Feller Semigroups Obtained by Variable Order Subordination

Kristian P. Evans and Niels Jacob

Department of Mathematics
University of Wales Swansea
Singleton Park
Swansea SA2 8PP — United Kingdom
makpe@swan.ac.uk N.Jacob@swan.ac.uk

Received: August 25, 2006
Accepted: December 18, 2006

ABSTRACT

For certain classes of negative definite symbols $q(x,\xi)$ and state space dependent Bernstein function $f(x,s)$ we prove that $-p(x,D)$, the pseudo-differential operator with symbol $-p(x,\xi) = -f(x,q(x,\xi))$, extends to the generator of a Feller semigroup. Our result extends previously known results related to operators of variable (fractional) order of differentiation, or variable order fractional powers. New concrete examples are given.

Key words: Feller semigroups, subordination in the sense of Bochner, pseudo-differential operators with negative definite symbols of variable order, Hoh's symbolic calculus.

2000 Mathematics Subject Classification: 47D07, 47D06, 35S05, 46E35, 60J35.

Introduction

In the early days of the theory of pseudo-differential operators, pseudo differential operators of variable order had already been studied, compare A. Unterberger and J. Bokobza [21]. These considerations were taken up by H.-G. Leopold [16, 17] who gave more emphasis on the function space point of view. On the other hand, also in the early days of the theory of pseudo-differential operators Ph. Courrège [2] pointed out that (most) generators of Feller semigroups are pseudo-differential operators, but

The first named author is grateful for support obtained from Swansea University and EPSRC.

http://dx.doi.org/10.5209/rev_REMA.2007.v20.n2.16482

their symbols do not belong to "nice" or "classical" symbol classes. Indeed, on $S(\mathbb{R}^n)$ the generator of a Feller semigroup has the representation

$$Au(x) = -q(x,D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)\, d\xi$$

where the symbol $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is measurable and locally bounded and for $x \in \mathbb{R}^n$ fixed $q(x,\cdot)$ is a continuous negative definite function, i.e., we have the Lèvy-Khinchin representation

$$q(x,\xi) = c(x) + id(x)\xi + \sum_{k,l=1}^{n} a_{k,l}(x)\xi_k\xi_l + \int_{\mathbb{R}^n\setminus\{0\}} \Big(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+|y|^2}\Big)\nu(x,dy)$$

with $c(x) \geq 0$, $d(x) \in \mathbb{R}^n$, $a_{kl}(x) = a_{lk}(x) \in \mathbb{R}$ and $\sum_{k,l=1}^{n} a_{kl}(x)\xi_k\xi_l \geq 0$, and $\int_{\mathbb{R}^n\setminus\{0\}}(1\wedge|y|^2)\nu(x,dy) < \infty$. Thus these symbols need not to be smooth with respect to ξ nor do they need to have a nice expansion into homogeneous functions. Maybe the fact that these symbols are a bit exotic is the reason why Courrège's result was almost ignored for around 25 years. In [10], see also [9], Courrège's idea was taken up and a systematic study of pseudo-differential operators generating Markov processes was initiated, see also [11–13].

The fact that the composition of a Bernstein function f with a continuous negative definite function ψ is again a continuous negative definite function gives a powerful tool to construct new (Feller) semigroups from given ones. If $q(x,\xi)$ is a suitable symbol such that $-q(x,D)$ generates a Feller semigroup, then $(f\circ q)(x,\xi) = f(q(x.\xi))$ is a symbol with the property that $\xi \to (f\circ q)(x,\xi)$ is a continuous negative definite function and therefore $-(f\circ q)(x,D)$ is a candidate for being a generator of a Feller semigroup. Of course, this procedure is closely linked to subordination in the sense of Bochner.

In a joint paper [14] with H.-G. Leopold it was suggested to study Feller semigroups obtained by subordination of variable order, more precisely, to consider "fractional powers of variable order" in case of the symbol $(1+|\xi|^2)$, i.e., to study $(x,\xi) \to (1+|\xi|^2)^{\alpha(x)}$. These ideas were taken up and further investigations on fractional powers of variable order are due to A. Negoro [20], K. Kikuchi and A. Negoro [15], as well as F. Baldus [1]. Finally, W. Hoh in [7] could combine his symbolic calculus [5] with these ideas, compare W. Hoh [6, 8].

The purpose of this note is twofold. First we suggest a method to study "variable order subordination" for more general Bernstein functions than $f_\alpha(s) = s^\alpha$, $0 < \alpha < 1$. More precisely, we consider symbols of the form

$$p(x,\xi) = f(x, q(x,\xi))$$

where q is a suitable symbol from Hoh's class and $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a smooth function such that for fixed $x \in \mathbb{R}^n$ the function $s \to f(x,s)$ is a Bernstein function.

Our method uses some ideas from the theory of t-coercive (differential) operators as investigated by I. S. Louhivaara and C. Simader [18,19] in order to establish the result that $-p(x, D)$ generates a Feller semigroup. Secondly, we enrich the class of examples by studying the Bernstein function

$$s \to s^{\frac{\alpha}{2}}(1 - e^{-4s^{\frac{\alpha}{2}}}).$$

Since we depend on Hoh's symbolic calculus we recollect some basic facts of this calculus in our first section. All our methods are standard, i.e., they are as in [11–13].

1. Hoh's symbolic calculus

Before starting with our main considerations we need to recollect some basic results from Hoh's symbolic calculus, see W. Hoh [5] or [6], compare also [12].

Definition 1.1. A continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ belongs to the class Λ if for all $\alpha \in {\mathbb{N}_0}^n$ it satisfies

$$|\partial_\xi^\alpha (1 + \psi(\xi))| \leq c_{|\alpha|}(1 + \psi(\xi))^{\frac{2-\rho(|\alpha|)}{2}},$$

where $\rho(k) = k \wedge 2$ for $k \in {\mathbb{N}_0}^n$.

Definition 1.2.

(i) Let $m \in \mathbb{R}$ and $\psi \in \Lambda$. We then call a C^∞-function $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ a symbol in the class $S_\rho^{m,\psi}(\mathbb{R}^n)$ if for all $\alpha, \beta \in {\mathbb{N}_0}^n$ there are constants $c_{\alpha,\beta} \geq 0$ such that

$$|\partial_x^\beta \partial_\xi^\alpha q(x, \xi)| \leq c_{\alpha,\beta}(1 + \psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}}$$

holds for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. We call $m \in \mathbb{R}$ the order of the symbol $q(x, \xi)$.

(ii) Let $\psi \in \Lambda$ and suppose that for an arbitrarily often differentiable function $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ the estimate

$$|\partial_\xi^\alpha \partial_x^\beta q(x, \xi)| \leq \tilde{c}_{\alpha,\beta}(1 + \psi(\xi))^{\frac{m}{2}}$$

holds for all $\alpha, \beta \in {\mathbb{N}_0}^n$ and $x, \xi \in \mathbb{R}^n$. In this case we call q a symbol of the class $S_0^{m,\psi}(\mathbb{R}^n)$.

Note that $S_\rho^{m,\psi}(\mathbb{R}^n) \subset S_0^{m,\psi}(\mathbb{R}^n)$. For $q \in S_0^{m,\psi}(\mathbb{R}^n)$, hence also for $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$, we can define on $S(\mathbb{R}^n)$ the pseudo-differential operator $q(x, D)$ by

$$q(x, D)u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x, \xi)\hat{u}(\xi)\, d\xi$$

and we denote the classes of these operators by $\Psi_\rho^{m,\psi}(\mathbb{R}^n)$ and $\Psi_0^{m,\psi}(\mathbb{R}^n)$, respectively.

Theorem 1.3. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ then $q(x,D)$ maps $S(\mathbb{R}^n)$ continuously into itself.*

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function. For $s \in \mathbb{R}$ and $u \in S(\mathbb{R}^n)$ (or $u \in S'(\mathbb{R}^n)$) we define the norm

$$\|u\|_{\psi,s}^2 = \|(1+\psi(D))^{\frac{1}{2}} u\|_0^2 = \int_{\mathbb{R}^n} (1+\psi(s))^s |\hat{u}(\xi)|^2 \, d\xi.$$

The space $H^{\psi,s}(\mathbb{R}^n)$ is defined as

$$H^{\psi,s}(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n); \|u\|_{\psi,s} < \infty\}.$$

The scale $H^{\psi,s}(\mathbb{R}^n)$, $s \in \mathbb{R}^n$, and more general spaces have been systematically investigated in [3, 4], see also [12]. In particular we know that if for some $\rho_1 > 0$ and $\tilde{c}_1 > 0$ the estimate $\psi(\xi) \geq \tilde{c}_1 |\xi|^{\rho_1}$ holds for all $\xi \in \mathbb{R}^n$, $|\xi| \geq R$, $R \geq 0$, then the space $H^{\psi,s}(\mathbb{R}^n)$ is continuously embedded into $C_\infty(\mathbb{R}^n)$ provided $s > \frac{n}{2\rho_1}$.

Theorem 1.4. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ and let $q(x,D)$ be the corresponding pseudo-differential operator. For all $s \in \mathbb{R}$ the operator $q(x,D)$ maps the space $H^{\psi,m+s}(\mathbb{R}^n)$ continuously into the space $H^{\psi,s}(\mathbb{R}^n)$, and for all $u \in H^{\psi,m+s}(\mathbb{R}^n)$ we have the estimate*

$$\|q(x,D)u\|_{\psi,s} \leq c\|u\|_{\psi,m+s}.$$

On $S(\mathbb{R}^n)$ we may define the bilinear form

$$B(u,v) := (q(x,D)u, v)_0, \quad q \in S_\rho^{m,\psi}(\mathbb{R}^n).$$

Theorem 1.5. *Let $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ be real valued and $m > 0$. It follows that*

$$|B(u,v)| \leq c\|u\|_{\psi,\frac{m}{2}} \|v\|_{\psi,\frac{m}{2}}$$

holds for all $u, v \in S(\mathbb{R}^n)$. Hence the bilinear form B has a continuous extension onto $H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$. If in addition for all $x \in \mathbb{R}^n$

$$q(x,\xi) \geq \delta_0 (1+\psi(\xi))^{\frac{m}{2}} \quad \text{for} \quad |\xi| \geq R \tag{1}$$

with some $\delta_0 > 0$ and $R \geq 0$, and

$$\lim_{|\xi|\to\infty} \psi(\xi) = \infty \tag{2}$$

holds, then we have for all $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ the Gårding inequality

$$Re B(u,u) \geq \frac{\delta_0}{2} \|u\|_{\psi,\frac{m}{2}}^2 - \lambda_0 \|u\|_0^2.$$

Furthermore we have

Theorem 1.6. *If we assume* (1) *and* (2) *then for* $s > -m$ *we have*

$$\frac{\delta_0}{2}\|u\|_{\psi,m+s} \leq \|q(x,D)u\|^2_{\psi,s} + \|u\|^2_{\psi,m+s-\frac{1}{2}}$$

for $q \in S^{m,\psi}_\rho(\mathbb{R}^n)$ *real-valued and all* $u \in H^{\psi,s+m}(\mathbb{R}^n)$.

From Theorem 1.5 and 1.6 one may deduce the following regularity result:

Theorem 1.7. *Let* $q \in S^{m,\psi}_\rho(\mathbb{R}^n)$ *be as in Theorem 1.6,* $m \geq 1$. *Further suppose that for* $f \in H^{\psi,s}(\mathbb{R}^n)$, $s \geq 0$, *there exists* $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ *such that*

$$B(u,\phi) = (f,\phi)_{L^2}$$

holds for all $\phi \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ *(or* $\phi \in S(\mathbb{R}^n)$*). Then* u *belongs already to the space* $H^{\psi,m+s}(\mathbb{R}^n)$.

So far we have used properties of symbols to establish mapping properties and estimates for operators. The real power of a symbolic calculus is that it reduces calculations for operators to calculations for symbols. The following result is most important for us

Theorem 1.8. *Let* $\psi \in \Lambda$. *For* $q_1 \in S^{m_1,\psi}_\rho(\mathbb{R}^n)$ *and* $q_2 \in S^{m_2,\psi}_\rho(\mathbb{R}^n)$ *the symbol* q *of the operator* $q(x,D) := q_1(x,D) \circ q_2(x,D)$ *is given by*

$$q(x,\xi) = q_1(x,\xi) \cdot q_2(x,\xi) + \sum_{j=1}^{n} \partial_{\xi_j} q_1(x,\xi) D_{x_j} q_2(x,\xi) + q_{r_1}(x,\xi) \tag{3}$$

with $q_{r_1} \in S^{m_1+m_2-2,\psi}_0(\mathbb{R}^n)$.

Remark 1.9. An easy calculation yields $q_1 \cdot q_2 \in S^{m_1+m_2,\psi}_\rho(\mathbb{R}^n)$, $\partial_{\xi_j} q_1 \in S^{m_1-1,\psi}_\rho(\mathbb{R}^n)$, and $D_{x_j} q_2 \in S^{m_2,\psi}_\rho(\mathbb{R}^n)$. Hence the second term on the right hand side in (3) belongs to $S^{m_1+m_2-1,\psi}_\rho(\mathbb{R}^n)$.

2. The formal background of our proof that $-p(x,D)$ generates a Feller semigroup

The proof that $-p(x,D)$ as described in the introduction, see also below, extends to a generator of a Feller semigroup depends on various estimates which might be different for different operators. However, once these estimates are established we only need to apply a piece of "soft" analysis. In this section we discuss this part of the proof, i.e., we will assume all crucial estimates hold. Let $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be an arbitrarily often differentiable function such that for $y \in \mathbb{R}^n$ fixed the function $s \to f(y,s)$ is a Bernstein function. Moreover we assume

$$\inf_{y\in\mathbb{R}^n} f(y,s) \geq f_0(s) \quad \text{for all} \quad s \in [0,\infty) \tag{4}$$

as well as

$$\sup_{y\in\mathbb{R}^n} f(y,s) \le f_1(s) \quad \text{for all} \quad s \in [0,\infty) \tag{5}$$

where f_0 and f_1 are Bernstein functions. For a given real-valued negative definite symbol $q(x,\xi)$ it follows that

$$p(y;x,\xi) \coloneqq f(y,q(x,\xi))$$

give rise to a further negative definite symbol by defining

$$p(x,\xi) \coloneqq p(x;x,\xi). \tag{6}$$

In case where $q(x,\xi)$ is comparable with a fixed continuous negative definite function ψ, i.e.,

$$0 < c_0 \le \frac{q(x,\xi)}{\psi(\xi)} \le c_1, \quad c_1 \ge 1, \tag{7}$$

for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, we find using [11, Lemma 3.9.34.B]

$$p(x,\xi) \le f(y_1,q(x,\xi)) \le c_1 f_1(\psi(\xi))$$

and we define

$$\psi_1(\xi) \coloneqq c_1 f_1(\psi(\xi)). \tag{8}$$

Moreover it holds

$$p(x,\xi) \ge f(y_0,q(x,\xi)) \ge c_0' f_0(\psi(\xi))$$

and we set

$$\psi_0(\xi) \coloneqq c_0' f_0(\psi(\xi)). \tag{9}$$

Clearly, ψ_0 and ψ_1 are continuous negative definite functions. Later on we assume that for $|\xi|$ large

$$\psi(\xi) \ge \tilde{c}_1 |\xi|^{\rho_1}, \qquad \tilde{c}_1 > 0 \quad \text{and} \quad \rho_1 > 0 \tag{10}$$

holds as well as

$$f(y_0,s) \ge \tilde{c}_0 s^{\rho_0}, \qquad \tilde{c}_0 > 0 \quad \text{and} \quad \rho_0 > 0. \tag{11}$$

This implies for $|\xi|$ large that

$$\psi_0(\xi) \ge \tilde{c}_2 |\xi|^{\rho_0\rho_1}, \quad \tilde{c}_2 > 0, \tag{12}$$

holds. Since $\psi_0(\xi) \le \psi_1(\xi)$ we have

$$H^{\psi_1,1}(\mathbb{R}^n) \hookrightarrow H^{\psi_0,1}(\mathbb{R}^n).$$

We add the assumption that there exists $0 < \sigma < \frac{1}{2}$ such that

$$(1+\psi_1)^{\frac{1}{2}} \in S_\rho^{1+\sigma,\psi_0}(\mathbb{R}^n). \tag{13}$$

This will imply that

$$H^{\psi_0,m(1+\sigma)}(\mathbb{R}^n) \hookrightarrow H^{\psi_1,m}(\mathbb{R}^n) \tag{14}$$

holds for $m \geq 0$. Further, (13) implies that if $p_1(x,\xi)$ is any symbol belonging to $S_\rho^{m,\psi_1}(\mathbb{R}^n)$ then it also belongs to $S_\rho^{m(1+\sigma),\psi_0}(\mathbb{R}^n)$ which follows from

$$\begin{aligned}|\partial_\xi^\alpha \partial_x^\beta p_1(x,\xi)| &\leq c_{\alpha,\beta}(1+\psi_1(\xi))^{\frac{m-\rho(|\alpha|)}{2}}\\ &\leq \tilde{c}_{\alpha,\beta}(1+\psi_0(\xi))^{\frac{m-\rho(|\alpha|)(1+\sigma)}{2}}\\ &\leq \tilde{c}_{\alpha,\beta}(1+\psi_0(\xi))^{\frac{(1+\sigma)m-\rho(|\alpha|)}{2}}.\end{aligned}$$

The pseudo-differential operator $q(x,D)$ has the symbol $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$. We assume that the pseudo-differential operator $p(x,D)$, defined on $S(\mathbb{R}^n)$ by

$$\begin{aligned}p(x,D)u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\,d\xi\\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x,q(x,\xi))\hat{u}(\xi)\,d\xi\end{aligned}$$

has a symbol $p \in S_\rho^{2+\tau_1,\psi_1}(\mathbb{R}^n)$ for some appropriate $\tau_1 \geq 0$. This implies together with (13) that the operator $p(x,D)$ is continuous from $H^{\psi_0,2+\tau_1+2\sigma+\tau_1\sigma+s}(\mathbb{R}^n)$ to $H^{\psi_0,s}(\mathbb{R}^n)$, in particular it is continuous from $H^{\psi_0,1}(\mathbb{R}^n)$ to $H^{\psi_0,-1-\tau_1-2\sigma-\tau_1\sigma}(\mathbb{R}^n)$. With $p(x,D)$ we can associate the bilinear form

$$B(u,v) := (p(x,D)u,v)_0, \quad u,v \in S(\mathbb{R}^n).$$

Assuming the estimate

$$|B(u,v)| \leq \kappa \|u\|_{\psi_1,1}\|v\|_{\psi_1,1}, \quad \kappa \geq 0,$$

to hold for all $u,v \in S(\mathbb{R}^n)$, we may extend B to a continuous bilinear form on $H^{\psi_1,1}(\mathbb{R}^n)$. This extension is again denoted by B. For $u \in H^{\psi_1,1}(\mathbb{R}^n)$ we assume in addition

$$B(u,u) \geq \gamma \|u\|_{\psi_0,1}^2 - \lambda_0\|u\|_0^2, \quad f\lambda_0 \geq 0, \quad \gamma > 0. \tag{15}$$

Following ideas from I. S. Louhivaara and Ch. Simader, [18, 19], we consider an intermediate space associated with

$$B_{\lambda_0}(u,v) := B(u,v) + \lambda_0(u,v)_0,$$

namely the space $H^{p_{\lambda_0}}(\mathbb{R}^n)$ defined as a completion of $S(\mathbb{R}^n)$ (or $H^{\psi_1,1}(\mathbb{R}^n)$) with respect to the scalar product B_{λ_0}. Obviously we have

$$H^{\psi_1,1}(\mathbb{R}^n) \hookrightarrow H^{p_{\lambda_0}}(\mathbb{R}^n) \hookrightarrow H^{\psi_0,1}(\mathbb{R}^n) \tag{16}$$

in the sense of continuous embeddings. Moreover, by the Lax-Milgram theorem, for every $g \in (H^{p_{\lambda_0}}(\mathbb{R}^n))^*$ exists a unique element $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ satisfying

$$B_{\lambda_0}(u,v) = \langle g, v\rangle \tag{17}$$

for all $v \in H^{p_{\lambda_0}}(\mathbb{R}^n)$. This element we call the variational solution to the equation $p(x,D)u + \lambda_0 u = g$.

From (16) we derive

$$H^{\psi_0,-1}(\mathbb{R}^n) = (H^{\psi_0,1}(\mathbb{R}^n))^* \hookrightarrow (H^{p_{\lambda_0}}(\mathbb{R}^n))^*,$$

hence for $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ there exists a unique $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ satisfying (17). We claim now that for every $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ there exists a unique $u \in H^{\psi_0,1}(\mathbb{R}^n)$ such that

$$p_{\lambda_0}(x,D)u = p(x,D)u + \lambda_0 u = g \tag{18}$$

holds. Denote by $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ the unique solution to (17) for $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ given and take a sequence $(u_k)_{k\in\mathbb{N}}$, $u_k \in S(\mathbb{R}^n)$, converging in $H^{p_{\lambda_0}}(\mathbb{R}^n)$ to u. It follows from

$$(p_{\lambda_0}(x,D)u_k, v)_0 = B_{\lambda_0}(u_k, v), \quad v \in S(\mathbb{R}^n),$$

and the continuity of $p_{\lambda_0}(x,D)$ from $H^{\psi_0,1}(\mathbb{R}^n)$ into $H^{\psi_0,(-1-2\sigma)}(\mathbb{R}^n)$ that for $k \to \infty$

$$\langle p_{\lambda_0}(x,D)u, v\rangle = B_{\lambda_0}(u,v) = \langle g, v\rangle$$

for all $v \in S(\mathbb{R}^n)$. Thus $p_{\lambda_0}(x,D)u = g$. The uniqueness follows of course once again from (15).

In order to get more regularity for variational solutions or equivalently for solutions to (18) we assume that for $\lambda \geq \lambda_0$ the function $p_\lambda^{-1}(x,\xi) := \frac{1}{p(x,\xi)+\lambda}$ belongs to $S_\rho^{-2+\tau_0,\psi_0}(\mathbb{R}^n)$ for some $\tau_0 > 0$. In this case we can prove

Theorem 2.1. *Let $p(x,\xi)$ be given by* (6) *where we assume for q condition* (7) *and for f we require* (4), (5) *to hold. In addition we suppose that $p \in S_\rho^{2+\tau_1,\psi_1}(\mathbb{R}^n) \subset S_\rho^{2+\tau_1+2\sigma+\tau_1\sigma,\psi_0}(\mathbb{R}^n)$ and $p_\lambda^{-1} \in S_\rho^{-2+\tau_0,\psi_0}(\mathbb{R}^n)$, $\tau_1 + \tau_0 + 2\sigma + \tau_1\sigma < 1$. Let $u \in H^{p_{\lambda_0}}(\mathbb{R}^n) \subset H^{\psi_0,1}(\mathbb{R}^n)$ be the solution to* (18) *for $g \in H^{\psi_0,k}(\mathbb{R}^n)$, $k \geq 0$. Then it follows that $u \in H^{\psi_0,2+k-\tau_0}(\mathbb{R}^n)$.*

Proof. From Theorem 1.8 it follows that

$$p_{\lambda_0}^{-1}(x,D) \circ p_{\lambda_0}(x,D) = id + r(x,D) \tag{19}$$

with $r \in S_0^{-1+\tau_1+\tau_0+2\sigma+\tau_1\sigma,\psi_0}(\mathbb{R}^n)$. Since $p_{\lambda_0}(x,D)u = g$ we deduce from (19) that

$$\begin{aligned} u &= p_{\lambda_0}^{-1}(x,D) \circ p_{\lambda_0}(x,D)u - r(x,D)u \\ &= p_{\lambda_0}^{-1}(x,D)g - r(x,D)u. \end{aligned}$$

Now, $p_{\lambda_0}^{-1}(x,D)g \in H^{\psi_0,k+2-\tau_0}(\mathbb{R}^n)$ and $r(x,D)u \in H^{\psi_0,2-\tau_1-\tau_0-2\sigma-\tau_1\sigma}(\mathbb{R}^n)$ implying that $u \in H^{\psi_0,t}(\mathbb{R}^n)$ for $t = (k+2-\tau_0) \wedge (2-\tau_1-\tau_0-2\sigma-\tau_1\sigma) > 1$. With a finite number of iterations we arrive at $u \in H^{\psi_0,2+k-\tau_0}(\mathbb{R}^n)$. □

Remark 2.2. From $\tau_1 + \tau_0 + 2\sigma + \tau_1\sigma < 1$ the necessary condition $\sigma < \frac{1}{2}$ follows.

Corollary 2.3. *In the situation of Theorem 2.1, if* $2 + k - \tau_0 > \frac{n}{2\rho_0\rho_1}$*, compare* (12)*, then* $u \in C_\infty(\mathbb{R}^n)$*.*

Finally we can collect all preparatory material to prove

Theorem 2.4. *Let* $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *be an arbitrarily often differentiable function such that for* $y \in \mathbb{R}^n$ *fixed, the function* $s \to f(y,s)$ *is a Bernstein function. Moreover assume* (4)*,* (5)*, and* (11)*. In addition let* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be a continuous negative definite function in the class* Λ *which satisfies in addition* (10)*. For an elliptic symbol* $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ *satisfying* (7) *we define* $p(x,\xi)$ *by* (6)*. For* ψ_1 *and* ψ_2 *defined by* (8) *and* (9)*, respectively we assume* (14)*. Suppose that* $p \in S_\rho^{2+\tau_1,\psi_1}(\mathbb{R}^n)$ *and* $\frac{1}{p+\lambda} \in S_\rho^{-2+\tau_0,\psi_0}(\mathbb{R}^n)$*. If* $\tau_1 + \tau_0 + \sigma(2+\tau_1) < 1$*,* σ *as in* (14)*, then* $-p(x,D)$ *extends to a generator of a Feller semigroup on* $C_\infty(\mathbb{R}^n)$*.*

Proof. We want to apply the Hille-Yosida-Ray theorem, compare [11, Theorem 4.5.3]. We know that $p(x,D)$ maps $H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$ into $H^{\psi_0,k}(\mathbb{R}^n)$. Hence if $k > \frac{n}{2\rho_0\rho_1}$ the operator $(-p(x,D), H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n))$ is densely defined on $C_\infty(\mathbb{R}^n)$ with range in $C_\infty(\mathbb{R}^n)$. That $-p(x,D)$ satisfies the positive maximum principle on $H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$ follows from [12, Theorem 2.6.1]. Now, for $\lambda \geq \lambda_0$ we know that for $g \in H^{\psi_0,k+1}(\mathbb{R}^n)$ we have a unique solution to $p_\lambda(x,D)u = g$ belonging to $H^{\psi_0,2+k+1-\tau_0}(\mathbb{R}^n)$. But $\tau_1 + \tau_0 + 2\sigma + \tau_1\sigma < 1$ implies that $H^{\psi_0,2+k+1-\tau_0}(\mathbb{R}^n) \subset H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$, hence for $g \in H^{\psi_0,k+1}(\mathbb{R}^n)$ we always have a (unique) solution $u \in H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$ implying the theorem. □

3. Some concrete examples

The first part of this section will consider the work W. Hoh has done on pseudo-differential operators with variable order of differentiation. We will consider the case where the Bernstein function $s \to f(s)$ is substituted by $(x,s) \to s^{r(x)}$ with $r : \mathbb{R}^n \to \mathbb{R}$ being a continuous function such that $0 \leq r(x) \leq 1$ holds. Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that $\xi \to q(x,\xi)$ is a continuous negative definite function. It then follows that

$$\xi \to q(x,\xi)^{r(x)}$$

is once again a continuous negative definite function implying that the pseudo-differential operator

$$Au(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)^{r(x)} \hat{u}(\xi)\, d\xi$$

is a candidate for a generator of a Feller semigroup. We now meet Hoh's result:

Theorem 3.1. *Let* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be a fixed continuous negative definite function such that its Lévy measure has a compact support and that*

$$\psi(\xi) \geq c_0|\xi|^r, \quad |\xi| \text{ large and } r > 0,$$

holds. Let $q \in S^{2,\psi}_{\rho}(\mathbb{R}^n)$ *be a real-valued negative definite symbol which is elliptic, i.e., we have*

$$q(x,\xi) \geq \delta_0(1+\psi(\xi)).$$

Further let $m : \mathbb{R}^n \to (0,1]$ *be an element in* $C^{\infty}_b(\mathbb{R}^n)$ *satisfying*

$$M - \mu < \frac{1}{2}$$

where $M := \sup m(x)$ *and* $0 < \mu := \inf m(x)$*. Consider the symbol*

$$(x,\xi) \to p(x,\xi) := q(x,\xi)^{m(x)}$$

which has the property that $\xi \to p(x,\xi)$ *is a continuous negative definite function. The operator*

$$-p(x,D)u(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi$$

maps $C^{\infty}_0(\mathbb{R}^n)$ *into* $C_{\infty}(\mathbb{R}^n)$*, is closeable in* $C_{\infty}(\mathbb{R}^n)$ *and its closure is a generator of a Feller semigroup.*

For a proof see W. Hoh [7], compare also [6].

We are now going to consider a further example. First note that the function $s \to \sqrt{s}(1-e^{-4\sqrt{s}})$ is a Bernstein function. Hence, using [11, Corollary 3.9.36], it follows that for $0 \leq \alpha \leq 1$ the function $s \to s^{\frac{\alpha}{2}}(1-e^{-4s^{\frac{\alpha}{2}}})$ is also a Bernstein function. Thus, given a negative definite symbol $q \in S^{2,\psi}_{\rho}(\mathbb{R}^n)$ we may consider the new symbol

$$p(x,\xi) = (1+q(x,\xi))^{\frac{\alpha(x)}{2}}\left(1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\right)$$

for $\alpha(\cdot)$ being an appropriate function.

Lemma 3.2. *Let* $q \in S^{2,\psi}_{\rho}(\mathbb{R}^n)$ *be a real-valued negative definite symbol which is elliptic, i.e.,*

$$q(x,\xi) \geq \delta_0(1+\psi(\xi)).$$

Also let $\alpha(\cdot) : \mathbb{R}^n \to (0,1]$ *be an element in* $C^{\infty}_b(\mathbb{R}^n)$ *satisfying*

$$m - \mu < \frac{1}{2}$$

where $m = \sup \frac{\alpha(x)}{2}$ *and* $\mu = \inf \frac{\alpha(x)}{2} > 0$*.*

Now if we let $p(x,\xi) = (1+q(x,\xi))^{\frac{\alpha(x)}{2}}\left(1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\right)$*, then we have for all* $\epsilon > 0$ *the estimates*

$$|\partial^{\alpha}_{\xi}\partial^{\beta}_{x} p(x,\xi)| \leq c_{\alpha,\beta,\epsilon}\, p(x,\xi)(1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}} \tag{20}$$

i.e., $p \in S^{2m+\epsilon,\psi}_{\rho}(\mathbb{R}^n)$*.*

Proof. We have to estimate

$$\begin{aligned}\partial_\xi^\alpha \partial_x^\beta p(x,\xi) &= \partial_\xi^\alpha \partial_x^\beta \Big((1+q(x,\xi))^{\frac{\alpha(x)}{2}} \Big(1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)\Big) \\ &= \partial_\xi^\alpha \partial_x^\beta \Big(e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))} \Big(1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)\Big).\end{aligned}$$

Using [11, (2.19)] we get

$$\begin{aligned}&\partial_\xi^\alpha \partial_x^\beta \Big(e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))} \Big(1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)\Big) \\ &\qquad = \sum_{\alpha'\le\alpha}\sum_{\beta'\le\beta} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} (\partial_\xi^{\alpha'}\partial_x^{\beta'} e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}) \\ &\qquad\qquad \times \Big(\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}\Big(1 - e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)\Big). \qquad (21)\end{aligned}$$

First consider

$$|(\partial_\xi^{\alpha'}\partial_x^{\beta'} e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))})|.$$

By [11, (2.28)] with $l = |\alpha'| + |\beta'|$ we get

$$\begin{aligned}&|(\partial_\xi^{\alpha'}\partial_x^{\beta'} e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))})| \\ &\qquad \le e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))} \sum_{\substack{\alpha'^1+\cdots+\alpha'^{l'}=\alpha' \\ \beta'^1+\cdots+\beta'^{l'}=\beta' \\ l'=0,1,\dots,l}} \Big| c_{\{\alpha'^j,\beta'^j\}} \prod_{j=1}^{l'} q_{\alpha'^j\beta'^j}(x,\xi)\Big|, \qquad (22)\end{aligned}$$

where

$$\begin{aligned}q_{\alpha'^j\beta'^j}(x,\xi) &= \partial_\xi^{\alpha'^j}\partial_x^{\beta'^j}\Big(\frac{\alpha(x)}{2}\log(1+q(x,\xi))\Big) \\ &= \sum_{\bar\beta'^j\le\beta'^j} \binom{\beta'^j}{\bar\beta'^j}\Big(\partial_x^{\beta'^j-\bar\beta'^j}\frac{\alpha(x)}{2}\Big)\,\partial_\xi^{\alpha'^j}\partial_x^{\bar\beta'^j}\log(1+q(x,\xi)).\end{aligned}$$

Now, using [11, (2.26)] with $k = |\alpha'^j| + |\bar\beta'^j| > 0$ we get

$$\partial_\xi^{\alpha'^j}\partial_x^{\bar\beta'^j}\log(1+q(x,\xi)) = \sum_{\substack{\tilde\alpha'^1+\cdots+\tilde\alpha'^{l'} \\ \tilde\beta'^1+\cdots+\tilde\beta'^{l'}=\bar\beta'^j}} c_{\{\tilde\alpha'^j,\tilde\beta'^j\}} \prod_{i=1}^{k} \frac{\partial_\xi^{\tilde\alpha'^i}\partial_x^{\tilde\beta'^i}(1+q(x,\xi))}{(1+q(x,\xi))}.$$

Since we assume that $q(x,\xi)$ is an elliptic symbol in $S_\rho^{2,\psi}(\mathbb{R}^n)$, we get

$$\begin{aligned}\left|\partial_\xi^{\alpha'^j}\partial_x^{\bar\beta'^j}\log(1+q(x,\xi))\right| &\le c_{\alpha'^j,\bar\beta'^j}\sum_{\substack{\tilde\alpha'^1+\cdots+\tilde\alpha'^{l'}\\ \tilde\beta'^1+\cdots+\tilde\beta'^{l'}=\bar\beta'^j}}\prod_{i=1}^{k}(1+\psi(\xi))^{\frac{-\rho(|\tilde\alpha'^i|)}{2}}\\ &\le c_{\alpha^j,\bar\beta^j}(1+\psi(\xi))^{\frac{-\rho(|\alpha'^j|)}{2}},\end{aligned}$$

where we used the subadditivity of ρ. We always have

$$|\log(1+q(x,\xi))|\le c_\epsilon(1+\psi(\xi))^{\frac{\epsilon}{2l}}.$$

It follows for $\alpha\in C_b^\infty(\mathbb{R}^n)$ that

$$|q_{\alpha'^j,\beta'^j}(x,\xi)|\le c_{\alpha'^j,\beta'^j,\epsilon}\begin{cases}(1+\psi(\xi))^{\frac{-\rho(|\alpha'^j|)}{2}}, & \alpha'^j\neq 0\\ (1+\psi(\xi))^{\frac{\epsilon}{2l}}, & \alpha'^j=0.\end{cases}\tag{23}$$

Putting (22) and (23) together we get

$$\left|\left(\partial_\xi^{\alpha'}\partial_x^{\beta'}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}\right)\right|\le c_{\alpha',\beta',\epsilon}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}(1+\psi(\xi))^{\frac{-\rho(|\alpha'|)+\epsilon}{2}}.\tag{24}$$

For the desired result we need

$$\begin{aligned}&\left|\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}\left(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\right)\right|\\ &\qquad\le c_{\alpha',\beta',\alpha,\beta,\epsilon}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}}.\end{aligned}$$

When $\alpha-\alpha'=0$ and $\beta-\beta'=0$ there is nothing to prove.

Otherwise, by [11, (2.28)] with $l_2=|\alpha-\alpha'|+|\beta-\beta'|$, we get

$$\begin{aligned}&\left|\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})\right|\\ &\qquad\le e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\left|\sum c_{\{(\alpha-\alpha')^j,(\beta-\beta')^j\}}\prod_{j=1}^{l_2'}q_{(\alpha-\alpha')^j(\beta-\beta')^j}(x,\xi)\right|,\end{aligned}\tag{25}$$

where the sum is such that

$$\begin{gathered}(\alpha-\alpha')^1+\cdots+(\alpha-\alpha')^{l_2'}=(\alpha-\alpha'),\\ (\beta-\beta')^1+\cdots+(\beta-\beta')^{l_2'}=(\beta-\beta'),\\ l_2'=1,\dots,l_2,\end{gathered}$$

and where

$$q_{(\alpha-\alpha')^j(\beta-\beta')^j}(x,\xi)=\partial_\xi^{(\alpha-\alpha')^j}\partial_x^{(\beta-\beta')^j}(4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}).$$

Since $q(x,\xi)$ is an elliptic symbol in the class $S_\rho^{2,\psi}(\mathbb{R}^n)$ we have the estimate

$$|q_{(\alpha-\alpha')^j(\beta-\beta')^j}(x,\xi)| \le \tilde{L}(1+q(x,\xi)) \quad \text{for all} \quad (\alpha-\alpha')^j, (\beta-\beta')^j \in \mathbb{N}_0^n,$$

where $\tilde{L}(\lambda)$ is a suitable polynomial ≥ 0 which might depend on $(\alpha-\alpha')^j$ and $(\beta-\beta')^j$. Now returning to (25) we get

$$\begin{aligned}
&\Big|\partial_\xi^{(\alpha-\alpha')}\partial_x^{(\beta-\beta')}\Big(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)\Big| \\
&\quad \le \tilde{L}(1+q(x,\xi))e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \\
&\quad = \frac{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\cdot\frac{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\tilde{L}(1+q(x,\xi))e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \\
&\qquad \times(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}}(1+\psi(\xi))^{\frac{\rho(|\alpha-\alpha'|)}{2}} \\
&\quad \le \frac{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}}\cdot c_0
\end{aligned}$$

since

$$\left|\frac{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}(1+\psi(\xi))^{\frac{\rho(|\alpha-\alpha'|)}{2}}\tilde{L}(1+q(x,\xi))e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\right| \le c_0.$$

Now using [12, (2.7)], i.e., for all $a \ge 0$ and $t \ge 0$ the estimate

$$\frac{at}{1+at} \le 1-e^{-at},$$

we get

$$\begin{aligned}
&\Big|\partial_\xi^{(\alpha-\alpha')}\partial_x^{(\beta-\beta')}\Big(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)\Big| \\
&\qquad\qquad \le c_0\Big(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}}. \qquad (26)
\end{aligned}$$

Substituting (24) and (26) into (21)

$$\begin{aligned}
&\Big|\partial_\xi^\alpha\partial_x^\beta\Big(e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}\Big(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)\Big)\Big| \\
&\qquad \le \sum_{\alpha'\le\alpha}\sum_{\beta'\le\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}c_{\alpha',\beta',\epsilon}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))} \\
&\qquad\quad \times(1+\psi(\xi))^{\frac{-\rho(|\alpha'|)+\epsilon}{2}}\Big(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big)(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}} \\
&\qquad \le c_{\alpha,\beta,\epsilon}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}\Big(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\Big) \\
&\qquad\quad \times(1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}} \\
&\qquad \le c_{\alpha,\beta,\epsilon}p(x,\xi)(1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}}.
\end{aligned}$$

The proof now follows from the estimate $p(x,\xi) \le (1+\psi(\xi))^m$. □

Lemma 3.3. *The function* $p_\lambda^{-1}(x,\xi) = \frac{1}{p(x,\xi)+\lambda}$ *belongs to the class* $S_\rho^{-2\mu+\epsilon,\psi}(\mathbb{R}^n)$.

Proof. Using [11, (2.27)] we find with $l = |\alpha| + |\beta|$ that

$$|\partial_\xi^\alpha \partial_x^\beta p_\lambda^{-1}(x,\xi)| \le \frac{1}{p_\lambda(x,\xi)} \sum_{\substack{\alpha^1+\dots+\alpha^l=\alpha \\ \beta^1+\dots+\beta^l=\beta}} c_{\{\alpha^j,\beta^j\}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} p_\lambda(x,\xi)}{p_\lambda(x,\xi)} \right|.$$

For any $\epsilon > 0$ we find using (20)

$$\left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} p_\lambda(x,\xi)}{p_\lambda(x,\xi)} \right| \le \tilde{c}_{\alpha^j,\beta^j} (1+\psi(\xi))^{\frac{-\rho(|\alpha^j|)+\epsilon}{2}}$$

and the ellipticity assumption of $p(x,\xi)$ together with the subadditivity of ρ yields

$$|\partial_\xi^\alpha \partial_x^\beta p_\lambda^{-1}(x,\xi)| \le \tilde{c}_{\alpha,\beta,\epsilon} (1+\psi(\xi))^{-\mu} (1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}}$$

which proves the lemma. □

References

[1] F. Baldus, *Application of the Weyl-Hörmander calculus to generators of Feller semigroups*, Math. Nachr. **252** (2003), 3–23.

[2] Ph. Courrège, *Sur la forme intégro-différentielle des opérateurs de C_K^∞ dans C satisfaisant au principle du maximum*, Sém. Théorie du Potentiel (1965/66). Exposé 2, 38 pp.

[3] W. Farkas, N. Jacob, and R. L. Schilling, *Function spaces related to continuous negative definite functions*: *ψ-Bessel potential spaces*, Dissertationes Math. **393** (2001), 1–62.

[4] ______, *Feller semigroups, L^p-sub-Markovian semigroups, and applications to pseudo-differential operators with negative definite symbols*, Forum Math. **13** (2001), no. 1, 51–90.

[5] W. Hoh, *A symbolic calculus for pseudo-differential operators generating Feller semigroups*, Osaka J. Math. **35** (1998), no. 4, 789–820.

[6] ______, *Pseudo differential operators generating Markov processes*, Habilitationsschrift, Universität Bielefeld, Bielefeld, 1998.

[7] ______, *Pseudo differential operators with negative definite symbols of variable order*, Rev. Mat. Iberoamericana **16** (2000), no. 2, 219–241.

[8] ______, *Perturbations of pseudodifferential operators with negative definite symbol*, Appl. Math. Optim. **45** (2002), no. 3, 269–281.

[9] N. Jacob, *Dirichlet forms and pseudo differential operators*, Exposition. Math. **6** (1988), no. 4, 313–351.

[10] ______, *Feller semigroups, Dirichlet forms, and pseudo differential operators*, Forum Math. **4** (1992), no. 5, 433–446.

[11] ______, *Pseudo differential operators and Markov processes*, I: *Fourier analysis and semigroups*, Imperial College Press, London, 2001.

[12] ———, *Pseudo differential operators and Markov processes*, II: *Generators and their potential theory*, Imperial College Press, London, 2002.

[13] ———, *Pseudo differential operators and Markov processes*, III: *Markov processes and applications*, Imperial College Press, London, 2005.

[14] N. Jacob and H.-G. Leopold, *Pseudo-differential operators with variable order of differentiation generating Feller semigroups*, Integral Equations Operator Theory **17** (1993), no. 4, 544–553.

[15] K. Kikuchi and A. Negoro, *On Markov process generated by pseudodifferential operator of variable order*, Osaka J. Math. **34** (1997), no. 2, 319–335.

[16] H.-G. Leopold, *Pseudodifferentialoperatoren und Funktionenräume variabler Glattheit*, Dissertation B, Friedrich-Schiller-Universität Jena, Jena, 1987.

[17] ———, *On function spaces of variable order of differentiation*, Forum Math. **3** (1991), no. 1, 1–21.

[18] I. S. Louhivaara and C. G. Simader, *Fredholmsche verallgemeinerte Dirichletprobleme für koerzitive lineare partielle Differentialoperatoren*, Proceedings of the Rolf Nevanlinna Symposium on Complex Analysis (Math. Res. Inst., Univ. Istanbul, Silivri, 1976), Publ. Math. Res. Inst. Istanbul, vol. 7, Univ. Istanbul, Istanbul, 1978, pp. 47–57.

[19] ———, *Über koerzitive lineare partielle Differentialoperatoren*: *Fredholmsche verallgemeinerte Dirichletprobleme und deren Klasseneinteilung*, Complex analysis and its applications: A collection of papers dedicated to I. N. Vekua on his 70$^{\text{th}}$ birthday, N. N. Boboljuvov et al, eds., Izdat. Nauka, Moscow, 1978, pp. 342–345.

[20] A. Negoro, *Stable-like processes*: *construction of the transition density and the behavior of sample paths near* $t = 0$, Osaka J. Math. **31** (1994), no. 1, 189–214.

[21] A. Unterberger and J. Bokobza, *Les opérateurs pseudo-différentiels d'ordre variable*, C. R. Acad. Sci. Paris **261** (1965), 2271–2273.