Invertibility of Operators in Spaces of Real Interpolation

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ABSTRACT

Let A be a linear bounded operator from a couple $\vec{X} = (X_0, X_1)$ to a couple $\vec{Y} = (Y_0, Y_1)$ such that the restrictions of A on the spaces X_0 and X_1 have bounded inverses. This condition does not imply that the restriction of A on the real interpolation space $(X_0, X_1)_{\theta,q}$ has a bounded inverse for all values of the parameters θ and q. In this paper under some conditions on the kernel of A we describe all spaces $(X_0, X_1)_{\theta,q}$ such that the operator $A : (x_0, X_1)_{\theta,q} \to (Y_0, Y_1)$ has a bounded inverse.

Key words: real interpolation, invertible operators.

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Introduction

In the area of partial differential equations, the importance of invertibility of operators in scales of spaces was first observed by Alberto Calderón in 1985 [5], who considered the case of L^p scale and an operator bounded in L^2. New applications of invertibility of operators to PDE were recently obtained by Kalton and Mitrea [10]. These applications are closely connected to interpolation theory and, in particular, to the remarkable theorem proved by I. Ya. Shneiberg (see [16, 17]). This theorem in its simplest form claims that if a linear bounded operator A from a couple $\vec{X} = (X_0, X_1)$ to itself is invertible on a complex interpolation space $[X_0, X_1]_{\theta_0}$, then it is also invertible on the spaces $[X_0, X_1]_\theta$ when θ is close to θ_0: $|\theta - \theta_0| < \varepsilon$. Later on different

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generalizations and applications of Shneiberg's results were obtained by various authors (see, for example, [2, 6, 7, 11, 19, 20]). In particular, in the work [11] a general theory of Shneiberg-type theorems was proposed.

The above mentioned applications are closely connected to the following problem. Let A be a linear bounded operator from a Banach couple $\vec{X} = (X_0, X_1)$ to a Banach couple $\vec{Y} = (Y_0, Y_1)$. Let also Ω_q be the set of all θ for which the restriction of the operator A on the space $(X_0, X_1)_{\theta,q}$ has a bounded inverse defined on the space $(Y_0, Y_1)_{\theta,q}$. Then it follows from an analog of Shneiberg theorem (proved for the case $q < \infty$ in [20] and proved for the general case, including $q = \infty$, in [11]) that the set Ω_q is open. To describe the set Ω_q, the following problem has to be solved:

Problem. *Suppose that the restrictions of the operator A on the spaces X_0 and X_1 have bounded inverses defined on the spaces Y_0 and Y_1, respectively. How can we describe all real interpolation spaces $(X_0, X_1)_{\theta,q}$ such that the restriction of the operator A on a space $(X_0, X_1)_{\theta,q}$ has a bounded inverse on the space $(Y_0, Y_1)_{\theta,q}$?*

Two different but complimentary approaches to this problem are possible. The first approach consists of a complete and, if possible, explicit description of the set Ω_q. In the general case, this task is rather complicated, even in the case when the kernel of the operator A is of dimension one. Let us also note that the proofs known for this case are based on Hahn-Banach theorem and are not constructive (see [1, 9]).

The second approach consists of finding sufficiently simple and easily tested conditions that would allow for a complete solution of the problem. A constructive solution is preferable since the problem can, in fact, be reduced to the problem of solving the equation

$$Ax = y,$$

where $y \in (Y_0, Y_1)_{\theta,q}$ and θ does not belong to the set Ω_q.

The present work takes the first step in developing the second approach. Our main result is the following

Theorem A. *Let A be a bounded linear operator from a Banach couple $\vec{X} = (X_0, X_1)$ to a Banach couple $\vec{Y} = (Y_0, Y_1)$ such that A is invertible on the spaces X_0 and X_1. Suppose also that its kernel $\operatorname{Ker} A \subset X_0 + X_1$ is finite-dimensional and has a basis $e_1, \dots, e_n$ such that*

$$K(t, e_i; X_0, X_1) \approx t^{\theta_i} \qquad (\theta_i \in (0,1), \theta_i \neq \theta_j \quad \text{for } i \neq j).$$

Then the operator A is invertible on the space $(X_0, X_1)_{\theta,q}$ if and only if $\theta \neq \theta_i$ ($i = 1, \dots, n$).

A direct constructive proof of this result will be presented below. It is easy to see, especially in the case when the kernel is one-dimensional, how the algorithm for constructing the solution to the equation $Ax = y$, $y \in (Y_0, Y_1)_{\theta,q}$, changes as the parameter θ passes a critical value θ_i.

The following example, taken from [12], illustrates this theorem. Let $L_1(t^{-\alpha}, \frac{dt}{t})$ be a space of functions on $(0,\infty)$ defined by the norm

$$\|f\|_{L_1(t^{-\alpha},\frac{dt}{t})} = \int_0^\infty |f(t)|t^{-\alpha}\frac{dt}{t} < \infty$$

and let us consider an operator $A = I - H$ (Identity minus Hardy) which is defined by the formula $(Af)(t) = f(t) - \frac{1}{t}\int_0^t f(s)ds$. Let also $(X_0, X_1) = (L_1(\sqrt{t}, \frac{dt}{t}), L_1(\frac{1}{\sqrt{t}}, \frac{dt}{t}))$. It is easy to verify that the operator $A = I - H$ has a one-dimensional kernel in $X_0 + X_1$ which consists of constant functions $f(x) \equiv C$. Note that for $f(x) \equiv C$ holds

$$K(t, f; X_0, X_1) = \int_0^\infty C \min\Big(\sqrt{s}, \frac{t}{\sqrt{s}}\Big)\frac{ds}{s} \approx C\sqrt{t}.$$

As the operator A is bounded and invertible on the spaces X_0 and X_1 (see [12]), therefore the conditions of Theorem A are fulfilled. Hence Theorem A describes all spaces $(X_0, X_1)_{\theta,q}$ on which $A = I - H$ is invertible.

We will prove the theorem in two steps. In the first step we reduce the theorem to the case when the kernel of the operator A is one-dimensional and in the second step we consider the case of a one-dimensional kernel.

1. Reduction to the case of a one-dimensional kernel

First of all let us note that it is sufficient to consider the case when A is a quotient operator. Indeed, if we denote by $\bar{A} : \vec{X} \to \vec{X}/\operatorname{Ker} A$ the quotient operator then we have $A = B\bar{A}$, where $B : \vec{X}/\operatorname{Ker} A \to \vec{Y}$ is invertible on the end spaces and has no kernel. Therefore, B is an invertible operator for all interpolation spaces $(X_0, X_1)_{\theta,q}$, and it is sufficient to prove the theorem for the operator $\bar{A}$. Note that $\bar{A}$ can be represented as a product $\bar{A} = A_n A_{n-1} \cdots A_1$, where A_1 is an operator with the kernel $\operatorname{Ker} A_1 = \operatorname{Span}\{e_1\}$ and A_i $(i = 2, \dots, n)$ is an operator with a one-dimensional kernel generated by the element $A_{i-1} \cdots A_1 e_i$. Therefore, Theorem A can be easily proved by induction using the following result.

Theorem 1.1. *If an operator A from a couple $\vec{X}$ to a couple $\vec{Y}$ is invertible on the spaces X_0 and X_1 and has a one-dimensional kernel $\operatorname{Ker} A = \{\lambda e\}$ such that $K(t, e; \vec{X}) \approx t^{\theta_0}$, then from $K(t, x; \vec{X}) \approx t^{\theta}$ with $\theta \neq \theta_0$ it follows that*

$$K(t, Ax; \vec{Y}) \approx t^{\theta}.$$

The proof of the theorem is based on the following lemma.

Lemma 1.2. *Suppose that the operator $A : \vec{X} \to \vec{Y}$ is such that $A(X_i) = Y_i$ $(i = 0, 1)$. Then for any $x \in X_0 + X_1$ holds*

$$K(t, Ax; \vec{Y}) \approx \inf_{u \in \operatorname{Ker} A} K(t, x - u; \vec{X})$$

with the constant of equivalence independent of x and t.

Proof. Let $u \in \operatorname{Ker} A$ and let $x_0 \in X_0$ and $x_1 \in X_1$ be some decomposition of $x - u$, i.e., $x - u = x_0 + x_1$. Then

$$Ax = Ax_0 + Ax_1$$

and

$$K(t, Ax; \vec{Y}) \leq \|Ax_0\|_{Y_0} + t\|Ax_1\|_{Y_1} \leq \|A\|(\|x_0\|_{X_0} + t\|x_1\|_{X_1}).$$

Hence

$$K(t, Ax; \vec{Y}) \leq \|A\| \inf_{u \in \operatorname{Ker} A} K(t, x - u; \vec{X}).$$

To prove the opposite inequality let us consider a decomposition $Ax = y_0 + y_1$ with $y_0 \in Y_0$ and $y_1 \in Y_1$. Since $A(X_i) = Y_i$ $(i = 0, 1)$ we can find such elements $x_0 \in X_0$ and $x_1 \in X_1$ that $Ax_i = y_i$ $(i = 0, 1)$ and $\|x_i\|_{X_i} \leq c\|y_i\|_{Y_i}$ $(i = 0, 1)$ with the constant $c > 0$ independent of y_0, y_1, and x. Then from the equality

$$Ax = y_0 + y_1 = Ax_0 + Ax_1$$

it follows that $x - x_0 - x_1 = u \in \operatorname{Ker} A$ and

$$K(t, x - u; \vec{X}) \leq \|x_0\|_{X_0} + t\|x_1\|_{X_1} \leq c(\|y_0\|_{Y_0} + t\|y_1\|_{Y_1}).$$

Hence

$$\inf_{u \in \operatorname{Ker} A} K(t, x - u; \vec{X}) \leq cK(t, Ax; \vec{Y}). \qquad \square$$

Let us now return to the proof of Theorem 1.1.

Proof. From Lemma 1.2 it follows that it is sufficient to prove that the conditions

$$c_0 t^{\theta_0} \leq K(t, e; \vec{X}) \leq c_1 t^{\theta_0},$$
$$d_0 t^{\theta} \leq K(t, x; \vec{X}) \leq d_1 t^{\theta}$$

imply

$$\inf_{\lambda} K(t, x - \lambda e; \vec{X}) \approx t^{\theta}.$$

Here c_0, c_1, d_0, and d_1 are some positive constants.

As

$$K(t, Ax, \overrightarrow{Y}) \approx \inf_{\lambda} K(t, x - \lambda e; \vec{X}) \leq K(t, x; \vec{X}) \leq d_1 t^{\theta}$$

it is sufficient to prove the estimate from below

$$\inf_{\lambda} K(t, x - \lambda e; \vec{X}) \geq \delta t^{\theta}.$$

Let us fix a number $t > 0$. From the inequality

$$K(t, x - \lambda e; \vec{X}) \geq K(t, \lambda e; \vec{X}) - K(t, x; \vec{X}) \geq |\lambda| c_0 t^{\theta_0} - d_1 t^{\theta}$$

it follows that if

$$|\lambda| \geq \frac{2d_1}{c_0 t^{\theta_0 - \theta}}$$

then $K(t, x - \lambda e; \vec{X}) \geq d_1 t^\theta$ and it is sufficient to consider the case when

$$|\lambda| < \frac{2d_1}{c_0 t^{\theta_0 - \theta}}.$$

Now we will consider the two cases $\theta > \theta_0$ and $\theta < \theta_0$ separately. In the case of $\theta > \theta_0$ from the concavity of the K-functional it follows that for any $T \geq t$ we have

$$\begin{aligned} K(t, x - \lambda e; \vec{X}) &\geq \frac{t}{T} K(T, x - \lambda e; \vec{X}) \geq \frac{t}{T}(K(T, x; \vec{X}) - |\lambda| K(T, e; \vec{X})) \\ &\geq \frac{t}{T}\Big(d_0 T^\theta - \frac{2d_1}{c_0 t^{\theta_0 - \theta}} c_1 T^{\theta_0}\Big). \end{aligned}$$

If $T = \gamma t$ $(\gamma > 1)$ then

$$K(t, x - \lambda e; \vec{X}) \geq \frac{1}{\gamma}\Big(d_0 \gamma^\theta t^\theta - \frac{2d_1}{c_0 t^{\theta_0 - \theta}} c_1 \gamma^{\theta_0} t^{\theta_0}\Big).$$

Let now γ be such that

$$d_0 \gamma^\theta = \frac{3d_1}{c_0} c_1 \gamma^{\theta_0}.$$

Since $\theta > \theta_0$, $d_1 \geq d_0$, and $c_1 \geq c_0$, therefore $\gamma > 1$ and we have

$$K(t, x - \lambda e; \vec{X}) \geq \Big(\frac{1}{\gamma} \frac{d_1}{c_0} c_1 \gamma^{\theta_0}\Big) t^\theta = \delta t^\theta,$$

with the constant $\delta > 0$ dependent only on the constants θ, θ_0, d_1, d_0, c_1, and c_0. In the case of $\theta < \theta_0$ we take $T = \gamma t$ with $\gamma < 1$. From the properties of the K-functional we obtain the inequalities

$$\begin{aligned} K(t, x - \lambda e; \vec{X}) &\geq K(T, x - \lambda e; \vec{X}) \geq K(T, x; \vec{X}) - |\lambda|\, K(T, e; \vec{X}) \\ &\geq d_0 T^\theta - \frac{2d_1}{c_0 t^{\theta_0 - \theta}} c_1 T^{\theta_0} = t^\theta \Big(d_0 \gamma^\theta - \frac{2d_1}{c_0} c_1 \gamma^{\theta_0}\Big). \end{aligned}$$

Since $\theta < \theta_0$ we can choose such $\gamma < 1$ that

$$d_0 \gamma^\theta = \frac{3d_1}{c_0} c_1 \gamma^{\theta_0}.$$

For such γ we have

$$K(t, x - \lambda e; \vec{X}) \geq \frac{d_1}{c_0} c_1 \gamma^{\theta_0} t^\theta = \delta t^\theta,$$

with the constant $\delta > 0$ dependent only on the constants θ, θ_0, d_1, d_0, c_1, and c_0. □

2. The case of a one-dimensional kernel

Let $A : \vec{X} \to \vec{Y}$ be a bounded linear operator which is invertible on spaces X_0 and X_1. Suppose also that A has in $X_0 + X_1$ a one-dimensional kernel $\operatorname{Ker} A = \{\lambda e\}$ with $K(t, e; \vec{X}) \approx t^{\theta_0}$. We need to prove that A is invertible on the space $(X_0, X_1)_{\theta,q}$ if and only if $\theta \neq \theta_0$.

We start with the case when $\theta \neq \theta_0$. Since $K(t, e; \vec{X}) \approx t^{\theta_0}$, therefore $\operatorname{Ker} A \cap (X_0, X_1)_{\theta,q} = \{0\}$ and it is sufficient to show that for a given $y \in (Y_0, Y_1)_{\theta,q}$ it is possible to construct an element $x \in (X_0, X_1)_{\theta,q}$ such that $Ax = y$ and $\|x\|_{(X_0,X_1)_{\theta,q}} \leq \gamma\|y\|_{(Y_0,Y_1)_{\theta,q}}$ with γ independent of y. From the equivalence theorem of the K- and J-methods (see [4]) it follows that there exists a sequence of elements $y_n \in Y_0 \cap Y_1$, $n \in \mathbb{Z}$, such that

$$\left(\sum_{n\in\mathbb{Z}} \left(2^{-\theta n} J(2^n, y_n; \vec{Y})\right)^q\right)^{\frac{1}{q}} \leq \gamma\|y\|_{(Y_0,Y_1)_{\theta,q}}, \tag{1}$$

where $J(2^n, y_n; \vec{Y}) = \max\{\|y_n\|_{Y_0}, 2^n\|y_n\|_{Y_1}\}$. As the operator A has inverses on the spaces X_0 and X_1 defined on the spaces Y_0 and Y_1, respectively, therefore we can find two sequences $x_0^n \in X_0$, $x_1^n \in X_1$, $n \in \mathbb{Z}$, such that

$$Ax_0^n = Ax_1^n = y_n \text{ and } \|x_0^n\|_{X_0} \leq \gamma\|y_n\|_{Y_0}, \quad \|x_1^n\|_{X_1} \leq \gamma\|y_n\|_{Y_1}. \tag{2}$$

Now we can define the required element $x \in (X_0, X_1)_{\theta,q}$ as

$$x = \sum_n x_1^n \qquad \text{for } \theta > \theta_0,$$

and

$$x = \sum_n x_0^n \qquad \text{for } \theta < \theta_0.$$

Let us first consider the case of $\theta > \theta_0$. We note that if the series $x = \sum_n x_1^n$ converges in $X_0 + X_1$ then we have $Ax = \sum_n Ax_1^n = \sum_n y_n = y$. To prove the convergence we need the inequality

$$\left\|\sum_n x_1^n\right\|_{(X_0,X_1)_{\theta,q}} \leq \gamma\|y\|_{(Y_0,Y_1)_{\theta,q}}. \tag{3}$$

As $Ax_0^n = Ax_1^n = y_n$, then $x_0^n - x_1^n \in \operatorname{Ker} A$ and hence $x_0^n - x_1^n = \lambda_n e$. Moreover, from $K(2^k, \lambda_k e; \vec{X}) \approx |\lambda_k| 2^{k\theta_0}$ and (2) it follows that

$$|\lambda_k| \leq \gamma 2^{-k\theta_0} K(2^k, \lambda_k e; \vec{X}) \leq \gamma 2^{-k\theta_0}\left(\|x_0^k\|_{X_0} + 2^k \left\|x_1^k\right\|_{X_1}\right) \leq \gamma 2^{-k\theta_0} J(2^k, y_k; \vec{Y}).$$

(By γ and γ_1 we will denote different positive constants in different contexts.) Hence

$$\begin{aligned} K\Big(2^n, \sum_k x_1^k; \vec{X}\Big) &\le K\Big(2^n, \sum_{k<n} x_0^k + \sum_{k\ge n} x_1^k; \vec{X}\Big) + K\Big(2^n, \sum_{k<n} -\lambda_k e; \vec{X}\Big) \\ &\le \Big\| \sum_{k<n} x_0^k \Big\|_{X_0} + 2^n \Big\| \sum_{k\ge n} x_1^k \Big\|_{X_1} + \sum_{k<n} |\lambda_k| K(2^n, e; \vec{X}) \\ &\le \sum_{k<n} \|x_0^k\|_{X_0} + 2^n \sum_{k\ge n} \|x_1^k\|_{X_1} + \gamma 2^{\theta_0 n} \sum_{k<n} |\lambda_k| \\ &\le \gamma \Big(\sum_k \min\Big(1, \frac{2^n}{2^k}\Big) J(2^k, y_k; \vec{Y}) \Big) + \gamma 2^{\theta_0 n} \sum_{k<n} 2^{-k\theta_0} J(2^k, y_k; \vec{Y}). \end{aligned}$$

Therefore, the proof of the inequality (3) (and also the convergence of $\sum_n x_1^n$ in $X_0 + X_1$) follows from (1) and the boundedness of the operators S and S_{θ_0} in the space $l_q(\{2^{-n\theta}\}_{n\in\mathbb{Z}})$. Here S and S_{θ_0} are defined by the formulas

$$(S\{a_k\})_n = \sum_k \min\Big(1, \frac{2^n}{2^k}\Big) a_k, \qquad (S_{\theta_0}\{a_k\})_n = 2^{\theta_0 n} \sum_{k<n} 2^{-k\theta_0} a_k. \tag{4}$$

The boundedness of the first operator in the space $l_q(\{2^{-n\theta}\}_{n\in\mathbb{Z}})$ follows from the fact that this operator is a discrete analog of the Calderón operator

$$(Sf)(t) = \int_0^t f(s) \frac{ds}{s} + t \int_t^\infty s^{-1} f(s) \frac{ds}{s},$$

which is bounded in $L_q(t^{-\theta}, \frac{dt}{t})$ for all $\theta \in (0,1)$.

The second operator S_{θ_0} is a discrete analog of the operator

$$(S_{\theta_0} f)(t) = t^{\theta_0} \int_0^t s^{-\theta_0} f(s) \frac{ds}{s},$$

which is bounded in $L_q(t^{-\theta}, \frac{dt}{t})$ for $\theta > \theta_0$. Indeed, from the Minkovskii inequality we have

$$\begin{aligned}\left(\int_0^\infty \left(t^{-\theta}(S_{\theta_0}f)(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} &= \left(\int_0^\infty \left(t^{-(\theta-\theta_0)}\int_0^t s^{-\theta_0} f(s)\frac{ds}{s}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(t^{-(\theta-\theta_0)}\int_0^1 (tu)^{-\theta_0} f(tu)\frac{du}{u}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(t^{-\theta}\int_0^1 u^{-\theta_0} f(tu)\frac{du}{u}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\le \int_0^1 u^{-\theta_0}\left(\int_0^\infty \left(t^{-\theta} f(tu)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \frac{du}{u} \\ &\le \int_0^1 u^{-\theta_0}\left(\int_0^\infty \left(\left(\frac{s}{u}\right)^{-\theta} f(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} \frac{du}{u} \\ &\le \left(\int_0^\infty \left(s^{-\theta} f(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} \cdot \int_0^1 u^{\theta-\theta_0}\frac{du}{u} \\ &\le \gamma\left(\int_0^\infty \left(s^{-\theta} f(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}}.\end{aligned}$$

This concludes the proof for the case of $\theta > \theta_0$.

The case of $\theta < \theta_0$ can be considered in a similar way, we only need to define $x = \sum_n x_0^n$ and to prove that

$$\Big\|\sum_n x_0^n\Big\|_{(X_0,X_1)_{\theta,q}} \le \gamma \|y\|_{(Y_0,Y_1)_{\theta,q}}.$$

This inequality is proved similarly to (3). We have

$$\begin{aligned}K\Big(2^n, \sum_k x_0^k; \vec{X}\Big) &\le K\Big(2^n, \sum_{k<n} x_0^k + \sum_{k\ge n} x_1^k; \vec{X}\Big) + K\Big(2^n, \sum_{k\ge n} \lambda_k e; \vec{X}\Big) \\ &\le \Big\|\sum_{k<n} x_0^k\Big\|_{X_0} + 2^n \Big\|\sum_{k\ge n} x_1^k\Big\|_{X_1} + \sum_{k\ge n} |\lambda_k| K(2^n, e; \vec{X}) \\ &\le \sum_{k<n} \|x_0^k\|_{X_0} + 2^n \sum_{k\ge n} \|x_1^k\|_{X_1} + \gamma 2^{\theta_0 n} \sum_{k\ge n} |\lambda_k| \\ &\le \gamma\Big(\sum_k \min\Big(1, \frac{2^n}{2^k}\Big) J(2^k, y_k; \vec{Y})\Big) \\ &\quad + \gamma 2^{\theta_0 n} \sum_{k\ge n} 2^{-k\theta_0} J(2^k, y_k; \vec{Y}).\end{aligned}$$

Therefore, the inequality (3) follows from (1) and the boundedness of the operators S (see (4)) and S^{θ_0} in $l_p(\{2^{-n\theta}\}_{n\in\mathbb{Z}})$. Here S^{θ_0} is defined by the formula

$$(S^{\theta_0}\{a_k\})_n = 2^{\theta_0 n}\sum_{k\geq n} 2^{-k\theta_0} a_k.$$

We already know that the operator S is bounded in $l_q(\{2^{-n\theta}\}_{n\in\mathbb{Z}})$ for all $\theta\in(0,1)$. The operator S^{θ_0} is a discrete analog of the operator

$$(S^{\theta_0}f)(t) = t^{\theta_0}\int_t^\infty s^{-\theta_0} f(s)\frac{ds}{s}.$$

Its boundedness in $L_q(t^{-\theta},\frac{dt}{t})$ for $\theta<\theta_0$ follows from the Minkovskii inequality:

$$\begin{aligned}
\Big(\int_0^\infty \big(t^{-\theta}(S^{\theta_0}f)(t)\big)^q\,\frac{dt}{t}\Big)^{\frac1q} &= \Big(\int_0^\infty \Big(t^{-(\theta-\theta_0)}\int_t^\infty s^{-\theta_0} f(s)\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac1q}\\
&= \Big(\int_0^\infty \Big(t^{-(\theta-\theta_0)}\int_0^1 \Big(\frac{t}{u}\Big)^{-\theta_0} f\Big(\frac{t}{u}\Big)\frac{du}{u}\Big)^q\frac{dt}{t}\Big)^{\frac1q}\\
&= \Big(\int_0^\infty \Big(t^{-\theta}\int_0^1 u^{\theta_0} f\Big(\frac{t}{u}\Big)\frac{du}{u}\Big)^q\frac{dt}{t}\Big)^{\frac1q}\\
&\leq \int_0^1 u^{\theta_0}\Big(\int_0^\infty \Big(t^{-\theta} f\Big(\frac{t}{u}\Big)\Big)^q\frac{dt}{t}\Big)^{\frac1q}\frac{du}{u}\\
&\leq \int_0^1 u^{\theta_0}\Big(\int_0^\infty \big((su)^{-\theta} f(s)\big)^q\frac{ds}{s}\Big)^{\frac1q}\frac{du}{u}\\
&\leq \Big(\int_0^\infty (s^{-\theta}f(s))^q\frac{ds}{s}\Big)^{\frac1q}\cdot\int_0^1 u^{-(\theta-\theta_0)}\frac{du}{u}\\
&\leq \gamma\Big(\int_0^\infty (s^{-\theta}f(s))^q\frac{ds}{s}\Big)^{\frac1q}.
\end{aligned}$$

This completes the case of $\theta<\theta_0$, and it only remains to consider the case of $\theta=\theta_0$.

We need to show that the operator A does not have an inverse on the space $(X_0,X_1)_{\theta_0,q}$. As the element $e\in\operatorname{Ker}A$ belongs to $(X_0,X_1)_{\theta_0,\infty}$, therefore A does not have an inverse on $(X_0,X_1)_{\theta_0,\infty}$.

Let us consider the case of $(X_0,X_1)_{\theta_0,q}$ with $q<\infty$. In this case the kernel of A does not intersect with $(X_0,X_1)_{\theta_0,q}$, but we will show that it is possible to construct a family of elements $x_\varepsilon\in(X_0,X_1)_{\theta_0,q}$ such that $\sup_\varepsilon\|Ax_\varepsilon\|_{(Y_0,Y_1)_{\theta_0,q}}<\infty$ and $\lim_{\varepsilon\to0}\|x_\varepsilon\|_{(X_0,X_1)_{\theta_0,q}}=\infty$. Hence the restriction of the operator A on $(X_0,X_1)_{\theta_0,q}$ does not have an inverse.

To construct the family of elements $x_\varepsilon\in(X_0,X_1)_{\theta_0,q}$ we fix an arbitrary $\varepsilon\in(0,1)$ and consider the K-functional of the element e on the three intervals $(0,\varepsilon]$, $(\varepsilon,\varepsilon^{-1})$,

$[\varepsilon^{-1},\infty)$. Let us denote by φ_0^ε, φ_1^ε, and φ_2^ε the concave majorants of $K(\cdot,e;\vec{X})\chi_{(0,\varepsilon]}$, $K(\cdot,e;\vec{X})\chi_{(\varepsilon,\varepsilon^{-1})}$, and $K(\cdot,e;\vec{X})\chi_{[\varepsilon^{-1},\infty)}$ on $(0,\infty)$, i.e.,

$$\begin{aligned}\varphi_0^\varepsilon &= K(\cdot,e;\vec{X})\chi_{(0,\varepsilon)} + K(\varepsilon,e;\vec{X})\chi_{[\varepsilon,\infty)},\\ \varphi_1^\varepsilon &= \frac{t}{\varepsilon}K(\varepsilon,e;\vec{X})\chi_{(0,\varepsilon]} + K(\cdot,e;\vec{X})\chi_{(\varepsilon,\varepsilon^{-1})} + K(\varepsilon^{-1},e;\vec{X})\chi_{[\varepsilon^{-1},\infty)},\\ \varphi_2^\varepsilon &= \frac{t}{\varepsilon^{-1}}K(\varepsilon^{-1},e;\vec{X})\chi_{(0,\varepsilon^{-1}]} + K(\cdot,e;\vec{X})\chi_{(\varepsilon^{-1},\infty)}.\end{aligned} \tag{5}$$

Then $K(\cdot,e;\vec{X}) \le \varphi_0^\varepsilon+\varphi_1^\varepsilon+\varphi_2^\varepsilon$ and from the K-divisibility theorem (see [3]) it follows that there exists a decomposition $e = x_0^\varepsilon + x_1^\varepsilon + x_2^\varepsilon$ such that

$$K(\cdot,x_i^\varepsilon;\vec{X}) \le \gamma\varphi_i^\varepsilon, \quad i=0,1,2,$$

with the constant $\gamma>0$ independent of ε. Let us take $x_\varepsilon = x_1^\varepsilon$. Then we only need to prove that

$$\lim_{\varepsilon\to0}\|x_1^\varepsilon\|_{(X_0,X_1)_{\theta_0,q}} = \infty \tag{6}$$

and

$$\sup_\varepsilon \|Ax_1^\varepsilon\|_{(Y_0,Y_1)_{\theta_0,q}} < \infty. \tag{7}$$

To prove (6) we note that from $K(t,e;\vec{X}) \approx t^{\theta_0}$ and from the formulas (5) for $t\in[\varepsilon,\varepsilon^{-1}]$ it follows that

$$K(t,x_1^\varepsilon;\vec{X}) \ge K(t,e;\vec{X}) - K(t,x_0^\varepsilon;\vec{X}) - K(t,x_2^\varepsilon;\vec{X}) \ge \gamma t^{\theta_0} - \gamma_1\varepsilon^{\theta_0} - \gamma_1\frac{t}{\varepsilon^{-1}}\varepsilon^{-\theta_0}.$$

Let us now fix a number $\delta\in(0,1)$. Then from the above inequality we have

$$\lim_{\varepsilon\to0}\|x_1^\varepsilon\|_{(X_0,X_1)_{\theta_0,q}} \ge \left(\int_\delta^{\delta^{-1}} (t^{-\theta_0}\gamma t^{\theta_0})^q\frac{dt}{t}\right)^{\frac1q} = \gamma\Big(2\ln\frac1\delta\Big)^{\frac1q}.$$

Since $\delta\in(0,1)$ is arbitrary, we have $\lim_{\varepsilon\to0}\|x_1^\varepsilon\|_{(X_0,X_1)_{\theta_0,q}} = \infty$. To prove (7) it is sufficient to prove the following estimate

$$K(t,Ax_1^\varepsilon;\vec{Y}) \le \gamma\varepsilon^{\theta_0}\min\Big(1,\frac{t}{\varepsilon}\Big) + \gamma(\varepsilon^{-1})^{\theta_0}\min\Big(1,\frac{t}{\varepsilon^{-1}}\Big). \tag{8}$$

The proof of (8) outside of the interval $[\varepsilon,\varepsilon^{-1}]$ follows from $K(t,e;\vec{X}) \approx t^{\theta_0}$ and

$$K(t,Ax_1^\varepsilon;\vec{Y}) \le \gamma K(t,x_1^\varepsilon;\vec{X}) \le \gamma\varphi_1^\varepsilon \le \gamma\frac{t}{\varepsilon}K(\varepsilon,e;\vec{X})\chi_{(0,\varepsilon]} + \gamma K(\varepsilon^{-1},e;\vec{X})\chi_{[\varepsilon^{-1},\infty)},$$

and its proof inside the interval $[\varepsilon,\varepsilon^{-1}]$ follows from Lemma 1.2:

$$\begin{aligned}K(t,Ax_1^\varepsilon;\vec{Y}) &\approx \inf_\lambda K(t,x_1^\varepsilon-\lambda e;\vec{X}) \le K(t,x_1^\varepsilon-e;\vec{X}) \le K(t,x_0^\varepsilon;\vec{X}) + K(t,x_2^\varepsilon;\vec{X})\\ &\le \gamma K(\varepsilon,e;\vec{X}) + \gamma\frac{t}{\varepsilon^{-1}}K(\varepsilon^{-1},e;\vec{X}).\end{aligned}$$

Thus the case of $\theta=\theta_0$ and the proof of Theorem A are complete.

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