Interpolation of the Measure of Non-Compactness between Quasi-Banach Spaces

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ABSTRACT

We study the behavior of the ball measure of non-compactness under several interpolation methods. First we deal with methods that interpolate couples of spaces, and then we proceed to extend the results to methods that interpolate finite families of spaces. We will need an approximation hypothesis on the target family of spaces.

Key words: measure of non-compactness, interpolation theory, quasi-Banach spaces.

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Introduction

The study of interpolation of compact operators has its origins in the paper by Krasnoselskii [17] in which it is proved that the compactness of an operator can be interpolated between L_p-spaces. Following this paper, a number of authors have studied the interpolation properties of compact operators between abstract Banach spaces. Among then J. L. Lions and J. Peetre [19], A. Calderón [3], A. Persson [25], S. G. Krein

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and Yu. I. Petunin [18], and K. Hayakawa [15]. The proof of the fact that real interpolation preserves compactness (only one restriction of the operator being compact) was given by Cwikel in [10], see also the paper by Cobos, Kühn, and Schonbek [7]. This problem remains open for the complex interpolation method. Significant contributions are due to Cwikel [10], Cwikel and Kalton [11], Cobos, Kühn, and Schonbek [7], Cobos, Fernández-Cabrera, and Martínez [4] among other authors.

The measure of non-compactness of an operator is a quantity closely related to compactness. Actually, if $T : A \longrightarrow B$ is a bounded linear operator acting between quasi-normed spaces, the ball measure of non-compactness $\beta(T)$ of T is given by the infimum of all $r > 0$ such that there exists a finite number of elements $b_1, \dots, b_s \in B$ so that

$$T(\mathcal{U}_A) \subseteq \bigcup_{j=1}^{s} \{b_j + r\mathcal{U}_B\}.$$

As it is easily seen $0 \leq \beta(T) \leq \|T\|$, and the operator T is compact if and only if $\beta(T) = 0$. Thus the measure of non-compactness gives an estimate of the grade of compactness of the operator. If $T : (A_0, A_1) \longrightarrow (B_0, B_1)$ is a bounded linear operator acting between two (quasi)-Banach couples and $\mathcal{M}$ is an interpolation method, it is natural to ask what is the relation among $\beta_0(T) = \beta(T : A_0 \longrightarrow B_0)$, $\beta_1(T) = \beta(T : A_1 \longrightarrow B_1)$ and $\beta(T : \mathcal{M}(A_0, A_1) \longrightarrow \mathcal{M}(B_0, B_1))$. The behavior of the measure of non-compactness under real interpolation has been studied by different authors. The first results are due to Teixeira and Edmunds in 1981, see [27]. In that paper they studied two very different types of results: when some of the couples degenerates in a single space, either $A_0 = A_1$ or $B_0 = B_1$, and the much more interesting situation when no couple is degenerated. In order to prove a theorem in this latter case they had to impose an approximation hypothesis on the target couple $\bar{B} = (B_0, B_1)$. The estimate obtained by Teixeira and Edmunds is the following logarithmically convex inequality:

$$\beta(T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}) \leq C\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}.$$

Years later, in 1999, Cobos, Martínez, and the present author proved a completely general result removing the approximation hypothesis from Teixeira and Edmunds result, see [5]. The technique they used to establish this result strongly depends on the construction of the real interpolation method. This allowed to establish similar results for interpolations methods that extend the classical real interpolation method and that work with more than two spaces, see [6]. Unfortunately we have not been able to use this approach to establish estimates for other methods than the real. Nevertheless, if we impose an approximation hypothesis on the target couple, as Teixeira and Edmunds did, we are able to give estimates of the measure of non-compactness of the interpolated operator for a variety of interpolation methods.

This paper is developed in the context of quasi-Banach spaces. The interpolation couples, or N-tuples, that will be handled are formed by quasi-Banach spaces. Let us recall the fact that if $(A, \| \|_A)$ is a quasi-Banach space with constant C, $\|x+y\| \leq C(\|x\|+\|y\|)$, and we choose $\rho > 0$ such that $(2C)^\rho = 2$, then there exists a norm $\| \|_A^*$ such that

$$\| \|_A^* \leq \| \|_A^\rho \leq 2\| \|_A^*,$$

see [2] for more details. Given a quasi-Banach couple $\bar{A} = (A_0, A_1)$, A_i quasi-Banach space of constant C_{A_i}, we have the quantities ρ_{A_0} and ρ_{A_1} satisfying $(2C_{A_0})^{\rho_{A_0}} = (2C_{A_1})^{\rho_{A_1}} = 2$. The spaces $A_0 \cap A_1$ and $A_0 + A_1$ are also quasi-Banach spaces of constant $\max\{C_{A_0}, C_{A_1}\}$. If we take $\rho_{\bar{A}} = \min\{\rho_{A_0}, \rho_{A_1}\}$, we have the equality $\big(2\max\{C_{A_0}, C_{A_1}\}\big)^{\rho_{\bar{A}}} = 2$, and the $\rho_{\bar{A}}$-powers of the norms on $A_0 \cap A_1$ or on $A_0 + A_1$ are equivalent to some norms on either space respectively.

The structure of the paper is the following: Section 1 is devoted to the study of the behavior of the measure of non-compactness under interpolation methods for couples. Section 2 treats the case of some multidimensional methods. Section 3 extends the result in [5] to the quasi-Banach case.

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1. Interpolation of the measure of non-compactness for Operators between couples.

Let us start with some basic concepts of interpolation theory, for a more complete reading see [1,2]. We say that $\bar{A} = (A_0, A_1)$ is a quasi-Banach couple if A_0 and A_1 are quasi-Banach spaces both continuously embedded in the same Hausdorff topological space $\mathcal{U}$. In this conditions we can consider the intersection

$$\Delta(\bar{A}) = A_0 \cap A_1 \quad \text{with quasi-norm } \|a\|_{\Delta(\bar{A})} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\},$$

and the sum

$$\Sigma(\bar{A}) = A_0 + A_1 \quad \text{with quasi-norm } \|a\| = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1}; a = a_0 + a_1\}.$$

A quasi-Banach space A is said to be intermediate with respect to the couple $\bar{A}$ if

$$\Delta(\bar{A}) \hookrightarrow A \hookrightarrow \Sigma(\bar{A}).$$

Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two quasi-Banach couples and let $T : \Sigma(\bar{A}) \longrightarrow \Sigma(\bar{B})$ be a bounded linear operator such that T maps A_0 into B_0 and A_1 into B_1 boundedly. In that case we say T is a bounded linear operator between the couples $\bar{A}$ and $\bar{B}$ and we write $T : \bar{A} \longrightarrow \bar{B}$. The norm of the operator is defined as

$$\|T\|_{\bar{A},\bar{B}} = \max\{\|T\|_0, \|T\|_1\}$$

where $\|T\|_i = \|T : A_i \longrightarrow B_i\|$ for $i = 0, 1$.

An interpolation method $\mathcal{M}$ associates to each couple $\bar{A}$ an intermediate quasi-Banach space $\mathcal{M}(\bar{A})$ with the following interpolation property: If $T : \bar{A} \longrightarrow \bar{B}$ is a bounded linear operator between two couples then $T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})$ is a bounded linear operator. Moreover

$$\|T\|_{\mathcal{M}(\bar{A}),\mathcal{M}(\bar{B})} \leq C \max\{\|T\|_0, \|T\|_1\},$$

where the constant C does not depends on $\bar{A}, \bar{B}$ or T.

Let $\mathcal{M}$ be an interpolation method. Closely related to $\mathcal{M}$ there is the function

$$\varphi_{\mathcal{M}}(t,s) = \sup\{\, \|T\|_{\mathcal{M}(\bar{A}),\mathcal{M}(\bar{B})} \text{ s.t. } \|T\|_0 \leq t,\ \|T\|_1 \leq s\,\}$$

that already appears in [20], and among whose most immediate properties must be mentioned that $\varphi_{\mathcal{M}}$ is homogeneous of degree one, non-decreasing in each variable and there is also the inequality

$$\|T\|_{\mathcal{M}(\bar{A}),\mathcal{M}(\bar{B})} \leq \varphi_{\mathcal{M}}(\|T\|_0, \|T\|_1).$$

The approximation hypothesis used by Teixeira and Edmunds in [27] is the following

Approximation Hypothesis. *We say a couple $\bar{B} = (B_0, B_1)$ satisfies the Approximation Hypothesis if there are positive constants C_0, C_1 such that for every $\varepsilon > 0$ and finite sets $F_0 \subset B_0$ and $F_1 \subset B_1$ there exists an operator $P \in \mathcal{L}(\bar{B}, \bar{B})$ such that*

(i) *$P : B_k \longrightarrow B_k$ is compact, $k = 0$ or 1.*

(ii) *$P(B_k) \subset \Delta(\bar{B})$ for $k = 0, 1$.*

(iii) *$\|I - P\|_k \leq C_k$ and $\|x - Px\|_{B_k} < \varepsilon$ for all $x \in F_k$, $k = 0, 1$,*

Using the Approximation Hypothesis we can establish the following Persson type compactness theorem. A similar theorem, restricted to the Banach case and with a weaker approximation hypothesis, can be found in [20].

Theorem 1.1. *Let $\mathcal{M}$ be any interpolation method, $\bar{A}$ and $\bar{B}$ quasi-Banach couples, $\bar{B}$ satisfying the Approximation Hypothesis. Let $T : \bar{A} \longrightarrow \bar{B}$ be a bounded linear operator. If either*

$$T : A_0 \longrightarrow B_0 \text{ is compact and } \varphi_{\mathcal{M}}(t,1) \to 0 \text{ as } t \to 0$$

or

$$T : A_1 \longrightarrow B_1 \text{ is compact and } \varphi_{\mathcal{M}}(1,t) \to 0 \text{ as } t \to 0$$

then the operator $T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})$ is compact

Now we are in condition to establish the main theorem of this section.

Theorem 1.2. *Let $\bar{B} = (B_0, B_1)$ a quasi-Banach couple with the Approximation Hypothesis, and let $\mathcal{M}$ be an interpolation method satisfying*

$$\varphi_{\mathcal{M}}(t^{1-k}, t^k) \to 0 \text{ as } t \to 0, \quad \text{for } k = 0, 1.$$

Then for any bounded linear operator $T : \bar{A} \longrightarrow \bar{B}$ the following inequality holds

$$\beta(T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})) \leq C\varphi_{\mathcal{M}}(\beta_0(T), \beta_1(T))$$

for some constant C independent of T and $\bar{A}$.

Proof. The proof uses the ideas of Teixeira and Edmunds in [27]. Given any $\varepsilon > 0$, the Approximation Hypothesis guarantees there exists $P \in \mathcal{L}(\bar{B}, \bar{B})$ such that:

(i) $P : B_k \longrightarrow B_k$ is compact, $k = 0$ or 1.

(ii) $P(B_k) \subset \Delta(\bar{B})$ for $k = 0, 1$.

(iii) $\|T - PT\|_k \leq 2^{1/\rho_{B_k}} C_k \beta_k(T) + \varepsilon$, for $k = 0, 1$.

In fact, choose $\delta > 0$ such that, for $k = 0, 1$,

$$2^{1/\rho_{B_k}} \left[\left(C_k(\beta_k(T) + \delta)\right)^{\rho_{B_k}} + \delta^{\rho_{B_k}}\right]^{1/\rho_{B_k}} < 2^{1/\rho_{B_k}} C_k \beta_k(T) + \varepsilon. \tag{1}$$

By definition of measure of non-compactness there exist finite sets $F_0 \subset B_0$ and $F_1 \subset B_1$ such that

$$T(\mathcal{U}_{A_k}) \subseteq \bigcup_{y \in F_k} \left\{y + (\beta_k(T) + \delta)\mathcal{U}_{B_k}\right\} \quad \text{for } k = 0, 1.$$

By the Approximation Hypothesis there exists $P \in \mathcal{L}(\bar{B}, \bar{B})$ satisfying conditions (i), (ii), and such that $\|y - Py\|_{B_k} \leq \delta$ for any $y \in F_k$ and $\|I - P\|_k \leq C_k$, for $k = 0, 1$. Let's check condition (iii). Given $a \in \mathcal{U}_{A_k}$,

$$\begin{aligned} \|Ta - PTa\|_{B_k}^{\rho_{B_k}} &= \|(I - P)Ta\|_{B_k}^{\rho_{B_k}} \leq 2[\|(I - P)(Ta - y)\|_{B_k}^{\rho_{B_k}} + \|(I - P)y\|_{B_k}^{\rho_{B_k}}] \\ &\leq 2[\left(C_k(\beta_k(T) + \delta)\right)^{\rho_{B_k}} + \delta^{\rho_{B_k}}] \leq \left(2^{1/\rho_{B_k}} C_k \beta_k(T) + \varepsilon\right)^{\rho_{B_k}}, \end{aligned}$$

where the last inequality follows from equation (1). In order to estimate the measure of non-compactness split the operator T as $T = (T - PT) + PT$ and thus, recalling that $\mathcal{M}(\bar{B})$ is within the class of quasi-Banach spaces and so for some $\rho > 0$ $\| \ \|_{\mathcal{M}(\bar{B})}$ is a ρ-norm, obtain the inequality

$$\beta(T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B}))^\rho \leq 2[\beta(T - PT)^\rho + \beta(PT)^\rho].$$

The operator $PT : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})$ is compact by Theorem 1.1, hence $\beta(PT) = 0$. The measure of non-compactness of the other term can be estimated by its norm, so

$$\begin{aligned} \beta(T - PT : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})) &\leq \|T - PT\|_{\mathcal{M}(\bar{A}), \mathcal{M}(\bar{B})} \\ &\leq \varphi_{\mathcal{M}}(\|T - PT\|_0, \|T - PT\|_1) \\ &\leq \varphi_{\mathcal{M}}(2^{1/\rho_{B_0}} C_0 \beta_0(T) + \varepsilon, 2^{1/\rho_{B_1}} C_1 \beta_1(T) + \varepsilon). \end{aligned}$$

Taking infimum in $\varepsilon > 0$ and using the properties of $\varphi_{\mathcal{M}}$

$$\beta(T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})) \leq 2^{1/\rho} 2^{1/\rho_{\bar{B}}} \max\{C_0, C_1\} \varphi_{\mathcal{M}}(\beta_0(T), \beta_1(T)). \qquad \square$$

Now we show some examples where the theorem applies.

Example 1.3 (The complex method). It is also well known the complex interpolation method is an exact interpolation method of exponent $0 < \theta < 1$. As a consequence

$$\varphi_{[\theta]}(t, s) = t^{1-\theta} s^{\theta}.$$

Then for any operator $T : \bar{A} \longrightarrow \bar{B}$ acting between Banach couples, with $\bar{B}$ satisfying the Approximation Hypothesis, the measure of non-compactness fulfills the inequality

$$\beta(T : \bar{A}_{[\theta]} \longrightarrow \bar{B}_{[\theta]}) \leq C_0^{1-\theta} C_1^{\theta} \beta_0(T)^{1-\theta} \beta_1(T)^{\theta}.$$

Example 1.4 (The Peetre and Gustavsson-Peetre methods). These methods are also known as the $\pm$-methods, they depend on a functional parameter and have similar definitions. Let $\rho : (0, \infty) \longrightarrow (0, \infty)$ be a quasi-concave function ($\rho(t)$ increases from 0 to ∞ and $\frac{\rho(t)}{t}$ decreases from ∞ to 0). Assume also that $s_\rho(\lambda) = \circ(\max\{1, \lambda\})$ as $\lambda \to 0$ or ∞. Given a quasi-Banach couple $\bar{A} = (A_0, A_1)$ Peetre defined in [24] the space $\langle \bar{A} \rangle_\rho$ as the set of all those sums $\sum_{m \in \mathbb{Z}} u_m$ convergent in $\Sigma(\bar{A})$ such that for any bounded sequence of scalars, $(\varepsilon_m)_{-\infty}^{\infty}$, the inequalities

$$\Big\| \sum_{-\infty}^{\infty} \varepsilon_m 2^{jm} \tfrac{u_m}{\rho(2^{jm})} \Big\|_{A_j} \leq C \sup_{m \in \mathbb{Z}} |\varepsilon_m|$$

hold for some constant C and $j = 0, 1$.

A few years later Gustavsson and Peetre introduced in [14] the space $\langle \bar{A}, \rho \rangle$ in a very similar way as Peetre did. See also [13, 21]. This time they required the sequences $(u_m)_{m \in \mathbb{Z}}$ to satisfy, besides the convergence of $\sum_{m \in \mathbb{Z}} u_m$ in $\sum(\bar{A})$, that for any bounded sequence of scalars, $(\varepsilon_m)_{-\infty}^{\infty}$, the inequalities

$$\Big\| \sum_{\mathcal{F}} \varepsilon_m 2^{jm} \tfrac{u_m}{\rho(2^{jm})} \Big\|_{A_j} \leq C \sup_{m \in \mathbb{Z}} |\varepsilon_m|$$

hold for some constant C, any finite set $\mathcal{F}$ and $j = 0, 1$.

Both interpolation methods are of genus s_ρ. That is to say, if $T : \bar{A} \longrightarrow \bar{B}$ is a bounded linear operator between two quasi-Banach couples then

$$\|T : \langle \bar{A} \rangle_\rho \longrightarrow \langle \bar{B} \rangle_\rho\| \leq C \|T\|_0 \, s_\rho\left(\frac{\|T\|_1}{\|T\|_0} \right),$$
$$\|T : \langle \bar{A}, \rho \rangle \longrightarrow \langle \bar{B}, \rho \rangle\| \leq C \|T\|_0 \, s_\rho\left(\frac{\|T\|_1}{\|T\|_0} \right),$$

where the constant C is independent of T, $\bar{A}$ and $\bar{B}$. These estimates provide upper bounds for the corresponding φ functions, actually

$$\varphi_{\langle\cdot\rangle_\rho}(t,s) \leq Ct\, s_\rho\Big(\frac{s}{t}\Big),$$
$$\varphi_{\langle\cdot,\rho\rangle}(t,s) \leq Ct\, s_\rho\Big(\frac{s}{t}\Big).$$

Now let $\bar{B}$ be a quasi-Banach couple with the Approximation Hypothesis, then for any bounded linear operator $T : \bar{A} \longrightarrow \overline{B}$ the following inequalities hold:

$$\beta(T : \langle\bar{A}\rangle_\rho \longrightarrow \langle\bar{B}\rangle_\rho) \leq C\beta_0(T)\, s_\rho\Big(\frac{\beta_1}{\beta_0}\Big),$$
$$\beta(T : \langle\bar{A},\rho\rangle \longrightarrow \langle\bar{B},\rho\rangle) \leq C\beta_0(T)\, s_\rho\Big(\frac{\beta_1}{\beta_0}\Big).$$

Example 1.5 (The Ovchinnikov Method). Let $\rho : (0,\infty) \longrightarrow (0,\infty)$ be as in the previous example. Given a Banach couple $\bar{A}$ consider the space

$$H_1(\bar{A}) = \mathrm{Corb}_{\ell_1(\frac{1}{\rho(2^n)})}[\ell_1, \ell_1(2^{-n})](\bar{A}),$$

that is to say the space of all elements $a \in \Sigma(\bar{A})$ such that

$$\sup_{\|T\|_{\overline{\ell_1},\bar{A}} \leq 1} \big\{\|Ta\|_{\ell_1(\frac{1}{\rho(2^n)})}\big\} < \infty,$$

where $\overline{\ell_1} = (\ell_1, \ell_1(2^{-n}))$. This interpolation method defined by Ovchinnikov in 1976 can be seen as the "dual" of Gustavsson-Peetre method, see [22, 23] and also [16] for more information.

Ovchinnikov's method is of genus s_ρ. Actually, if $T : \bar{A} \longrightarrow \bar{B}$ is a bounded linear operator then

$$\|T : H_1(\bar{A}) \longrightarrow H_1(\bar{B})\| \leq 2\|T\|_0\, s_\rho\left(\frac{\|T\|_1}{\|T\|_0}\right).$$

Hereby it follows $\varphi_{H_1}(t,s) \leq 2t\, s_\rho(\frac{s}{t})$. If in addition the couple $\bar{B}$ fulfills the Approximation Hypothesis, then

$$\beta(T : H_1(\bar{A}) \longrightarrow H_1(\bar{B})) \leq C\beta_0(T)\, s_\rho\Big(\frac{\beta_1(T)}{\beta_0(T)}\Big).$$

2. Interpolation of the measure of non-compactness for Operators between N-tuples.

This section is devoted to interpolation methods for N-tuples of spaces. In order to establish some previous and essential compactness results we will need a K-functional over the N-tuples of spaces we handle. Following Cobos and Peetre in [8] we choose a convex polygon $\Pi = \overline{P_1, \dots, P_N}$ in $\mathbb{R}^2$ with vertices $P_j = (x_j, y_j)$. Let $\bar{A} = \{A_1, \dots, A_N\}$be a quasi-Banach N -tuple. We can imagine each space of the N-tuple located on the corresponding vertex of the polygon Π. Now, for any $a \in \sum(\bar{A})$ and $t, s > 0$ define

$$K(t, s, a; \bar{A}) = \inf\left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} \, ; \, a = \sum_{j=1}^{N} a_j \right\}.$$

Let $\mathcal{M}$ be an interpolation method for N-tuples of spaces. Consider the function

$$\varphi_M(t_1, \dots, t_N) = \sup\{ \|T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})\| \}$$

where T runs over all those operators acting between interpolation N-tuples, $T : \bar{A} \longrightarrow \bar{B}$, satisfying that $\|T : A_j \longrightarrow B_j\| \leq t_j$ for $1 \leq j \leq N$.

Associated to the interpolation method $\mathcal{M}$ and the polygon Π we define the function

$$\varphi^{\Pi}_{\mathcal{M}}(t, s) = \varphi_{\mathcal{M}}(t^{x_1} s^{y_1}, t^{x_2} s^{y_2}, \dots, t^{x_N} s^{y_N}).$$

The following inequality relates the K-functional, the norm on $\mathcal{M}(\bar{A})$ and the function $\varphi^{\Pi}_{\mathcal{M}}$: for all $a \in \mathcal{M}(\bar{A})$ and $t, s > 0$

$$K(t, s, a : \bar{A}) \leq \varphi^{\Pi}_{\mathcal{M}}(t, s) \|a\|_{\mathcal{M}(\bar{A})}.$$

That is to say, the interpolation space $\mathcal{M}(\bar{A})$ is of class $\mathcal{C}(\varphi^{\Pi}_{\mathcal{M}}; \bar{A})$. This puts us in conditions to establish the following Lions-Peetre compactness theorem:

Theorem 2.1. *Let $\bar{A} = \{A_1, A_2, \dots, A_N\}$ be a quasi-Banach N-tuple, B a quasi-Banach space and $T : \bar{A} \longrightarrow B$ a bounded linear operator. If for some nonempty subset $I \subseteq \{1, 2, \dots, N\}$*

(i) *$T : A_j \longrightarrow B_j$ is compact for every $j \in I$ and*

(ii) *there exist sequences (t_n) and (s_n) in $\mathbb{R}^+$ such that $\forall j \notin I$*

$$\frac{\varphi^{\Pi}_{\mathcal{M}}(t_n, s_n)}{t_n^{x_j} s_n^{y_j}} \to 0 \text{ as } n \to \infty,$$

then the interpolated operator $T : \mathcal{M}(\bar{A}) \longrightarrow B$ is compact.

This Lions-Peetre type compactness theorem is a first step to interpolation of measure of non-compactness. However a more general result, in which the target N-tuple does not degenerated into a single space, will be needed. We will state a Persson type compactness theorem using an approximation hypothesis on the target N-tuple $\bar{B}$.

Subsequently, and as in the previous theorem, I will stand for a fixed nonempty subset of $\{1, 2, \dots, N\}$. Given $T : \bar{A} \longrightarrow \bar{B}$ a bounded linear operator, one may think of I as the set of indexes for which the corresponding restrictions, $T : A_j \longrightarrow B_j$, are compact. We will consider interpolations methods for which there exist sequences (t_n) and (s_n) in $\mathbb{R}^+$ such that for every $j \notin I$

$$\frac{\varphi^{\Pi}_{\mathcal{M}}(t_n, s_n)}{t_n^{x_j} s_n^{y_j}} \to 0 \text{ as } n \to \infty. \tag{2}$$

The following approximation hypothesis will be used to establish a Persson type compactness theorem as well as to estimate the measure of non-compactness of the interpolated operator.

Approximation Hypothesis. *We say that a quasi-Banach N-tuple $\bar{B} = \{B_1, B_2, \dots, B_N\}$ satisfies the Approximation Hypothesis if there exist positive constants $C_1, C_2, \dots, C_N$ such that for every $\varepsilon > 0$ and finite sets $F_1 \subset B_1$, $F_2 \subset B_2, \dots, F_N \subset B_N$, there exists an operator $P \in \mathcal{L}(\bar{B}, \bar{B})$ such that:*

(i) *$P : B_j \longrightarrow B_j$ is compact for every $j \in I$.*

(ii) *$P(B_k) \subset \Delta(\bar{B})$ for $k = 1, 2, \dots, N$.*

(iii) *$\|I - P\|_k \le C_k$ and $\|x - Px\|_{B_k} < \varepsilon$ for all $x \in F_k$, $k = 1, 2, \dots, N$.*

As an example of a N-tuple that satisfies the Approximation Hypothesis we may mention that if X is a locally compact space endowed with a positive measure μ and $1 \le p_1, p_2, \dots, p_N < \infty$, then the N-tuple

$$\{L_{p_1}(X, \mu), L_{p_2}(X, \mu), \dots, L_{p_N}(X, \mu)\}$$

satisfies the Approximation Hypothesis. In order to check this statement just follow the ideas of Persson in [25] and those of Teixeira and Edmunds in [27].

Our next compactness theorem concerns interpolation methods that not only satisfy condition (2) but whose function $\varphi_{\mathcal{M}}$ also satisfies

$$\varphi_{\mathcal{M}}(\bar{\varepsilon}) \to 0 \quad \text{as } \varepsilon \to 0 \tag{3}$$

where $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)$ with

$$\varepsilon_j = \begin{cases} \varepsilon & \text{if } j \in I, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2.2. *Let $\bar{A} = \{A_1, \dots, A_N\}$ and $\bar{B} = \{B_1, \dots, B_N\}$ be quasi-Banach N-tuples, $\bar{B}$ satisfying the Approximation Hypothesis. Let $T : \bar{A} \longrightarrow \bar{B}$ be a bounded linear operator such that for every $j \in I$, $T : A_j \longrightarrow B_j$ is compact. Then, if the interpolation method $\mathcal{M}$ fulfills conditions* (2) *and* (3)*, the operator $T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})$ is compact.*

Once established these compactness theorem we focus on interpolation of measure of non-compactness.

Theorem 2.3. *Let $\mathcal{M}$ be an interpolation method that fulfills conditions* (2) *and* (3)*. Let $T : \bar{A} \longrightarrow \bar{B}$ be a bounded linear operator between quasi-Banach N-tuples, $\bar{B}$ satisfying the Approximation Hypothesis. Then*

$$\beta(T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})) \leq C\varphi_{\mathcal{M}}(\beta_1(T), \dots, \beta_N(T))$$

for some constant C independent of T and $\bar{A}$.

Proof. Given any $\varepsilon > 0$, use the ideas of Theorem 1.2 to choose an operator $P \in \mathcal{L}(\bar{B}, \bar{B})$ such that

(i) $P : B_j \longrightarrow B_j$ is compact for every $j \in I$.

(ii) $P(B_k) \subset \Delta(\bar{B})$ for $k = 1, 2, \dots, N$.

(iii) $\|T - TP\|_k \leq 2^{1/\rho_{B_k}} C_k \beta_k(T) + \varepsilon$ for $k = 1, 2, \dots, N$.

Decompose $T : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})$ as $T = (T - PT) + PT$. Since $\mathcal{M}(\bar{B})$ is within the class of quasi-Banach spaces its norm, $\| \, \|_{\mathcal{M}(\bar{B})}$, is a ρ-norm for some $\rho > 0$. Then

$$\beta(T)^\rho \leq 2(\beta(T - PT)^\rho + \beta(PT)^\rho).$$

The operator $PT : \mathcal{M}(\bar{A}) \longrightarrow \mathcal{M}(\bar{B})$ is compact by Theorem 2.2. So $\beta(PT) = 0$, which leads us to the inequalities

$$\begin{aligned}\beta(T) &\leq 2^{1/\rho}\beta(T - PT) \leq 2^{1/\rho}\|T - PT\|_{\mathcal{M}} \\ &\leq 2^{1/\rho}\varphi_{\mathcal{M}}(\|T - PT\|_1, \|T - PT\|_2, \dots, \|T - PT\|_N) \\ &\leq 2^{1/\rho}\varphi_{\mathcal{M}}\big(2^{1/\rho_{B_1}} C_1\beta_1(T) + \varepsilon, 2^{1/\rho_{B_2}} C_2\beta_2(T) + \varepsilon, \dots, 2^{1/\rho_{B_N}} C_N\beta_N(T) + \varepsilon\big).\end{aligned}$$

Taking infimum on $\varepsilon > 0$, and using the facts that $\varphi_{\mathcal{M}}$ is homogeneous and non-decreasing in each variable, we get that

$$\begin{aligned}\beta(T) &\leq 2^{1/\rho}\varphi_{\mathcal{M}}\big(2^{1/\rho_{B_1}} C_1\beta_1(T), 2^{1/\rho_{B_2}} C_2\beta_2(T), \dots, 2^{1/\rho_{B_N}} C_N\beta_N(T)\big) \\ &\leq 2^{1/\rho}2^{1/\rho_{\bar{B}}} \max\{C_1, C_2, \dots, C_N\}\varphi_{\mathcal{M}}(\beta_1(T), \dots, \beta_N(T)).\end{aligned}$$

□

Remark 2.4. In case we treat with interpolation methods associated to polygons, such as J and K interpolation methods, see [8], or the Gustavsson-Peetre methods for N-tuples, see [12], the hypotheses on the function $\varphi_{\mathcal{M}}$, conditions (2) and (3), can be easily verified.

These methods use a convex polygon $\Pi = \overline{P_1, \dots, P_N}$ and an interior point (α, β). The known estimates for the norm of the interpolated operator establish the inequality

$$\varphi_{\mathcal{M}}(t_1, \dots, t_N) \leq C \max_{\mathcal{P}_{(\alpha,\beta)}} \left\{ t_i^{c_i} t_j^{c_j} t_k^{c_k} \right\}$$

where $\mathcal{P}_{(\alpha,\beta)}$ stand for the set of all those triangles $\overline{P_iP_jP_k}$ that contain the point (α,β) and (c_i, c_j, c_k) are the barycentric coordinates of (α,β) with respect to the vertices P_i,P_j, and P_k. So, in order to satisfy hypothesis (2) we need all restrictions of the operator, but at most two consecutive ones, to be compact. In fact, suppose the non-compact restrictions of the operator are $T : A_k \longrightarrow B_k$ and $T : A_{k+1} \longrightarrow B_{k+1}$. By [26, Lemma 1.4, chapter 4] there exist sequences (t_n) and (s_n) in $\mathbb{R}$ such that

$$\lim_n t_n^{x_j-x_k} s_n^{y_j-y_k} = 0 = \lim_n t_n^{x_j-x_{k+1}} s_n^{y_j-y_{k+1}}$$

and

$$t_n^{x_k-x_{k+1}} s_n^{y_k-y_{k+1}} = 1 \quad \forall n \in \mathbb{N}.$$

In particular

$$\begin{aligned}\frac{\varphi^{\Pi}_{\mathcal{M}}(t_n, s_n)}{t_n^{x_k} s_n^{y_k}} &= \frac{\varphi^{\Pi}_{\mathcal{M}}(t_n^{x_1} s_n^{y_1}, \dots, t_n^{x_N} s_n^{y_N})}{t_n^{x_k} s_n^{y_k}}\\ &\leq C \max_{\mathcal{P}_{(\alpha,\beta)}} \left\{ (t_n^{x_i-x_k} s_n^{y_i-y_k})^{c_i} (t_n^{x_j-x_k} s_n^{y_j-y_k})^{c_j} (t_n^{x_l-x_k} s_n^{y_l-y_k})^{c_l} \right\} \to 0\end{aligned}$$

as $n \to \infty$. So, for these methods it suffices to ask all restrictions, except two consecutive ones, to be compact.

3. Interpolation of the Measure of Noncompactness by the real method. The quasi-Banach case.

The real interpolation method does not depend heavily on all properties of the norms involved. In fact, in most cases the triangle inequality can be replace by the more general quasi-triangle inequality, $\|x+y\| \leq C(\|x\|+\|y\|)$. One can think that regarding the real interpolation method those results that hold for Banach spaces must also hold in the quasi-Banach spaces context. As an example we may recall that the well known result of interpolation of compact operators between Banach couples by the real method also holds for operators acting between quasi-Banach couples, see [9]. This is also the case of interpolation of measure of non-compactness, whose behavior under interpolation by the real method was studied by Cobos, Martínez, and the present author in [5], see [6] for the polygon methods. The former mentioned paper is developed in the Banach spaces context, however the result can be extended to the quasi-Banach space framework.

For $0 < \rho_{\bar{A}} < \infty$ consider the spaces $\ell_{\rho_{\bar{A}}}(G_m)$ and $\ell_{\rho_{\bar{A}}}(2^{-m}G_m)$ as the collection of all those sequences, $(u_m)_{m\in\mathbb{Z}}$ in $A_0 \cap A_1$, for which the respective quasi-norms

$$\|(u_m)\|_{\ell_{\rho_{\bar{A}}}(G_m)} = \Big(\sum_{m\in\mathbb{Z}} J(2^m, u_m; \bar{A})^{\rho_{\bar{A}}}\Big)^{1/\rho_{\bar{A}}}$$

or

$$\|(u_m)\|_{\ell_{\rho_{\bar{A}}}(2^{-m}G_m)} = \Big(\sum_{m\in\mathbb{Z}} (2^{-m} J(2^m, u_m; \bar{A}))^{\rho_{\bar{A}}}\Big)^{1/\rho_{\bar{A}}}$$

are finite. We will also deal with the spaces $\ell_\infty(F_m)$ and $\ell_\infty(2^{-m}F_m)$ defined as usually.

Our next result will need the real interpolation spaces $(A_0, A_1)_{\theta,q;J}$ and $(B_0, B_1)_{\theta,q;K}$ endowed with the corresponding discrete quasi-norms:

$$\|a\|_{\theta,q;J} = \inf_{a=\sum u_m} \Big\{ \Big(\sum_{m\in\mathbb{Z}} (2^{-\theta m} J(2^m, u_m; \bar{A}))^q \Big)^{1/q} \Big\}$$

$$\|a\|_{\theta,q;K} = \Big(\sum_{m\in\mathbb{Z}} (2^{-\theta m} K(2^m, a; \bar{B}))^q \Big)^{1/q}.$$

Now we are in conditions to formulate the following result:

Theorem 3.1. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $T : \bar{A} \longrightarrow \bar{B}$ be a bounded linear operator. Then for any $0 < q \le \infty$ and $0 < \theta < 1$ we have*

$$\beta(T : \bar{A}_{\theta,q;J} \longrightarrow \bar{B}_{\theta,q;K}) \le C\beta(T : A_0 \longrightarrow B_0)^{1-\theta}\beta(T : A_1 \longrightarrow B_1)^{\theta}$$

Proof. The operator j, that associates to each element b in $B_0 + B_1$ the constant sequence $(\dots, b, b, b, \dots)$, acts boundedly into $\ell_\infty(F_m)$ when it is restricted to B_0, and into $\ell_\infty(2^{-\theta m}F_m)$ when it is restricted to B_1. Actually $\|j : B_0 \longrightarrow \ell_\infty(F_m)\| \le 1$, $\|j : B_1 \longrightarrow \ell_\infty(2^{-m}F_m)\| \le 1$ and

$$j : (B_0, B_1)_{\theta,q;K} \longrightarrow \ell_q(2^{-\theta m}F_m)$$

is a metric injection.

The operator π, that associates every sequence $(u_m)_{m\in\mathbb{Z}} \subset A_0 \cap A_1$ with the element $\pi\big((u_m)\big) = \sum_{m\in\mathbb{Z}} u_m$, is bounded from $\ell_{\rho_{\bar{A}}}(G_m) + \ell_{\rho_{\bar{A}}}(2^{-m}G_m)$ into $A_0 + A_1$. It is easy to check that $\|\pi : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow A_0\| \le 2\max\{C_{A_0}, C_{A_1}\}$, $\|\pi : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow A_1\| \le 2\max\{C_{A_0}, C_{A_1}\}$ and that

$$\pi : \ell_q(2^{-\theta m}G_m) \longrightarrow (A_0, A_1)_{\theta,q;J}$$

is a metric surjection. We have the following diagram of bounded operators:

$$\begin{array}{ccccccc}
\ell_{\rho_{\bar{A}}}(G_m) & \xrightarrow{\ \pi\ } & A_0 & \xrightarrow{\ T\ } & B_0 & \xrightarrow{\ j\ } & \ell_q(2^{-\theta m}F_m) \\
\ell_{\rho_{\bar{A}}}(2^{-m}G_m) & \xrightarrow{\ \pi\ } & A_1 & \xrightarrow{\ T\ } & B_1 & \xrightarrow{\ j\ } & \ell_\infty(2^{-m}F_m) \\
\hline\hline
\ell_q(2^{-\theta m}G_m) & \xrightarrow{\ \pi\ } & (A_0, A_1)_{\theta,q} & \xrightarrow{\ T\ } & B_0, B_1)_{\theta,q} & \xrightarrow{\ j\ } & \ell_\infty(F_m)
\end{array}$$

Put $\widehat{T} = jT\pi$. Properties of π and j yield that

$$\begin{aligned}
\beta_0(\widehat{T}) &= \beta\big(\widehat{T} : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow \ell_\infty(F_m)\big) \\
&\le \big\|\pi : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow A_0\big\|\,\beta\big(T : A_0 \longrightarrow B_0\big)\,\big\|j : B_0 \longrightarrow \ell_\infty(F_m)\big\| \\
&\le 2\max\{C_{A_0}, C_{A_1}\}\beta\big(T : A_0 \longrightarrow B_0\big).
\end{aligned}$$

Similarly

$$\beta_1(\widehat{T}) = \beta(\widehat{T}\ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_\infty(2^{-m}F_m)) \le 2\max\{C_{A_0}, C_{A_1}\}\beta(T : A_1 \longrightarrow B_1),$$

and finally, since $\max\{1, 2^{\frac{1-q}{q}}\}\max\{C_{B_0}, C_{B_1}\}$ is a valid constant for the quasinorm of $(B_0, B_1)_{\theta,q;K}$,

$$\begin{aligned}\beta_{\theta,q}(T) &= \beta(T : (A_0, A_1)_{\theta,q;J} \longrightarrow (B_0, B_1)_{\theta,q;K}) \\ &= \beta(T\pi : \ell_q(2^{-\theta m}G_m) \longrightarrow (B_0, B_1)_{\theta,q;K}) \\ &\le 2\max\{1, 2^{\frac{1-q}{q}}\}\max\{C_{B_0}, C_{B_1}\}\beta(\widehat{T}).\end{aligned} \tag{4}$$

So, in order to establish the theorem it suffices to estimate the measure of non-compactness $\beta(\widehat{T} : \ell_q(2^{-\theta m}G_m) \longrightarrow \ell_q(2^{-\theta m}F_m))$. Let's start by introducing a family of operators on the sequence space $\ell_{\rho_{\bar{A}}}(G_m) + \ell_{\rho_{\bar{A}}}(2^{-m}G_m)$. Define

$$P_n, P_n^-, P_n^+ : \ell_{\rho_{\bar{A}}}(G_m) + \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(G_m) + \ell_{\rho_{\bar{A}}}(2^{-m}G_m)$$

as

$$\begin{aligned}P_n(u) &= (\dots, 0, 0, 0, u_{-n}, u_{-n+1}, \dots, u_{n-1}, u_n, 0, 0, 0, \dots) \\ P_n^+(u) &= (\dots, u_{-n-3}, u_{-n-2}, u_{-n-1}, 0, 0, 0, \dots, 0, 0, 0, \dots) \\ P_n^-(u) &= (\dots, 0, 0, 0, \dots, 0, 0, 0, u_{n+1}, u_{n+2}, u_{n+3}, \dots).\end{aligned}$$

Clearly these operators satisfy the following properties:

(I) The identity operator on $\ell_1(G_m) + \ell_1(2^{-m}G_m)$ can be decomposed as $I = P_n + P_n^+ + P_n^-$ for all $n \in \mathbb{N}$.

(II) The operators are uniformly bounded, in fact:

$$\begin{aligned}\|P_n : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(G_m)\| &= \|P_n : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(2^{-m}G_m)\| = 1 \\ \|P_n^+ : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(G_m)\| &= \|P_n^+ : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(2^{-m}G_m)\| = 1 \\ \|P_n^- : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(G_m)\| &= \|P_n^- : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(2^{-m}G_m)\| = 1\end{aligned}$$

(III) The norms of the operators acting from $\ell_{\rho_{\bar{A}}}(G_m)$ into $\ell_{\rho_{\bar{A}}}(2^{-m}G_m)$ or viceversa are small, precisely

$$\begin{aligned}&\|P_n^+ : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(2^{-m}G_m)\| \\ &\qquad = \|P_n^- : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(G_m)\| = 2^{-(n+1)}.\end{aligned}$$

On the sequence space $\ell_\infty(F_m) + \ell_\infty(2^{-m}F_m)$ we can defined analogous operators, Q_n, Q_n^+ and Q_n^-, satisfying the corresponding versions of (I), (II), and (III).

$$
\begin{array}{ccc}
\ell_q(2^{-\theta m}G_m) & \xrightarrow{\;Q_n^+\widehat{T}P_n^-\;} & \ell_q(2^{-\theta m}F_m) \\
\Big\downarrow i & & \Big\uparrow i \\
\big(\ell_{\rho_{\bar{A}}}(G_m), \ell_{\rho_{\bar{A}}}(2^{-m}G_m)\big)_{\theta,q;J} & \xrightarrow{\;Q_n^+\widehat{T}P_n^-\;} & \big(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m)\big)_{\theta,q;K}
\end{array}
$$

Figure 1

Use these families of operators to decompose $\widehat{T} : \ell_q(2^{-\theta m}G_m) \longrightarrow \ell_q(2^{-\theta m}F_m)$ as

$$
\begin{aligned}
\widehat{T} &= \widehat{T}(P_n + P_n^+ + P_n^-) = \widehat{T}P_n + (Q_n + Q_n^+ + Q_n^-)\widehat{T}(P_n^+ + P_n^-) \\
&= \widehat{T}P_n + Q_n\widehat{T}(P_n^+ + P_n^-) + Q_n^+\widehat{T}P_n^- + Q_n^-\widehat{T}P_n^+ + Q_n^+\widehat{T}P_n^+ + Q_n^-\widehat{T}P_n^-.
\end{aligned}
$$

Recall that $\max\{1, 2^{\frac{1-q}{q}}\}\max\{C_{B_0}, C_{B_1}\}$ is a valid constant for the quasinorm of $\ell_q(2^{-\theta m}F_m)$, and so $\|\cdot\|_{\ell_q(2^{-\theta m}F_m)}$ fulfills the inequality

$$\|\cdot\|^* \le \|\cdot\|^\rho_{\ell_q(2^{-\theta m})F_m} \le 2\|\cdot\|^*$$

where $\|\cdot\|^*$ is a norm on $\ell_q(2^{-\theta m}F_m)$ and ρ satisfies

$$\big(\max\{1, 2^{\frac{1-q}{q}}\}\max\{C_{B_0}, C_{B_1}\}\big)^\rho = 2. \tag{5}$$

Using the previous considerations it is easy to establish the estimate

$$
\begin{aligned}
\beta(\widehat{T})^\rho \le 2\big[&\beta(\widehat{T}P_n)^\rho + \beta(Q_n\widehat{T}(P_n^+ + P_n^-))^\rho + \beta(Q_n^+\widehat{T}P_n^-)^\rho \\
&+ \beta(Q_n^-\widehat{T}P_n^+)^\rho + \beta(Q_n^+\widehat{T}P_n^+)^\rho + \beta(Q_n^-\widehat{T}P_n^-)^\rho\big].
\end{aligned} \tag{6}
$$

Now we proceed to estimate each one of these terms. Let's start with the term $\beta(Q_n^+\widehat{T}P_n^-) = \beta(Q_n^+\widehat{T}P_n^- : \ell_q(2^{-\theta m}G_m) \longrightarrow \ell_q(2^{-\theta m}F_m))$. Factorize the operator according to the diagram in figure 1.

It is easily checked that

$$\big\|\ell_q(2^{-\theta m}G_m) \longrightarrow \big(\ell_{\rho_{\bar{A}}}(G_m), \ell_{\rho_{\bar{A}}}(2^{-m}G_m)\big)_{\theta,q;J}\big\| = 1$$

and

$$\big\|\big(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m)\big)_{\theta,q;K} \longrightarrow \ell_q(2^{-\theta m}F_m)\big\| = \max\{C_{B_0}, C_{B_1}\}.$$

We will also need the norm of the inclusion from a J-space into a K-space, $\|i : (\cdot,\cdot)_{\theta,q;J} \hookrightarrow (\cdot,\cdot)_{\theta,q;K}\| = \delta$, where $\delta = \frac{1}{3-2^\theta-2^{1-\theta}}$ in the Banach case and $\delta = 2^{1/\eta}\big[\frac{1-2^{-\eta}}{1-2^{-\theta\eta}-2^{(\theta-1)\eta}+2^{-\eta}}\big]^{1/\eta}$ with $\eta < \min\{\rho_{\bar{A}}, q\}$ in the quasi-Banach case.

Estimating $\beta(Q_n^+\widehat{T}P_n^-)$ through the factorization we have that

$$\begin{aligned}
&\beta(Q_n^+\widehat{T}P_n^-)\\
&\le \max\{C_{B_0},C_{B_1}\}\beta_{\theta,q}(Q_n^+\widehat{T}P_n^-) \le \max\{C_{B_0},C_{B_1}\}\\
&\quad\times \big\|Q_n^+\widehat{T}P_n^- : \big(\ell_{\rho_{\bar{A}}}(G_m),\ell_{\rho_{\bar{A}}}(2^{-m}G_m)\big)_{\theta,q;J} \longrightarrow \big(\ell_\infty(F_m),\ell_\infty(2^{-m}F_m)\big)_{\theta,q;K}\big\|\\
&\le \max\{C_{B_0},C_{B_1}\}\delta\|Q_n^+\widehat{T}P_n^-\|_{\theta,q;K}\\
&\le 2^\theta \max\{C_{B_0},C_{B_1}\}\delta\|Q_n^+\widehat{T}P_n^-\|_0^{1-\theta}\|Q_n^+\widehat{T}P_n^-\|_1^\theta.
\end{aligned}$$

The norm $\|Q_n^+\widehat{T}P_n^-\|_0$ is upper bounded by $\|\widehat{T}\|_0$ while for the norm $\|Q_n^+\widehat{T}P_n^-\|_1$ the factorization

$$\begin{array}{ccc}
\ell_{\rho_{\bar{A}}}(G_m) & \xrightarrow{\ \widehat{T}\ } & \ell_\infty(F_m)\\
{\scriptstyle P_n^-}\uparrow & & \downarrow{\scriptstyle Q_n^+}\\
\ell_{\rho_{\bar{A}}}(2^{-m}G_m) & & \ell_\infty(2^{-m}F_m)
\end{array}$$

can be used to establish the following inequalities:

$$\begin{aligned}
\|Q_n^+\widehat{T}P_n^-\|_1 &= \|Q_n^+\widehat{T}P_n^- : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_\infty(2^{-m}F_m)\|\\
&\le \|P_n^- : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_{\rho_{\bar{A}}}(G_m)\|\|\widehat{T} : \ell_{\rho_{\bar{A}}}(G_m) \longrightarrow \ell_\infty(F_m)\|\\
&\quad\times \|Q_n^+ : \ell_\infty(F_m) \longrightarrow \ell_\infty(2^{-m}F_m)\| \le 2^{-(n+1)}\|\widehat{T}\|_0 2^{-(n+1)} \to 0
\end{aligned}$$

as $n\to\infty$. Then,

$$\lim_n \beta(Q_n^+\widehat{T}P_n^-) = 0. \tag{7}$$

Similarly

$$\lim_n \beta(Q_n^-\widehat{T}P_n^+) = 0. \tag{8}$$

Next we estimate the term

$$\beta(Q_n^+\widehat{T}P_n^+) = \beta\big(Q_n^+\widehat{T}P_n^+ : \ell_q(2^{-\theta m}G_m) \longrightarrow \ell_q(2^{-\theta m}F_m)\big).$$

As it was done before we factorize the operator $Q_n^+\widehat{T}P_n^+$ as we did with $Q_n^+\widehat{T}P_n^-$ according to figure 1.

$$\begin{aligned}
&\beta(Q_n^+\widehat{T}P_n^+)\\
&\le \max\{C_{B_0},C_{B_1}\}\\
&\quad\times \beta_{\theta,q}\big(Q_n^+\widehat{T}P_n^+\big(\ell_{\rho_{\bar{A}}}(G_m),\ell_{\rho_{\bar{A}}}(2^{-m}G_m)\big)_{\theta,q;J} \longrightarrow \big(\ell_\infty(F_m),\ell_\infty(2^{-m}F_m)\big)_{\theta,q;K}\big)\\
&\le 2^\theta \max\{C_{B_0},C_{B_1}\}\delta\|Q_n^+\widehat{T}P_n^+\|_0^{1-\theta}\|Q_n^+\widehat{T}P_n^+\|_1^\theta.
\end{aligned}$$

Since $\|Q_n^+\widehat{T}P_n^+\|_0 \leq \|\widehat{T}P_n^+\|_0$ and $\big(\|\widehat{T}P_n^+\|_0\big)_n$ is a monotone decreasing sequence, lower bounded by zero, we can put $\lambda = \lim_n\|\widehat{T}P_n^+\|_0$. Choose a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}_{\ell_{\rho_{\bar{A}}}(G_m)}$ such that $\|\widehat{T}P_n^+u_n\|_{\ell_\infty(F_m)} \to \lambda$.

Given $\varepsilon > 0$, choose a finite collection of elements of B_0, say $b_1, b_2, \dots, b_s$, such that

$$T\pi(\mathcal{U}_{\ell_{\rho_{\bar{A}}}(G_m)}) \subseteq \bigcup_{r=1}^{s}\{b_r + (\beta_0(T)+\varepsilon)\mathcal{U}_{B_0}\}.$$

Hence, for some subsequence (n') of $\mathbb{N}$ and some $1 \leq r \leq s$, say $r = 1$,

$$T\pi P_{n'}^+u_{n'} \in \{b_1 + (\beta_0(T)+\varepsilon)\mathcal{U}_{B_0}\}$$

for all n'. Let's compute $\|j(b_1)\|_{\ell_\infty}$,

$$\begin{aligned}K(2^m, b_1; \bar{B}) &\leq \|b_1 - T\pi Q_{n'}^+u_{n'}\|_0 + 2^m\|T\pi Q_{n'}u_{n'}\|_1\\ &\leq (\beta_0(T)+\varepsilon) + 2^m\|T\|_1\,\|\pi\|_1\,\|Q_{n'}^+ : \ell_\infty(F_m) \longrightarrow \ell_\infty(2^{-m}F_m)\|\\ &\leq (\beta_0(T)+\varepsilon) + 2^m\|T\|_1\,\|\pi\|_1\,2^{-(n'+1)} \to \beta_0(T)+\varepsilon\end{aligned}$$

as $n' \to \infty$. So we have the following estimate for λ:

$$\begin{aligned}\lambda &= \lim_{n'}\|\widehat{T}P_{n'}^+u_{n'}\|_{\ell_\infty(F_m)} \leq \lim_{n'}\sup_m\|T\pi P_{n'}^+u_{n'}\|_{F_m}\\ &\leq \lim_{n'}\sup_m \max\{C_{B_0}, C_{B_1}\}\big(\|T\pi P_{n'}^+u_{n'} - b_1\|_{F_m} + \|b_1\|_{F_m}\big)\\ &\leq 2\max\{C_{B_0}, C_{B_1}\}(\beta_0(T)+\varepsilon).\end{aligned}$$

Taking infimum in $\varepsilon > 0$ establish that

$$\lambda = \lim_n\|\widehat{T}P_n^+\|_0 \leq 2\max\{C_{B_0}, C_{B_1}\}\beta_0(T).$$

Now we estimate the term

$$\|Q_n^+\widehat{T}P_n^+\|_1 = \|Q_n^+\widehat{T}P_n^+ : \ell_{\rho_{\bar{A}}}(2^{-m}G_m) \longrightarrow \ell_\infty(2^{-m}F_m)\|.$$

First of all choose an arbitrary $\varepsilon > 0$. Sequences having finitely many non-zero terms are dense in $\ell_{\rho_{\bar{A}}}(2^{-m}G_m)$, and since $\beta_1(\widehat{T}) \leq 2\max\{C_{A_0}, C_{A_1}\}\beta_1(T)$, for every $\varepsilon' > 0$ we can find a cover of $\widehat{T}(\mathcal{U}_{\ell_{\rho_{\bar{A}}}(2^{-m}G_m)})$ of the form

$$\begin{aligned}&\widehat{T}(\mathcal{U}_{\ell_{\rho_{\bar{A}}}(2^{-m}G_m)})\\ &\qquad\subseteq \bigcup_{j=1}^{s}\big\{\widehat{T}x_j + \big(2\max\{C_{B_0}, C_{B_1}\}2\max\{C_{A_0}, C_{A_1}\}\beta_1(T)+\varepsilon'\big)\mathcal{U}_{\ell_\infty(2^{-m}F_m)}\big\},\end{aligned}$$

where, for every j, $\widehat{T}x_j \in \ell_\infty(F_m)\bigcap \ell_\infty(2^{-m}F_m)$. Use (III) to find $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, and any $1 \leq j \leq s$, $\|Q_n^+\widehat{T}x_j\|_{\ell_\infty(2^{-m}F_m)} < \varepsilon'$. Then, if $u \in \mathcal{U}_{\ell_{\rho_{\bar{A}}}(2^{-m}G_m)}$

$$\begin{aligned}&\|Q_n^+\widehat{T}P_n^+u\|_{\ell_\infty(2^{-m}F_m)}\\ &\quad\leq \max\{C_{B_0},C_{B_1}\}\big(\|Q_n^+\widehat{T}P_n^+u - Q_n^+\widehat{T}x_j\|_{\ell_\infty(2^{-m}F_m)}+\|Q_n^+\widehat{T}x_j\|_{\ell_\infty(2^{-m}F_m)}\big)\\ &\quad\leq \max\{C_{B_0},C_{B_1}\}\big(2\max\{C_{B_0},C_{B_1}\}2\max\{C_{A_0},C_{A_1}\}\beta_1(T)+\varepsilon'\big)+\varepsilon'.\end{aligned}$$

That is to say, if n is large enough and ε' sufficiently small

$$\|Q_n^+\widehat{T}P_n^+\|_1 \leq 2\max\{C_{B_0},C_{B_1}\}^2 2\max\{C_{A_0},C_{A_1}\}\beta_1(T)+\varepsilon.$$

From the estimates of λ and $\|Q_n^+\widehat{T}P_n^+\|_1$ we conclude that for n large enough

$$\beta(Q_n^+\widehat{T}P_n^+) \leq 2^{1+2\theta}\max\{C_{B_0},C_{B_1}\}^{2+\theta}\max\{C_{A_0},C_{A_1}\}^{\theta}\delta\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}+\varepsilon. \quad (9)$$

The same estimate is obtained for the term $\beta(Q_n^-\widehat{T}P_n^-)$, namely

$$\beta(Q_n^-\widehat{T}P_n^-) \leq 2^{1+2\theta}\max\{C_{B_0},C_{B_1}\}^{2+\theta}\max\{C_{A_0},C_{A_1}\}^{\theta}\delta\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}+\varepsilon. \quad (10)$$

The estimates of the two remaining terms are based on the construction of the real interpolation method. Let's start with the term $\beta(\widehat{T}P_n)$.

The closed unit ball, $\mathcal{U}_{\ell_q^{2n+1}}$, of ℓ_q^{2n+1} ($\mathbb{R}^{2n+1}$ with the ℓ_q-norm) is compact. So given $\varepsilon > 0$ there exists a finite cover of $\mathcal{U}_{\ell_q^{2n+1}}$ by balls of radius ε, say

$$\mathcal{U}_{\ell_q^{2n+1}} \subseteq \bigcup_{r=1}^{s}\{\mu_r + \varepsilon\mathcal{U}_{\ell_q^{2n+1}}\}.$$

Given any $u \in \mathcal{U}_{\ell_q}(2^{-\theta m}G_m)$

$$\|(2^{-\theta m}J(2^m,u_m))_{-n}^{n}\|_{\ell_q^{2n+1}} \leq \Big(\sum_{m\in\mathbb{Z}}(2^{-\theta m}J(2^m,u_m))^q\Big)^{1/q} \leq 1.$$

So, $(2^{-\theta m}J(2^m,u_m))_{-n}^{n} \in \{\mu_r + \varepsilon\mathcal{U}_{\ell_q^{2n+1}}\}$ for some $1 \leq r \leq s$. That is to say $\big(\sum_{m\in\mathbb{Z}}|2^{-\theta m}J(2^m,u_m) - \mu_r(m)|^q\big)^{1/q} \leq \varepsilon$, which implies that for $-n \leq m \leq n$

$$2^{-\theta m}J(2^m,u_m) \leq |\mu_r(m)| + \varepsilon.$$

In particular, by taking any $k_i > \beta_i(T)$, for $i = 0,1$ and $\nu \in \mathbb{Z}$ such that $2^{\nu-1} \leq \frac{k_1}{k_0} \leq 2^{\nu}$ we obtain that

$$2^{-\theta}\Big(2^{m-\nu}\frac{k_1}{k_0}\Big)^{-\theta}J\Big(2^{m-\nu}\frac{k_1}{k_0},u_m\Big) \leq 2^{-\theta m}J(2^m,u_m) \leq |\mu_r(m)| + \varepsilon$$

which is equivalent to

$$\|u_m\|_{A_0}\leq 2^\theta\Big(2^{m-\nu}\frac{k_1}{k_0}\Big)^\theta(|\mu_r(m)|+\varepsilon)=\Psi_r^0(m)$$

and

$$\|u_m\|_{A_1}\leq 2^\theta\Big(2^{m-\nu}\frac{k_1}{k_0}\Big)^{\theta-1}(|\mu_r(m)|+\varepsilon)=\Psi_r^1(m).$$

Recall that $k_i>\beta_i(T)$. So we can cover $T(\mathcal{U}_{A_0})$ and $T(\mathcal{U}_{A_1})$ by finitely many balls of radius k_0 and k_1 respectively, say

$$T(\mathcal{U}_{A_0})\subseteq\bigcup_{j=1}^{n_0}\{b_j^0+k_0\mathcal{U}_{B_0}\}\quad\text{and}\quad T(\mathcal{U}_{A_1})\subseteq\bigcup_{j=1}^{n_1}\{b_j^1+k_1\mathcal{U}_{B_1}\}$$

Hence, for any $-n\leq m\leq n$ there exists $1\leq r\leq s$ such that

$$Tu_m\in\Psi_r^0(m)\{b_j^0+k_0\mathcal{U}_{B_0}\}\quad\text{and}\quad Tu_m\in\Psi_r^1(m)\{b_l^1+k_1\mathcal{U}_{B_1}\}$$

for some $1\leq j\leq n_0$ and $1\leq l\leq n_1$. Here r depends only on u while j and l depend on m.

Choose elements, $d_p(m)$, in all possible intersections of these sets, say

$$d_p(m)\in\Psi_r^0(m)\{b_j^0+k_0\mathcal{U}_{B_0}\}\bigcap\Psi_r^1(m)\{b_l^1+k_1\mathcal{U}_{B_1}\}$$

where the index $p=p(j,l,r)$ depends on j, l and r. The number of elements $d_p(m)$'s is always finite although it may change with m. These elements belongs to $B_0\cap B_1$ and by construction, given any $(u_m)\in\mathcal{U}_{\ell_q(2^{-\theta m}G_m)}$ we can find some $1\leq r\leq s$ and $\{d_p(m)\}_{-n}^n$ such that

$$\begin{aligned}J(2^{m-\nu},Tu_m-d_p(m);\bar{B})&=\max\{\|Tu_m-d_p(m)\|_{B_0},2^{m-\nu}\|Tu_m-d_p(m)\|_{B_1}\}\\&\leq\max\{2C_{B_0}\Psi_r^0(m)k_0,2^{m-\nu}2C_{B_1}\Psi_r^1(m)k_1\}\\&=\max\{C_{B_0},C_{B_1}\}2^{\theta+1}2^{(m-\nu)\theta}k_0^{1-\theta}k_1^\theta(|\mu_r(m)|+\varepsilon)\end{aligned}$$

and so we can estimate the norm $\|T\pi P_nu-\sum_{-n}^n d_p(m)\|_{(B_0,B_1)_{\theta,q;J}}$, actually

$$\begin{aligned}&\Big(\sum_{-n}^n\big(2^{-\theta(m-\nu)}J(2^{m-\nu},Tu_m-d_p(m))\big)^q\Big)^{\frac{1}{q}}\\&\qquad\leq\max\{C_{B_0},C_{B_1}\}2^{1+\theta}k_0^{1-\theta}k_1^\theta\max\big\{1,2^{\frac{1-q}{q}}\big\}\big(1+(2n+1)^{1/q}\varepsilon\big).\end{aligned}$$

Hence

$$\begin{aligned}&\Big\|T\pi P_nu-\sum_{-n}^n d_p(m)\Big\|_{(B_0,B_1)_{\theta,q;K}}\\&\qquad\leq\delta 2^{1+\theta}\max\{C_{B_0},C_{B_1}\}k_0^{1-\theta}k_1^\theta\max\big\{1,2^{\frac{1-q}{q}}\big\}\big(1+(2n+1)^{1/q}\varepsilon\big),\end{aligned}$$

establishing that

$$\Big\|\widehat{T}P_n u - j\Big(\sum_{-n}^{n} d_p(m)\Big)\Big\|_{\ell_q(2^{-\theta m}F_m)} \\ \leq \delta 2^{1+\theta}\max\{C_{B_0}, C_{B_1}\}k_0^{1-\theta}k_1^{\theta}\max\big\{1, 2^{\frac{1-q}{q}}\big\}\big(1+(2n+1)^{1/q}\varepsilon\big).$$

Since there are finitely many elements of the form $j\big(\sum_{-n}^{n} d_p(m)\big)$, this shows, after taking infimum on $\varepsilon > 0$ and on $k_i > \beta_i(T)$, that

$$\beta(\widehat{T}P_n) \leq 2^{1+\theta}\delta\max\{C_{B_0}, C_{B_1}\}\max\{1, 2^{\frac{1-q}{q}}\}\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}. \tag{11}$$

Finally, in order to estimate the term $\beta(Q_n\widehat{T}(P_n^+ + P_n^-))$, note that

$$\beta(Q_n\widehat{T}(P_n^+ + P_n^-)) = \beta(Q_n jT\pi(P_n^+ + P_n^-)) \\ \leq \beta(Q_n jT : (A_0, A_1)_{\theta,q;J} \longrightarrow \ell_q(2^{-\theta m}F_m)).$$

Given any $\varepsilon > 0$, find $\{\mu_r\}_{r=1}^{s} \subset \mathcal{U}_{\ell_q^{2n+1}}$ such that for any $\lambda \in \mathcal{U}_{\ell_q^{2n+1}}$

$$\min_{1\leq r\leq s}\{\|\lambda - \mu_r\|_{\ell_q^{2n+1}}\} \leq \varepsilon.$$

Choose also $k_i > \beta_i(T)$ $(i = 0, 1)$ and find $\nu \in \mathbb{Z}$ such that $2^{\nu-1} \leq \frac{k_1}{k_0} \leq 2^{\nu}$. Then for $a \in \mathcal{U}_{(A_0,A_1)_{\theta,q;J}}$

$$\Big(\sum_{m=-n}^{n}(2^{-\theta(m+\nu)}K(2^{m+\nu}, a))^q\Big)^{1/q} \leq \|a\|_{\theta,q;K} \leq \delta$$

and so there exists $1 \leq r \leq s$ such that

$$\Big\|\frac{1}{\delta}\Big(2^{-\theta(m+\nu)}K(2^{m+\nu}, a)\Big)_{-n}^{n} - \mu_r\Big\|_{\ell_q^{2n+1}} < \varepsilon.$$

In particular, for $m = -n, \dots, 0 \dots, n$,

$$2^{-\theta(m+\nu)}K(2^{m+\nu}, a) \leq \delta(\mu_r(m) + \varepsilon)$$

and it is easily deduced that

$$K\big(2^m\tfrac{k_1}{k_0}, a\big) \leq K(2^{m+\nu}, a) \leq 2^{\theta}\delta\big(2^m\tfrac{k_1}{k_0}\big)^{\theta}(\mu_r(m) + \varepsilon).$$

Choose decompositions of $a = a_{0,m} + a_{1,m}$ such that for $-n \leq m \leq n$

$$\|a_{0,m}\|_{A_0} + 2^m\frac{k_1}{k_0}\|a_{1,m}\|_{A_1} \leq 2^{\theta}\delta\Big(2^m\frac{k_1}{k_0}\Big)^{\theta}(\mu_r(m) + \varepsilon).$$

Then, by our choice of $k_i > \beta_i(T)$, there are finite sets $\{b^i_{m,j}\}_{j=1}^h \subset B_i$ such that for $i = 0, 1$

$$\min_{1\leq j\leq h}\{\|Ta_{i,m} - b^i_{m,j}\|_{B_i}\} \leq 2^\theta \delta 2^{m(\theta-i)} k_0^{1-\theta} k_1^\theta (\mu_r(m) + \varepsilon).$$

Let W stand for the collection of all vector-valued sequences $\big(z_w(m)\big)$ defined by

$$z_w(m) = \begin{cases} 0 & \text{if } m \notin [-n, n], \\ b^0_{m,j} + b^1_{m,l} & \text{if } -n \leq m \leq n. \end{cases}$$

(j and l run freely on their respective domains.) W is a finite subset of $\ell_q(2^{-\theta m} F_m)$ and, given any $a \in \mathcal{U}_{(A_0,A_1)_{\theta,q;J}}$ we can find $z_w \in W$ such that for, $-n \leq m \leq n$, $z_w(m) = b^0_{m,j} + b^1_{m,l}$ satisfying

$$\|T_{a_{0,m}} - b^0_{m,j}\|_{B_0} \leq 2^\theta \delta 2^{m\theta} k_0^{1-\theta} k_1^\theta (\mu_r(m) + \varepsilon)$$

and

$$\|T_{a_{1,m}} - b^1_{m,l}\|_{B_1} \leq 2^\theta \delta 2^{m(\theta-1)} k_0^{1-\theta} k_1^\theta (\mu_r(m) + \varepsilon)$$

where $a = a_{0,m} + a_{1,m}$ is the previously described decomposition of a. It follows that

$$\begin{aligned} \|Q_n jTa - z_w\|_{\ell_q(2^{-\theta m} F_m)} &= \bigg(\sum_{-n}^{n} \big(2^{-\theta m} K(2^m, Ta_{0,m} - b^0_{m,j} + Ta_{1,m} - b^1_{m,l})^q\bigg)^{1/q} \\ &\leq \bigg(\sum_{-n}^{n} \big(2^{1+\theta} \delta k_0^{1-\theta} k_1^\theta (\mu_r(m) + \varepsilon)\big)^q\bigg)^{1/q} \\ &\leq 2^{1+\theta} \delta k_0^{1-\theta} k_1^\theta \max\{1, 2^{\frac{1-q}{q}}\}\big(1 + (2n+1)^{1/q}\varepsilon\big). \end{aligned}$$

Therefore

$$\beta\big(Q_n jT : (A_0, A_1)_{\theta,q;J} \longrightarrow \ell_q(2^{-\theta m} F_m)\big) \leq 2^{1+\theta} \delta \max\big\{1, 2^{\frac{1-q}{q}}\big\} \beta_0(T)^{1-\theta} \beta_1(T)^\theta,$$

and so

$$\beta\big(Q_n \widehat{T}(P_n^+ + P_n^-)\big) \leq 2^{1+\theta} \delta \max\big\{1, 2^{\frac{1-q}{q}}\big\} \beta_0(T)^{1-\theta} \beta_1(T)^\theta. \tag{12}$$

Now, combine estimates (6), (7), (8), (9), (10), (11), (12), and take infimum in $\varepsilon > 0$, to obtain the inequality

$$\beta(\widehat{T}) \leq 2^{1+\frac{1}{\rho}+\theta} \delta C \beta_0(T)^{1-\theta} \beta_1(T)^\theta,$$

where here the constant

$$C = \big[2 + \max\{1, 2^{\frac{1-q}{q}}\}^\rho + 2\big(2^\theta \max\{C_{B_0}, C_{B_1}\}^{2+\theta} \max\{C_{A_0}, C_{A_1}\}^\theta\big)^\rho\big)\big]^{1/\rho}.$$

Equation (4), along with equation (5), establishes that

$$\beta_{\theta,q}(T) \leq 2^{2+\frac{2}{\rho}+\theta} \delta C \beta_0(T)^{1-\theta} \beta_1(T)^\theta. \qquad \square$$

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