\psi_i : D \to A_i, \psi_i = p_i \circ \psi$, $i = 1, 2$. Because

$$\rho_0 \circ \Phi_i = \phi_i' \circ \psi = \phi_i' \circ ((p_1\psi, p_2\psi)) = \phi_i \circ \psi_i, \quad i = 1, 2,$$

there exist $\Psi_i : D \to A_i I$, with $\rho_0 \circ \Psi_i = \psi_i$ and $(\phi_i I) \circ \Psi_i = \Phi_i$. Consider $\Psi_i' : D \to (A_1 \oplus A_2)I = A_1 I \oplus A_2 I$, $i = 1, 2$, defined by $\Psi_1'(d) = (\Psi_1(d), \psi_2(d))$ and $\Psi_2'(d) = (\psi_1(d), \Psi_2(d))$. Then we have

$$\rho_0 \circ \Psi_1' = (\rho_0 \circ \Psi_1, \psi_2) = (\psi_1, \psi_2) = \psi,$$
$$\rho_0 \circ \Psi_2' = (\psi_1, \rho_0 \circ \Psi_2) = (\psi_1, \psi_2) = \psi,$$

and

$$(\phi_1' I) \circ \Psi_1' = (\phi_1 I \circ p_1 I) \circ (\Psi_1, \psi_2) = \phi_1 I \circ \Psi_1 = \Phi_1,$$

and analogously $(\phi_2' I) \circ \Psi_2' = \Phi_2$. Moreover, we have

$$\begin{aligned} \phi_2' \circ \rho_t \circ \Psi_1' &= \phi_2 \circ p_2 \circ \rho_t \circ (\Psi_1, \psi_2) = \phi_2 \circ p_2 \circ (\rho_t \circ \Psi_1, \psi_2) \\ &= \phi_2 \circ \psi_2 = \phi_2 \circ p_2 \circ \psi = \phi_2' \circ \psi \end{aligned}$$

and analogously $\phi_1' \circ \rho_t \circ \Psi_2' = \phi_1' \circ \psi$.

Example 4.3. Let $\phi : A \to B$ be a $*$-homomorphism, M_ϕ the mapping cylinder of ϕ and the $\iota : M_\phi \to B, p_A : M_\phi \to A$ the maps $\iota((a, \beta)) = \beta(0)$ (Theorem 1.4), resp. $p_A((a, \beta)) = a$ (Example 1.12). Then the pair $A \xleftarrow{p_A} M_\phi \xrightarrow{\iota} B$ is a bicofibration.

To see this, suppose that $\psi : D \to M_\phi$ and $\Phi_A : D \to AI, \Phi : D \to BI$ are given such that $\rho_0 \circ \Phi_A = p_A \circ \psi$ and $\rho_0 \circ \Phi = \iota \circ \psi$. At first we denote by $\Psi : D \to M_\phi I$ the homotopy from the proof of Theorem 1.4. Then

$$(p_A \circ \rho_t \circ \Psi)(d) = p_A(\psi(d)(t)) = p_A((a, u_t)) = a = p_A((a, u)) = p_A(\psi(d)).$$

Hence $p_A \circ \rho_t \circ \Psi = p_A \circ \psi$.

Then if $\psi(d) = (a_d, \beta_d)$, define the homotopy $\Psi_A : D \to M_\phi I$ by $\Psi_A(d)(t) = (\Phi_A(d)(t), \beta_{d,t})$, with $\beta_{d,t} \in BI$ given by

$$\beta_{d,t}(\tau) = \begin{cases} \beta_d(0), & \text{if} \quad 0 \le \tau \le \frac{t}{3}, \\ \beta_d\left(\frac{3\tau - t}{3 - 2t}\right), & \text{if} \quad \frac{t}{3} \le \tau \le 1 - \frac{t}{3}, \\ \phi(\Phi_A(d)(t + 3\tau - 3)), & \text{if} \quad 1 - \frac{t}{3} \le \tau \le 1. \end{cases}$$

Then Ψ_A is a homotopy well defined which verifies the conditions

$$\begin{aligned}(\rho_0\circ\Psi_A)(d)&=\Psi_A(d)(0)=(\Phi_A(d)(0),\beta_{d,0})=(a_d,\beta_d)=\psi(d),\\(p_AI\circ\Psi_A)(d)(t)&=p_A(\Psi_A(d)(t))=p_A(\Phi_A(d)(t),\beta_{d,t})=\Phi_A(d)(t),\end{aligned}$$

and

$$\begin{aligned}(\iota\circ\rho_t\circ\Psi_A)(d)&=\iota(\Psi_A(d)(t))=\iota((\Phi_A(d)(t),\beta_{d,t}))=\beta_{d,t}(0)\\&=\beta_d(0)=(\iota\circ\psi)(d)\end{aligned}$$

Thus the homotopies Ψ and Ψ_A verify the conditions (i)–(iv) from Definition 4.1.

Proposition 4.4. *The pair of $*$-homomorphisms $A\xleftarrow{\rho_0}AI\xrightarrow{\rho_1}A$ is a bicofibration.*

Proof. Let $\psi: D\to AI$ be a $*$-homorphism and homotopy morphisms $\Phi_i: D\to AI$, $i=0,1$, with $\rho_0\circ\Phi_0=\rho_0\circ\psi$ and $\rho_0\circ\Phi_1=\rho_1\circ\psi$. At first, we define $\Psi_0: D\to(AI)I$ by

$$\Psi_0(d)(t)(\tau)=\begin{cases}\Phi_0(d)(t-2\tau), & 0\le\tau\le\frac{t}{2},\\ \psi(d)\left(\frac{2\tau-t}{2-t}\right), & \frac{t}{2}\le\tau\le 1.\end{cases}$$

This homotopy $*$-homomorphism verifies $\rho_0\circ\Psi_0=\psi$, $\rho_0I\circ\Psi_0=\Phi_0$, and

$$(\rho_1\circ\rho_t\circ\Psi_0)(d)=\rho_1(\Psi_0(d)(t))=\Psi_0(d)(t)(r)=\psi(d)(1)=(\rho_1\circ\psi)(d),\quad\forall d\in D.$$

Then we define $\Psi_1: D\to(AI)I$ as follows. At first consider $\Psi': D\to(AI)I$ the analogous to the morphism Ψ_0 defined for $\Upsilon\circ\psi: D\to AI$ instead of ψ, and Φ_1 instead of Φ_0, where $\Upsilon: AI\to AI$ is the morphism $\Upsilon(\alpha)=\hat{\alpha}$. For this we have $\rho_0\circ\Psi'=\Upsilon\circ\psi$, $\rho_0I\circ\Psi'=\Phi_1$, and $\rho_1\circ\rho_t\circ\Psi'=\rho_1\circ(\Upsilon\circ\psi)=\rho_0\circ\psi$. Then we define $\Psi_1=\Upsilon I\circ\Psi'$. For this we can verify the relations

$$\begin{aligned}\rho_0\circ\Psi_1&=\rho_0\circ\Upsilon I\circ\Psi'=\Upsilon\circ\rho_0\circ\Psi'=\Upsilon\circ\Upsilon\circ\psi=\psi,\\ \rho_1I\circ\Psi_1&=\rho_1I\circ\Upsilon I\circ\Psi'=(\rho_1\circ\Upsilon)I\circ\Psi'=\rho_0I\circ\Psi'=\Phi_1,\end{aligned}$$

and

$$\rho_0\circ\rho_t\circ\Psi_1=\rho_1\circ\Upsilon\circ\rho_t\circ\Psi_1=\rho_1\circ\Upsilon\circ\rho_t\circ\Upsilon I\circ\Psi'=\rho_1\circ\rho_t\circ\Psi'=\rho_0\circ\psi.$$

Thus we have verified all conditions from Definition 4.1. □

Remark 4.5. If we replace above ρ_1 by ρ_r with $r\in(0,1)$ then the condition $\rho_r\circ\rho_t\circ\Psi_0=\rho_r\circ\psi$ is not verified. Thus the pair $A\xleftarrow{\rho_0}AI\xrightarrow{\rho_r}A$ may not be a cofibration.

Proposition 4.6. *Let $B_1\xleftarrow{\varphi_1}A\xrightarrow{\varphi_2}B_2$ be $*$-homomorphisms. Consider the following C^*-algebra*

$$Z_{(\varphi_1,\varphi_2)}=\{\,(a,\beta_1,\beta_2)\in A\oplus B_1I\oplus B_2I\ :\ \beta_i(1)=\varphi_i(a),\ \ i=1,2\,\}.$$

and the $*$-homomorphisms $\phi_i: Z_{(\phi_1,\phi_2)}\to B_i$, $i=1,2$, with $\phi_i((a,\beta_1,\beta_2))=\beta_i(0)$. Then $B_1\xleftarrow{\phi_1}Z_{(\phi_1,\phi_2)}\xrightarrow{\phi_2}B_2$ is a bicofibration.

Proof. Let $\psi : D \to Z_{(\phi_1,\phi_2)}$ be an arbitrary $*$-homomorphism and $\Phi_i : D \to B_i I$, $i = 1, 2$, homotopy morphisms with $\rho_0 \circ \Phi_i = \phi_i \circ \psi$. We need to define some homotopies $\Psi_i : D \to Z_{(\phi_1,\phi_2)} I$, $i = 1, 2$, for ψ. If $\psi(d) = (a, \beta_1, \beta_2)$, we shall define $\Psi_1(d)(t) = (a, \beta_{1t}, \beta_2)$, $\Psi_2(d)(t) = (a, \beta_1, \beta_{2t})$, where

$$\beta_{it}(\tau) = \begin{cases} \Phi_i(d)((t - 2\tau)), & 0 \le \tau \le \frac{t}{2}, \\ \beta_i\left(\frac{2\tau - t}{2 - t}\right), & \frac{t}{2} \le \tau \le 1, \end{cases} \qquad i = 1, 2.$$

This path is well defined since $\Phi_i(d)(0) = \phi_i(\psi(d)) = \beta_i(0)$.

Moreover $(a, \beta_{1t}, \beta_2), (a, \beta_1, \beta_{2t}) \in Z_{(\phi_1,\phi_2)}$ since $\beta_{it}(1) = \beta_i(1) = \varphi_i(a)$ and $\beta_i(1) = \varphi_i(a)$. For these homotopy $*$-homomorphisms Ψ_i we have

$$\Psi_1(d)(0) = (a, \beta_{10}, \beta_2) = (a, \beta_1, \beta_2) = \psi(d),$$
$$(\phi_1 I) \circ \Psi_1(d)(t) = \phi_1((a, \beta_{1t}, \beta_2)) = \beta_{1t}(0) = \Phi_1(d)(t) \Longrightarrow (\phi_1 I) \circ \Psi_1 = \Phi_1.$$

Analogously $\rho_0 \circ \Psi_2 = \psi$ and $(\phi_2 I) \circ \Psi_2 = \Phi_2$.

Moreover $(\phi_2 \circ \rho_t \circ \Psi_1)(d) = \phi_2((a, \beta_{1t}, \beta_2)) = \beta_2(0) = \phi_2(\psi(d))$, i.e., $\phi_2 \circ \rho_t \circ \Psi_1 = \phi_2 \circ \psi$. Similarly $\phi_1 \circ \rho_t \circ \Psi_2 = \phi_1 \circ \psi$. □

Proposition 4.7. *If $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ is a bicofibration then every $*$-homomorphism ϕ_i, $i = 1, 2$, is a cofibration.*

Proof. Suppose that $\psi : D \to A$ is a $*$-homomorphism and $\Phi : D \to B_1 I$ a homotopy for $\phi_1 \circ \psi$. Consider $\Phi_1 = \Phi$ and $\Phi_2 : D \to B_2 I$, the constant homotopy, i.e., $\rho_t \circ \Phi_2 = \phi_2 \circ \psi$. Then there exists $\Psi_1 : D \to AI$, such that $\rho_0 \circ \Psi_1 = \psi$ and $\phi_1 I \circ \Psi_1 = \Phi_1 = \Phi$. □

Corollary 4.8. *A $*$-homomorphism $\phi : A \to B$ is a cofibration if and only if the pair $0 \leftarrow A \xrightarrow{\phi} B$ is a bicofibration. Thus every cofibration can be considered as a particular bicofibration.*

Proof. Apply Example 4.2 and Proposition 4.7. □

Remark 4.9. An example of pair of cofibrations which is not a bicofibration is a pair $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{\phi} B$ with ϕ an arbitrary cofibration.

Theorem 4.10. *A pair of $*$-homomorphisms $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ is a bicofibration if and only if there exist $*$-homorphisms $r_i : Z_{(\phi_1,\phi_2)} \to AI$, $i = 1, 2$, verifying the following conditions:*

(i) $r_i((a, \beta_1, \beta_2))(0) = a$, $i = 1, 2$.

(ii) $(\phi_i I \circ r_i)((a, \beta_1, \beta_2)) = \widehat{\beta_i}$, $\forall (a, \beta_1, \beta_2) \in Z_{(\phi_1,\phi_2)}$, $i = 1, 2$.

(iii) $(\phi_2 \circ \rho_t \circ r_1)((a, \beta_1, \beta_2)) = \phi_2(a)$, $\forall (a, \beta_1, \beta_2) \in Z_{(\phi_1,\phi_2)}$ *and* $(\phi_1 \circ \rho_t \circ r_2)((a, \beta_1, \beta_2)) = \phi_1(a)$, $\forall (a, \beta_1, \beta_2) \in Z_{(\phi_1,\phi_2)}$.

Proof. Suppose there exist $*$-homomorphisms $r_i : Z_{(\phi_1,\phi_2)} \to AI$, $i = 1, 2$, with the properties (i)–(iii). We proceed as in the proof of Theorem 2.1. Let $\psi : D \to A$ and homotopy morphisms $\Phi_i : D \to B_iI$, with $\rho_0 \circ \Phi_i = \phi_i \circ \psi$, $i = 1, 2$. Define $\Psi_i : D \to AI$, $i = 1, 2$, by $\Psi_i(d) = r_i((\psi(d), \widehat{\Phi_1(d)}, \widehat{\Phi_2(d)}))$. Then $\rho_0 \circ \Psi_i = \psi$ and $(\phi_i I) \circ \Psi_i = \Phi_i$.

Moreover,

$$(\phi_2 \circ \rho_t \circ \Psi_1)(d) = (\phi_2 \circ \rho_t \circ r_1)((\psi(d), \widehat{\Phi_1(d)}, \widehat{\Phi_2(d)})) = (\phi_2 \circ \psi)(d),$$

i.e., $\phi_2 \circ \rho_t \circ \Psi_1 = \phi_2 \circ \psi$ and analogously $\phi_1 \circ \rho_t \circ \Psi_2 = \phi_1 \circ \psi$.

Conversely, suppose that $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ is a bicofibration. Consider $D = Z_{(\phi_1,\phi_2)}$ and $\psi : D \to A$, $\Phi_i : D \to B_iI$, $i = 1, 2$, defined by $\psi((a, \beta_1, \beta_2)) = a$ and $\Phi_i((a, \beta_1, \beta_2)) = \widehat{\beta_i}$, $\forall (a, \beta_1, \beta_2) \in Z_{(\phi_1,\phi_2)}$. Then

$$\begin{aligned}(\rho_0 \circ \Phi_i)((a, \beta_1, \beta_2)) &= \Phi_i((a, \beta_1, \beta_2))(0) = \widehat{\beta_i}(0) \\ &= \beta_i(1) = \phi_i(a) = (\phi_i \circ \psi)((a, \beta_1, \beta_2)),\end{aligned}$$

i.e., $\rho_0 \circ \Phi_i = \psi$, $i = 1, 2$, and this implies that there exist $\Psi_i : Z_{(\phi_1,\phi_2)} \to AI$, $i = 1, 2$, with

$$\Psi_i((a, \beta_1, \beta_2))(0) = \psi((a, \beta_1, \beta_2)) = a,$$
$$(\phi_i I \circ \Psi_i)((a, \beta_1, \beta_2)) = \Phi_i((a, \beta_1, \beta_2)) = \widehat{\beta_i}.$$

Moreover

$$(\phi_2 \circ \rho_t \circ \Psi_1)((a, \beta_1, \beta_2)) = (\phi_2 \circ \psi)((a, \beta_1, \beta_2)) = \phi_2(a)$$

and

$$(\phi_1 \circ \rho_t \circ \Psi_2)((a, \beta_1, \beta_2)) = (\phi_1 \circ \psi)((a, \beta_1, \beta_2)) = \phi_1(a).$$

Thus if we put $r_i = \Psi_i, i = 1, 2$, the conditions (i)–(iii) are fulfilled. □

Corollary 4.11. *A pair of $*$-homomorphisms $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ is a bicofibration if and only if there exist canonical retracts $\gamma_i : M_{\phi_i} \to AI$, $i = 1, 2$, such that $(\phi_2 \circ \rho_t \circ \gamma_1)((a, \beta_1)) = \phi_2(a), \forall (a, \beta_1) \in M_{\phi_1}$ and $(\phi_1 \circ \rho_t \circ \gamma_2)((a, \beta_2)) = \phi_1(a)$, $\forall (a, \beta_2) \in M_{\phi_2}$.*

Proof. Suppose that $B_1 \xrightarrow{\phi_1} A \xleftarrow{\phi_2} B_2$ is a bicofibration and consider $r_i : Z_{(\phi_1,\phi_2)} \to AI$, $i = 1, 2$, as in Theorem 4.10.

Define $\gamma_i : M_{\phi_i} \to AI$, $i = 1, 2$, in the following way:

$$\gamma_1((a, \beta_1)) = r_1((a, \beta_1, \phi_2(a)), \quad \forall (a, \beta_1) \in M_{\phi_1}$$

and

$$\gamma_2((a,\beta_2)) = r_2((a,\phi_1(a),\beta_2)), \quad \forall (a,\beta_2) \in M_{\phi_2},$$

where $\phi_2(a)$ and $\phi_1(a)$ mean the constant paths here.

Then if $\varkappa_i : AI \to M_{\phi_i}$, $i = 1,2$, denote the $*$-homomorphisms $\varkappa_i(\alpha) = (\alpha(0), \phi_i \circ \hat{\alpha})$, we have

$$\begin{aligned}(\varkappa_1 \circ \gamma_1)((a,\beta_1)) &= \varkappa_1(r_1((a,\beta_1,\phi_2(a))) \\ &= (r_1((a,\beta_1,\phi_2(a))(0), \phi_1 \circ \widehat{r_1((a,\beta_1,\phi_2(a))}) \\ &= (a, \widehat{\phi_1 \circ r_1((a,\beta_1,\phi_2(a))}) = (a,\beta_1),\end{aligned}$$

i.e., $\varkappa_1 \circ \gamma_1 = 1_{M_{\phi_1}}$.

Analogously we deduce the equality $\varkappa_2 \circ \gamma_2 = 1_{M_{\phi i}}$. Thus M_{ϕ_i}, $i = 1,2$, are canonical retracts of AI. Moreover,

$$(\phi_2 \circ \rho_t \circ \gamma_1)((a,\beta_1)) = (\phi_2 \circ \rho_t \circ \gamma_1)((a,\beta_1,\phi_2(a))) = \phi_2(a)$$

and

$$(\phi_1 \circ \rho_t \circ \gamma_2)((a,\beta_2)) = (\phi_1 \circ \rho_t \circ \gamma_2)((a,\phi_1(a),\beta_2)) = \phi_1(a).$$

Conversely , suppose that the retractions γ_i, $i = 1,2$, are given. Then we have $\gamma_i((a,\beta_i))(0) = a$ and $\phi_i \circ \gamma_i((a,\beta_i)) = \widehat{\beta_i}$, $i = 1,2$. Define $r_i : Z_{(\phi_1,\phi_2)} \to AI$, $i = 1,2$, $r_i((a,\beta_1,\beta_2)) = \gamma_i((a,\beta_i))$, $\forall (a,\beta_1,\beta_2) \in Z_{(\phi_1,\phi_2)}$. Then

$$\begin{aligned}r_i((a,\beta_1,\beta_2))(0) &= \gamma_i((a,\beta_i))(0) = a, \\ (\phi_i I \circ r_i)((a,\beta_1,\beta_2)) &= \phi_i \circ \gamma_i((a,\beta_i)) = \widehat{\beta_i}\end{aligned}$$

and

$$\begin{aligned}(\phi_2 \circ \rho_t \circ r_1)((a,\beta_1,\beta_2)) &= (\phi_2 \circ \rho_t \circ \gamma_1)((a,\beta_1)) = \phi_2(a), \\ (\phi_1 \circ \rho_t \circ \Psi_2)((a,\beta_1,\beta_2)) &= (\phi_1 \circ \rho_t \circ \gamma_2)((a,\beta_2)) = \phi_1(a),\end{aligned}$$

for all $(a,\beta_1,\beta_2) \in Z_{(\phi_1,\phi_2)}$.

Thus the conditions from Theorem 4.10 are satisfied. □

Using Corollary 4.11 and the proof of Corollary 2.6 and of Corollary 2.7, we deduce:

Corollary 4.12. *If $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ is a bicofibration then $B_1I \xleftarrow{\phi_1 I} AI \xrightarrow{\phi_2 I} B_2I$ and $CB_1 \xleftarrow{C(\phi_1)} CA \xrightarrow{C(\phi_2)} CB_2$ are also bicofibrations.*

Corollary 4.13. *For a fixed nuclear C^*-algebra F, the functor $A \to A \otimes_{\min} F$ preserves bicofibrations.*

Proof. Suppose that $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ is a bicofibration. We have that $M_{\phi_i \otimes_{\min} 1_F} \cong M_{\phi_i} \otimes_{\min} F$ and if $\varkappa_i : AI \to M_{\phi_i}$ is the morphism $\varkappa(\alpha) = (\alpha(0), \phi_i \circ \hat{\alpha})$, then the morphism $\varkappa_i \otimes_{\min} 1_F : AI \otimes_{\min} F \to M_{\phi_i} \otimes_{\min} F$ can be identified with $\varkappa_i' : (A \otimes_{\min} F)I \to M_{\phi \otimes_{\min} 1_F}$, the corresponding morphism for $\phi_i \otimes_{\min} 1_F$. Then if $\gamma_i : M_{\phi_i} \to AI$, $i = 1, 2$, are canonical retracts such that $\phi_2 \circ \rho_t \circ \gamma_1 = \phi_2 \circ p_A$ and $\phi_1 \circ \rho_t \circ \gamma_2 = \phi_1 \circ p_A$, we can define $\gamma_i' : M_{\phi \otimes_{\min} 1_F} \to M_{\phi \otimes_{\min} 1_F}$ as $\gamma_i \otimes_{\min} 1_F : M_{\phi_i} \otimes_{\min} F \to AI \otimes_{\min} F$. Then since we can also identify $\rho_t : (A \otimes_{\min} F)I \to A \otimes_{\min} F$ with $\rho_t \otimes_{\min} 1_F : AI \otimes_{\min} F \to A \otimes_{\min} F$, the relations $(\phi_2 \otimes_{\min} 1_F) \circ \rho_t \circ \gamma_1' = (\phi_2 \otimes_{\min} 1_F) \circ p_{A \otimes_{\min} F}$ and $(\phi_1 \otimes_{\min} 1_F) \circ \rho_t \circ \gamma_2' = (\phi_1 \otimes_{\min} 1_F) \circ p_{A \otimes_{\min} F}$ follow. By Corollary 4.11 we conclude that $B_1 \otimes_{\min} F \xleftarrow{\phi_1 \otimes_{\min} 1_F} A \otimes_{\min} F \xrightarrow{\phi_2 \otimes_{\min} 1_F} B_2 \otimes_{\min} F$ is a bicofibration. □

Remark 4.14. The corresponding property for cofibrations is given in [9, Prop. 1.11].

Corollary 4.15. *If $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ is a bicofibration, the same property has the pair of the suspension morphisms $\Sigma B_1 \xleftarrow{\Sigma\phi_1} \Sigma A \xrightarrow{\Sigma\phi_2} \Sigma B_2$. Particularly if $\phi : A \to B$ is a cofibration then $\Sigma A \xrightarrow{\Sigma\phi} \Sigma B$ is a cofibration (see Proposition 4.7 and Corollary 4.8).*

Proof. For a C^*-algebra A, $\Sigma A := \{f \in AI; f(0) = f(1) = 0\} \simeq A\mathbb{R} \simeq C_0(\mathbb{R}) \otimes A$, (see [1, p. 24]). Then we can apply Corollary 4.13. □

5. Application: some results in connection with the Čerin's homotopy groups

This section refers to the homotopy groups for C^*-algebras in the sense of Z. Čerin. We recall the definition of these groups [1].

Let A and B be C^*-algebras. Let $n \geq 0$ be an integer. Let $F^n = F^n(A; B)$ denote the set of all $*$-homomorphisms from A into the C^*-algebra $C_\partial(I^n; B)$ of all continuous functions from the $n-$ dimensional cube I^n into B which map the boundary ∂I^n of I^n into the zero element 0_B of the algebra B. These $*$-homomorphisms are divided into homotopy classes and the set of these classes define a group $\pi_n(A; B)$ (if $n \geq 1$), called the *n-th (absolute) homotopy group of B over A*. The group structure is obtained as usual by an *addition* in $F^n(A; B)$ defined by means of one coordinate of I^n. This construction is functorial, covariant with respect to B and contravariant with respect to A. Particularly, if A is a C^*-algebra and $\phi : B \to C$ is a $*$-homomorphism, then a homomorphism of groups $\phi_* : \pi_n(A; B) \to \pi_n(A; C)$ is defined by $\phi_*[f] = [f']$, for $f \in F^n(A; B)$, with $f'(a)(t) = \phi(f(a)(t))$, for $a \in A$, $t \in I^n$.

The pointed set $\pi_0(A; B)$ is the pointed set of all homotopy classes of $*$-homomorphisms from A into B.

Theorem 5.1. *Let $\phi : A \to B$ be an arbitrary $*$-homomorphism of C^*-algebras, K a C^*-algebra and $n \geq 0$ an integer. If $i' : C_\phi \to M_\phi$ is the inclusion and $\iota : M_\phi \to B$*

is the cofibration from Theorem 1.4, then there exists an exact sequence of Čerin's homotopy groups over K

$$\pi_{n+1}(K;B) \xrightarrow{\partial_*} \pi_n(K;C_\phi) \xrightarrow{i'_*} \pi_n(K;M_\phi) \xrightarrow{\iota_*} \pi_n(K;B).$$

This is an immediate consequence of the following theorem.

Theorem 5.2. *For $\phi : A \to B$ a cofibration, K a C^*-algebra and $n \geq 0$ an integer, there exists an exact sequence of Čerin's homotopy groups over K*

$$\pi_{n+1}(K;B) \xrightarrow{\partial_*} \pi_n(K;C_\phi) \xrightarrow{\pi(\phi)_*} \pi_n(K;A) \xrightarrow{\phi_*} \pi_n(K;B).$$

The following two lemmas will be applied to prove this theorem.

Lemma 5.3. *Let A and B be C^*-algebras and $n \geq 0$ an integer. Then there exists an isomorphism of groups $\sigma : \pi_n(A;\Sigma B) \to \pi_{n+1}(A;B)$ (bijection for $n = 0$).*

Proof. If $f \in F^n(A;\Sigma B)$, i.e., $f : A \to C_\partial(I^n;\Sigma B)$, we can define $f' : A \to C_\partial(I^{n+1};B)$ in the following way. If $t \in I^{n+1}$ we write this as $t = (t', t_{n+1})$, with $t' \in I^n$ and $t_{n+1} \in I$ and then we take $f'(a)(t) = f(a)(t')(t_{n+1})$, $\forall a \in A, t \in I^{n+1}$. If $t \in \partial I^{n+1}$ we can have $t' \in \partial I^n$ or $t_{n+1} \in \partial I$. In the first case $f(a)(t') = 0$ and in the second case $f(a)(t')(t_{n+1}) = 0$ since $f(a)(t') \in \Sigma B$. Thus f' is well defined and $f' \in F^{n+1}(A;B)$. Moreover if $g \in F^n(A;\Sigma B)$ is in the same homotopy class as f then g' defines the same homotopy class as f'.

Indeed supose that $h : A \to C_\partial(I^n;\Sigma B)I$ is a homotopy satisfying $\rho_0 \circ h = f$, $\rho_1 \circ h = g$. Define $h' : A \to C_\partial(I^{n+1};B)I$, by $h'(a)(\tau)(t) = h(a)(\tau)(t')(t_{n+1})$. As above we can see that h' is well defined. Moreover

$$h'(a)(0)(t) = h(a)(0)(t')(t_{n+1}) = f(a)(t')(t_{n+1}) = f'(t),$$

i.e., $\rho_0 \circ h' = f'$ and analogously $\rho_1 \circ h' = g'$.

Thus we have a correspondence $\sigma : \pi_n(A;\Sigma B) \to \pi_{n+1}(A;B), \sigma([f]) = [f']$.

Conversely, if $f' \in F^{n+1}(A;B)$, define $f : A \to C_\partial(I^n;\Sigma B)$ by $f(a)(t')(s) = f'(a)((t', s))$, for $t' \in I^n, s \in I$.

First we have $f(a)(t') \in \Sigma B$ since if $s \in \{0,1\}$, $(t', s) \in \partial I^{n+1}$ such that $f(a)(t')(0) = f(a)(t')(1) = 0$. Then if $t' \in \partial I^n$, $(t', s) \in \partial I^{n+1}$ which implies $f(a)(t')(s) = 0$, $\forall s \in I$, i.e., $f(a)(t') = 0$. We deduce that $f \in F^n(A;\Sigma B)$. Then as above we deduce that the homotopy class of f depends only on the homotopy class of f'.

Thus we can conclude that σ is a bijection. Finally it is easy to verify if $n \geq 1$ then the above $[f] \to [f']$ correspondence is compatible with the additions in $F^n(A;\Sigma B)$ and $F^{n+1}(A;B)$, so that σ is an isomorphism. □

Lemma 5.4. *For a $*$-homomorphism $\phi : B \to C$, define $\phi^n_\partial : C_\partial(I^n;B) \to C_\partial(I^n;C)$, by $\phi^n_\partial(\alpha) = \phi \circ \alpha$, for any $\alpha \in C_\partial(I^n;B)$. If ϕ is a cofibration then ϕ^n_∂ is also a cofibration.*

Proof. We shall apply Theorem 2.1. For this we observe at first that the mapping cylinder algebra $M_{\phi^n_\partial} = \{(\beta,\theta) \in C_\partial(I^n;B) \oplus C_\partial(I^n;C)I : \phi^n_\partial(\beta) = \theta(1)\}$ can be identified with $C_\partial(I^n;M_\phi)$ by the following isomorphism $\chi : M_{\phi^n_\partial} \to C_\partial(I^n;M_\phi)$, $\chi((\beta,\theta))(t) = (\beta(t),\theta_t)$, with $\theta_t \in CI$ defined by $\theta_t(\tau) = \theta(\tau)(t)$, for any $\tau \in I$. It is easy to see that this definition is correct and that χ is an isomorphism. Similarly there is an isomorphism $\delta : C_\partial(I^n;B)I \to C_\partial(I^n,BI)$, $\delta(\theta)(t)(\tau) = \theta(\tau)(t)$, for $t \in I^n$ and $\tau \in I$. Now let $r : M_\phi \to BI$ be a canonical retract with $\varkappa : BI \to M_\phi$ satisfying $\varkappa \circ r = 1_{M_\phi}$. Then we define $r' = \delta^{-1} \circ r^n_\partial \circ \chi : M_{\phi^n_\partial} \to C_\partial(I^n;B)$ and $\varkappa' = \chi^{-1} \circ \varkappa^n_\partial \circ \delta : C_\partial(I^n;B)I \to M_{\phi^n_\partial}$. And since $\varkappa \circ r = 1_{M_\phi}$ implies $\varkappa^n_\partial \circ r^n_\partial = 1_{C_\partial(I^n;M_\phi)}$, it is immediate that $\varkappa' \circ r' = 1_{M_{\phi^n_\partial}}$. By Theorem 2.1 we conclude that ϕ^n_∂ is a cofibration. $\square$

Proof of Theorem 5.2. Since for the cofibration ϕ there exists a homotopy equivalence (over A) between C_ϕ and $J \coloneqq \ker\phi$, see [9, Prop. 2.4], we can formulate the exactness in the term $\pi_n(K;A)$ as the exactness of the sequence

$$\pi_n(K;J) \xrightarrow{j_*} \pi_n(K;A) \xrightarrow{\phi_*} \pi_n(K;B),$$

where j denotes the inclusion $J \hookrightarrow A$.

First it is obvious that $\operatorname{Im} j_* \subseteq \ker\phi_*$ since $\phi_* \circ j_* = (\phi \circ j)_* = 0$. Now let $[f] \in \ker\phi_*$. This means that f is a $*$-homomorphism $f : K \to C_\partial(I^n;A)$ such that there exists a homotopy $\Phi : K \to C_\partial(I^n;B)I$ satisfying $\rho_0 \circ \Phi = \phi^n_\partial \circ f$ and $\rho_1 \circ \Phi = 0$. By Lemma 5.4 there exists $\Psi : K \to C_\partial(I^n;A)I$ such that the following diagram is commutative

$$\begin{tikzcd}
C_\partial(I^n;A) \arrow[rr, "\phi^n_\partial"] & & C_\partial(I^n;B) \\
& K \arrow[ul, "f"'] \arrow[dl, "\Psi"] \arrow[dr, "\Phi"] & \\
C_\partial(I^n;A)I \arrow[u, "\rho_0"] \arrow[rr, "\phi^n_\partial"'] & & C_\partial(I^n;B)I \arrow[u, "\rho_0"']
\end{tikzcd}$$

Therefore we have $\rho_0 \circ \Psi = f$ and $\phi^n_\partial I \circ \Psi = \Phi$. If we denote $f' \coloneqq \rho_1 \circ \Psi \in F^n(K;A)$, then $\phi^n_\partial(f') = \rho_1 \circ \Phi = 0$, i.e., $\phi(f'(k)(t)) = 0, \forall k \in K, \forall t \in I^n$, which shows that $f' \in F^n(K;J)$. Thus we can conclude that $[f] = [f'] = j_*[f']$, i.e., $[f] \in \operatorname{Im} j_*$. Therefore $\ker\phi_* \subseteq \operatorname{Im} j_*$, which permits to conclude the exactness of the sequence

$$\pi_n(K;C_\phi) \xrightarrow{\pi(\phi)_*} \pi_n(K;A) \xrightarrow{\phi_*} \pi_n(K;B). \tag{8}$$

Now by Example 1.13, $\pi(\phi) : C_\phi \to A$ is a also a cofibration and $\ker\pi(\phi) = \Sigma B$. By applying the exact sequence already obtained for this cofibration we obtain the exact sequence $\pi_n(K; \Sigma B) \xrightarrow{i_*} \pi_n(K; C_\phi) \xrightarrow{\pi(\phi)_*} \pi_n(K; A)$, where $i : \Sigma B \to C_\phi$ is the inclusion $i(\beta) = (0, \beta)$. Now if we define $\partial_* : \pi_{n+1}(K; B) \to \pi_n(K; C_\phi), \partial_* = i_* \circ \sigma$, for σ the isomorphism from Lemma 5.3, we obtain the exact sequence

$$\pi_{n+1}(K; B) \xrightarrow{\partial_*} \pi_n(K; C_\phi) \xrightarrow{\pi(\phi)_*} \pi_n(K; A). \tag{9}$$

By joining sequences (8) and (9) we finish the proof. □

Proof. We apply Theorem 5.2 for the cofibration $\iota : M_\phi \to B$ and use the homotopy equivalence $C_\iota \overset{h}{\sim} \ker \iota = \{(a, \beta) \in M_\phi : \beta(0) = 0\} = C_\phi$ induced by the inclusion $\ker \iota \hookrightarrow C_\iota$, see [9, Prop. 2.4]. □

Remark 5.5. Unfortunately we have not succeeded to prove that the exact sequences from Theorems 5.1 and 5.2 are long exact sequences. But we can complete these sequences with the following semiexact sequences $\pi_n(K; A) \xrightarrow{\phi_*} \pi_n(K; B) \xrightarrow{\partial_*} \pi_{n-1}(K; C_\phi)$ and $\pi_n(K; M_\phi) \xrightarrow{\iota_*} \pi_n(K; B) \xrightarrow{\partial_*} \pi_{n-1}(K; C_\phi)$ respectively. It is sufficient to verify the semiexactness only for the first sequence. First we observe that $\partial_* : \pi_n(K; B) \to \pi_{n-1}(K; C_\phi)$ can be expressed by the following formula: $\partial_*([f]) = [h]$, where for $f \in F^n(K; B)$, $h \in F^{n-1}(K; C_\phi)$ is defined by $h(k)(t') = (0_A, \beta_{k,t'})$ with $\beta_{k,t'}(\tau) = f(k)((t', \tau))$, $k \in K, t' \in I^{n-1}, \tau \in I$. Now, if $[g] \in \pi_n(K; A)$ then $(\partial_* \circ \phi_*)([g]) = [l]$ with $l \in F^{n-1}(K; C_\phi)$ given by $l(k)(t') = (0_A, \beta'_{k,t'})$ and $\beta'_{k,t'}(\tau) = \phi(g(k)((t', \tau))$, $k \in K, t' \in I^{n-1}, \tau \in I$. Now we define the following homotopy $*$-homomorphism: $\Psi : K \to C_\partial(I^{n-1}; C_\phi)I$ by $\Psi(k)(\tau')(t') = (g(k)((t', \tau')), \beta_{k,\tau',t'})$ with $\beta_{k,\tau',t'}(\tau) = \phi(g(k)(t', \tau\tau'))$ for $k \in K, t' \in I^{n-1}$, $\tau, \tau' \in I$. This is well defined since $\beta_{k,\tau',t'}(0) = \phi(g(k)((t', 0)) = \phi(0_A) = 0_B$ and $\beta_{k,\tau',t'}(1) = \phi(g(k)((t', \tau'))$ and for $\overline{t'} \in \partial I^{n-1}$, $\Psi(k)(\tau')(\overline{t'}) = 0_{C_\phi}$. Then, for this $*$-homotopy we have

$$\Psi(k)(0)(t') = (g(k)((t', 0)), \beta_{k,0,t'}) = (0_A, \beta_{k,0,t'}),$$

$\beta_{k,0,t'}(\tau) = \phi(g(k)((t', 0)) = 0_B$, $\Psi(k)(1)(t') = (g(k)((t', 1)), \beta_{k,1,t'}) = (0_A, \beta_{k,1,t'})$, and $\beta_{k,1,t'}(\tau) = \phi(g(k)((t', \tau)) = \beta'_{k,t'}(\tau)$, i.e., $\Psi(k)(0)(t') = l(k)(t')$. So we have obtained that l is homotopy equivalent with the trivial $*$-homomorphism $z : K \to C_\partial(I^{n-1}; C_\phi)$, which means that $\partial_* \circ \phi_* = 0$, and this implies the inclusion $\operatorname{Im} \phi_* \subseteq \ker \partial_*$.

Lemma 5.6. *Let $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ be a bicofibration and $n \geq 0$ an integer. Then the pair of $*$-homomorphisms $C_\partial(I^n; B_1) \xleftarrow{\phi^n_{1\partial}} C_\partial(I^n; A) \xrightarrow{\phi^n_{2\partial}} C_\partial(I^n; B_2)$ is a bicofibration.*

Theorem 5.7. *Let $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} \to B_2$ be a bicofibration, K a C^*-algebra, and $n \geq 0$ an integer. If $[f] \in \pi_n(K; A)$ is an element which belongs to $\ker \phi_{1*} \cap \ker \phi_{2*}$, then there exist $f_i \in F^n(K; \ker \phi_i)$, $i = 1, 2$, satisfying the following conditions:*

(i) $[f] = [f_i]$ *in* $\pi_n(K; A)$, $i = 1, 2$.

(ii) $\phi^n_{1\partial} \circ f_2 = \phi^n_{1\partial} \circ f$ *and* $\phi^n_{2\partial} \circ f_1 = \phi^n_{2\partial} \circ f$.

Proof. By hypothesis $f : K \to C\partial(I^n; A)$ is a $*$-morphism for which two homotopies $\Phi_i : K \to C\partial(I^n; B_i)I$, $i = 1, 2$, with $\rho_0 \circ \Phi_i = \phi^n_{i\partial} \circ f$ and $\rho_1 \circ \Phi_i = 0$, $i = 1, 2$, exist.

$$
\begin{tikzcd}
C\partial(I^n;B_1) & C\partial(I^n;A) \arrow[l, "\phi^n_{1\partial}"'] \arrow[r, "\phi^n_{2\partial}"] & C\partial(I^n;B_2) \\
 & K \arrow[u, "\psi"'] \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] \arrow[d, shift right, "\Psi_1"'] \arrow[d, shift left, "\Psi_2"] & \\
C\partial(I^n;B_1)I \arrow[uu, "\rho_0"] & C\partial(I^n;A)I \arrow[l, "\phi^n_{1\partial}I"] \arrow[r, "\phi^n_{2\partial}I"'] \arrow[d, "\rho_t"] & C\partial(I^n;B_2)I \arrow[uu, "\rho_0"'] \\
C\partial(I^n;B_1) & C\partial(I^n;A) \arrow[l, "\phi^n_{1\partial}"'] \arrow[r, "\phi^n_{2\partial}"] & C\partial(I^n;B_2)
\end{tikzcd}
$$

By Lemma 5.6 there exist two homotopies $\Psi_i : K \to C\partial(I^n; A)$, $i = 1, 2$, with $\rho_0 \circ \Psi_i = f$, $\phi^n_{i\partial} I \circ \Psi_i = \Phi_i$, $i = 1, 2$, and $\phi^n_{1\partial} \circ \rho_t \circ \Psi_2 = \phi^n_{1\partial} \circ f$, $\phi^n_{2\partial} \circ \rho_t \circ \Psi_1 = \phi^n_{2\partial} \circ f$. Define $f_i = \rho_1 \circ \Psi_i : K \to C\partial(I^n; A)$, $i = 1, 2$. Then $\Psi_i : f \sim f_i$ in $F^n(K, A)$ and $f_i \in F^n(K; \ker \phi_i)$, $i = 1, 2$. Moreover, $\phi^n_{1\partial} \circ \rho_1 \circ \Psi_2 = \phi^n_{1\partial} \circ f \Rightarrow \phi^n_{1\partial} \circ f_2 = \phi^n_{1\partial} \circ f$ and $\phi^n_{2\partial} \circ \rho_1 \circ \Psi_1 = \phi^n_{2\partial} \circ f \Rightarrow \phi^n_{2\partial} \circ f_1 = \phi^n_{2\partial} \circ f$. Thus the conditions (i), (ii) have been verified. □

Corollary 5.8. *Let* $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ *be a bicofibration,* K *a* C^**-algebra, and* $n \geq 0$ *an integer. If* $f_1 \in F^n(K; \ker \phi_1)$ *and* $\phi_{2*}[f_1] = 0$*, then there exists* $f_2 \in F^n(K; \ker \phi_2)$ *satisfying the conditions:*

(i) $[f_1] = [f_2]$ *in* $\pi_n(K; A)$ *and*

(ii) $\phi^n_{1\partial} \circ f_2 = 0$.

Corollary 5.9. *Let* $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$ *be a bicofibration,* K *a* C^**-algebra and* $n \geq 0$ *an integer. Then* $\ker \phi_{1*} \subseteq \ker \phi_{2*}$ *if and only if for each* $f_1 \in F^n(K; \ker \phi_1)$*, the following properties are satisfied:*

(i) $\phi^n_{2\partial} \circ f_1 = 0$.

(ii) *There exists* $f_2 \in F^n(K; \ker \phi_2)$*, with* $[f_1] = [f_2]$ *in* $\pi_n(K; A)$ *and* $\phi^n_{1\partial} \circ f_2 = 0$.

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