Extension of Lipschitz Functions Defined on Metric Subspaces of Homogeneous Type

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ABSTRACT

If a metric subspace M^o of an arbitrary metric space M carries a doubling measure μ, then there is a simultaneous linear extension of all Lipschitz functions on M^o ranged in a Banach space to those on M. Moreover, the norm of this linear operator is controlled by logarithm of the doubling constant of μ.

Key words: metric space of homogeneous type, Lipschitz function, linear extension.

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1. Formulation of the main result

Let (M, d) be a metric space and X be a Banach space. The space $\mathrm{Lip}(M, X)$ consists of all X-valued Lipschitz functions on M. The Lipschitz constant

$$L(f) := \sup_{m \neq m'} \left\{ \frac{\|f(m) - f(m')\|}{d(m, m')} \ : \ m, m' \in M \right\}$$

of a function f from this space is therefore finite and the function $f \mapsto L(f)$ is a Banach seminorm on $\mathrm{Lip}(M, X)$.

Let M^o be a metric subspace of M, i.e., $M^o \subset M$ is a metric space endowed with the induced metric $d|_{M^o \times M^o}$.

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Convention. We mark all objects related to the subspace M^o by the upper "o".

A linear operator $E : \mathrm{Lip}(M^o, X) \to \mathrm{Lip}(M, X)$ is called a *simultaneous extension* if for all $f \in \mathrm{Lip}(M^o, X)$

$$Ef|_{M^o} = f$$

and, moreover, the norm

$$\|E\| := \sup\left\{ \frac{L(Ef)}{L(f)} \ : \ f \in \mathrm{Lip}(M^o, X) \right\}$$

is finite.

To formulate the main result we also need

Definition 1.1. A Borel measure μ on a metric space (M, d) is said to be doubling if the μ-measure of every open ball

$$B_R(m) := \{ m' \in M \ : \ d(m, m') < R \}$$

is strictly positive and finite and the doubling constant

$$D(\mu) := \sup\left\{ \frac{\mu(B_{2R}(m))}{\mu(B_R(m))} \ : \ m \in M,\ R > 0 \right\} \tag{1}$$

is finite.

A metric space carrying a fixed doubling measure is called of homogeneous type.

Our main result is

Theorem 1.2. *Let M^o be a metric subspace of an arbitrary metric space (M, d). Assume that (M^o, d^o) is of homogeneous type and μ^o is the corresponding doubling measure. Then there exists a simultaneous extension $E : \mathrm{Lip}(M^o, X) \to \mathrm{Lip}(M, X)$ satisfying*

$$\|E\| \le c(\log_2 D(\mu^o) + 1) \tag{2}$$

with some numerical constant $c > 1$.

Let us discuss relations of this theorem to some known results. First, a similar result holds for an arbitrary subspace M^o provided that the ambient space M is of *pointwise homogeneous type*, see [1, Theorem 2.21; 2, Theorem 1.14]. The class of metric spaces of pointwise homogeneous type contains, in particular, all metric spaces of homogeneous type, Riemannian manifolds $M_\omega \cong \mathbb{R}^n \times \mathbb{R}_+$ with the path metric defined by the Riemannian metric

$$ds^2 := \omega(x_{n+1})(dx_1^2 + \cdots + dx_{n+1}^2), \qquad (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+,$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nonincreasing function (e.g., the hyperbolic spaces $\mathbb{H}^n$ are in this class), and finite direct products of these objects.

The following problem is of a considerable interest.

Problem 1.3. *Is it true that Theorem 1.2 is valid for $M^o(\subset M)$ isometric to a subspace of a metric space $(\hat{M}, \hat{d})$ of pointwise homogeneous type with $\|E\| \le c(\hat{M})$? (Here $c(\hat{M})$ depends on some characteristics of $\hat{M}$ only.)*

It is proved in [2] that as such M^o one can take, e.g., finite direct products of Gromov hyperbolic spaces of bounded geometry and that the answer in Problem 1.3 is positive in this case.

Second, as a consequence of Theorem 1.2 we obtain a deep extension result due to Lee and Naor, see [4, Theorem 1.6]. The latter asserts that a simultaneous extension $E : \mathrm{Lip}(M^o, X) \to \mathrm{Lip}(M, X)$ exists whenever the subspace (M^o, d^o) of (M, d) has the finite doubling constant $\delta(M^o)$ and, moreover,

$$\|E\| \le c \log_2 \delta(M^o)$$

with some numerical constant $c > 1$.

Let us recall that the doubling constant $\delta(M)$ of a metric space (M, d) is the infimum of integers N such that every closed ball of M of radius R can be covered by N closed balls of radius $R/2$. The space M is said to be *doubling* if $\delta(M) < \infty$.

To derive the Lee-Naor theorem from our main result we first note that without loss of generality one may assume that (M^o, d^o) is complete. By the Koniagin-Vol'berg theorem [6] (see also [5]) a complete doubling space M carries a doubling measure μ such that

$$\log_2 D(\mu) \le c \log_2 \delta(M)$$

where $c \ge 1$ is a numerical constant. Together with (2) this implies the Naor-Lee result.

On the other hand, it was noted in [3] that if M carries a doubling measure μ, then this space is doubling and

$$\log_2 \delta(M) \le c \log_2 D(\mu) \tag{3}$$

with some numerical constant $c > 1$. Hence, Theorem 1.2 is, in turn, a consequence of (3) and the Lee-Naor theorem. However, the rather elaborated proof of the latter result is nonconstructive. (It exploits an appropriate stochastic metric decomposition of $M \setminus M^o$.) In contrast, our proof is constructive and is based on a simple average procedure. Therefore our proof can be also seen as a streamlining constructive method of the proof of the Lee-Naor theorem.

2. Proof of Theorem 1.2

We begin with the following remark reducing the required result to a special case.

Let M and M^o be isometric to subspaces of a new metric space $\hat{M}$ and its subspace $\hat{M}^o$, respectively. Assume that there exists a simultaneous extension $\hat{E}$:

$\text{Lip}(\hat{M}^o, X) \to \text{Lip}(\hat{M}, X)$. Then, after identification of M^o and M with the corresponding isometric subspaces of $\hat{M}$, the operator $\hat{E}$ gives rise to a simultaneous extension $E : \text{Lip}(M^o, X) \to \text{Lip}(M, X)$ satisfying

$$\|E\| \le \|\hat{E}\|.$$

If, in addition, $\|\hat{E}\|$ is bounded by the right-hand side of (2), then the desired result immediately follows.

We choose as the above pair $\hat{M}^o \subset \hat{M}$ metric spaces denoted by M_N^o and M_N where $N \ge 1$ is a fixed integer and defined as follows.

The underlying sets of these spaces are

$$M_N := M \times \mathbb{R}^N, \qquad M_N^o := M^o \times \mathbb{R}^N;$$

a metric d_N on M_N is given by

$$d_N((m, x), (m', x')) := d(m, m') + |x - x'|_1$$

where $m, m' \in M$ and $x, x' \in \mathbb{R}^N$, and $|x|_1 := \sum_{i=1}^N |x_i|$ is the l_1^N-metric of $x \in \mathbb{R}^N$. Further, d_N^o denotes the metric on M_N^o induced by d_N.

Finally, we define a Borel measure μ_N^o on M_N^o as the tensor product of the measure μ^o and the Lebesgue measure λ_N on $\mathbb{R}^N$:

$$\mu_N^o := \mu^o \otimes \lambda_N.$$

We extend this measure to the σ-algebra consisting of subsets $S \subset M_N$ such that $S \cap M_N^o$ is a Borel subset of M_N^o. Namely, we set for these S

$$\bar{\mu}_N(S) := \mu_N^o(S \cap M_N^o).$$

It is important for the subsequent part of the proof that every open ball $B_R((m, x)) \subset M_N$ belongs to this σ-algebra. In fact, its intersection with M_N^o is a Borel subset of this space, since the function $(m', x') \mapsto d_N((m, x), (m', x'))$ is continuous on M_N^o. Hence,

$$\bar{\mu}_N(B_R((m, x))) = \mu_N^o(B_R((m, x)) \cap M_N^o). \tag{4}$$

Auxiliary results. The measure μ_N^o is clearly doubling. Therefore its *dilation function* given for $l \ge 1$ by

$$D_N^o(l) := \sup\left\{ \frac{\mu_N^o(B_{lR}^o(\hat{m}))}{\mu_N^o(B_R^o(\hat{m}))} \;:\; \hat{m} \in M_N^o \ \text{ and } R > 0 \right\}$$

is finite.

Hereafter we denote by $\hat{m}$ the pair (m, x) with $m \in M$ and $x \in \mathbb{R}^N$, and by $B_R^o(\hat{m})$ the open ball in M_N^o centered at $\hat{m} \in M_N^o$ and of radius R. The open ball $B_R(\hat{m})$ of M_N relates to that by

$$B_R^o(\hat{m}) = B_R(\hat{m}) \cap M_N^o$$

provided $\hat{m} \in M_N^o$.

In [1] the value $D_N^o(1+1/N)$ is proved to be bounded by some numerical constant for all sufficiently large N. In the argument presented below we require a similar estimate for a (modified) dilation function D_N for the extended measure $\bar{\mu}_N$. This is given for $l \geq 1$ by

$$D_N(l) := \sup\left\{ \frac{\bar{\mu}_N(B_{lR}(\hat{m}))}{\bar{\mu}_N(B_R(\hat{m}))} \right\} \tag{5}$$

where the supremum is taken over all R satisfying

$$R > 4d(\hat{m}, M_N^o) := 4\inf\{ d_N(\hat{m}, \hat{m}') : \hat{m}' \in M_N^o \} \tag{6}$$

and then over all $\hat{m} \in M_N$.

Due to (4) and (6) the denominator in (5) is not zero and $D_N(l)$ is well defined.

Comparison of the above dilation functions shows that $D_N^o(l) \leq D_N(l)$. Nevertheless, the converse is also true for l close to 1.

Lemma 2.1. *Assume that N and the doubling constant $D := D(\mu^o)$, see* (1)*, are related by*

$$N \geq [3\log_2 D] + 5. \tag{7}$$

Then the following is true:

$$D_N(1+1/N) \leq \frac{6}{5}e^4.$$

Proof. In accordance with the definition of D_N, see (5), we must estimate the function

$$\frac{\bar{\mu}_N(B_{R_N}(\hat{m}))}{\bar{\mu}_N(B_R(\hat{m}))} \quad \text{where} \quad R_N := \Big(1 + \frac{1}{N}\Big)R. \tag{8}$$

Since the points $\hat{m}'$ of the ball $B_{R_N}(\hat{m})$ of M_N satisfy the inequality

$$d(m, m') + |x - x'|_1 < R_N,$$

the Fubini theorem and (4) yield

$$\bar{\mu}_N(B_{R_N}(\hat{m})) = \gamma_N \int_{M^o \cap B_{R_N}(m)} (R_N - d(m, m'))^N \, d\mu^o(m'). \tag{9}$$

Here γ_N is the volume of the unit l_1^N-ball.

We must estimate the integral in (9) from above under the condition

$$d_N(\hat{m}, M_N^o) < R/4. \tag{10}$$

To this end split the integral into one over $B_{3R/4}(m) \cap M^o$ and one over the remaining part $(B_{R_N}(m) \setminus B_{3R/4}(m)) \cap M^o$. Denote these integrals by I_1 and I_2. For I_2 we get

$$I_2 \leq \gamma_N (R_N - 3R/4)^N \mu^o(B_{R_N}(m) \cap M^o). \tag{11}$$

Further, from (10) we clearly have

$$d(m, M^o) < R/4.$$

Pick a point $\tilde{m} \in M^o$ so that

$$d(m, M^o) \le d(m, \tilde{m}) < R/4.$$

Then we have the following embeddings

$$B^o_{R_N/4}(\tilde{m}) \subset B_{R_N/2}(m) \cap M^o \subset B_{R_N}(m) \cap M^o \subset B^o_{5R_N/4}(\tilde{m}).$$

Applying the doubling inequality for the measure μ^o, see (1), we then obtain

$$\mu^o(B_{R_N}(m) \cap M^o) \le D^3 \mu^o(B_{R_N/2}(m) \cap M^o) \ .$$

Moreover, due to (7)

$$D^3 < 2^{[3\log_2 D]+1} \le 2^{N-4}.$$

Combining the last two inequalities with (11) we have

$$I_2 \le \gamma_N 2^{-N-4} \Big(1 + \frac{4}{N}\Big)^N R^N \mu^o(B_{R_N/2}(m) \cap M^o) \ . \tag{12}$$

To estimate the integral I_1 we rewrite its integrand as follows:

$$(R_N - d(m,m'))^N = \Big(1 + \frac{1}{N}\Big)^N (R - d(m,m'))^N \bigg(1 + \frac{d(m,m')}{(N+1)(R - d(m,m'))}\bigg)^N.$$

Since $m' \in B_{3/4R}(m)$, the last factor is at most $\big(1 + \frac{3R/4}{(N+1)R/4}\big)^N = \big(1 + \frac{3}{N+1}\big)^N$. This yields

$$\begin{aligned} I_1 &\le \gamma_N \Big(1 + \frac{1}{N}\Big)^N \Big(1 + \frac{3}{N+1}\Big)^N \int_{B_{3R/4}(m) \cap M^o} (R - d(m,m'))^N \, d\mu^o(m') \\ &\le e^4 \bar{\mu}_N(B_R(\hat{m})). \end{aligned}$$

Hence for the part of fraction (8) related to I_1 we have

$$\tilde{I}_1 := \frac{I_1}{\bar{\mu}_N(B_R(\hat{m}))} \le e^4. \tag{13}$$

To estimate the remaining part $\tilde{I}_2 := \frac{I_2}{\bar{\mu}_N(B_R(\hat{m}))}$ we note that its denominator is greater than

$$\gamma_N \int_{M^o \cap B_{R_N/2}(m)} (R - d(m,m'))^N \, d\mu^o(m').$$

Since here $d(m, m') \le R_N/2$, this, in turn, is bounded from below by

$$\gamma_N 2^{-N}\Big(1-\frac{1}{N}\Big)^N R^N \mu^o(B_{R_N/2}(m)\cap M^o).$$

Combining this with (12) and noting that $N \ge 5$ we get

$$\tilde{I}_2 \le 2^{-4}\Big(1-\frac{1}{N}\Big)^{-N}\Big(1+\frac{4}{N}\Big)^N < \frac{1}{5}e^4.$$

Hence the fraction (8) is bounded by $\tilde{I}_1 + \tilde{I}_2 \le \frac{6}{5}e^4$, see (13), and this immediately implies the required estimate of $D_N(1+1/N)$. □

In the next lemma we estimate $\bar{\mu}_N$-measure of the spherical layer $B_{R_2}(\hat{m})-B_{R_1}(\hat{m})$, $R_2 \ge R_1$, by a kind of a surface measure. For its formulation we set

$$A_N := \frac{12}{5}e^4 N. \tag{14}$$

Lemma 2.2. *Assume that*

$$N \ge [3\log_2 D] + 6.$$

Then for all $\hat{m} \in M_N$ *and* $R_1, R_2 > 0$ *satisfying*

$$R_2 \ge \max\{\, R_1, 8d_N(\hat{m}, M_N^o)\,\}$$

the following is true

$$\bar{\mu}_N(B_{R_2}(\hat{m}) \setminus B_{R_1}(\hat{m})) \le A_N \frac{\bar{\mu}_N(B_{R_2}(\hat{m}))}{R_2}(R_2 - R_1).$$

Proof. By definition $M_N = M_{N-1}\times\mathbb{R}$ and $\bar{\mu}_N = \bar{\mu}_{N-1}\otimes\lambda_1$. Then by the Fubini theorem we have for $R_1 \le R_2$ with $\hat{m} = (\tilde{m}, t)$

$$\begin{aligned}\bar{\mu}_N(B_{R_2}(\hat{m})) - \bar{\mu}_N(B_{R_1}(\hat{m})) &= 2\int_{R_1}^{R_2} \bar{\mu}_{N-1}(B_s(\tilde{m}))\,ds\\ &\le \frac{2R_2\bar{\mu}_{N-1}(B_{R_2}(\tilde{m}))}{R_2}(R_2 - R_1).\end{aligned}$$

We claim that for arbitrary $l > 1$ and $R \ge 8d_N(\hat{m}, M_N^o) := 8d_{N-1}(\tilde{m}, M_{N-1}^o)$

$$R\bar{\mu}_{N-1}(B_R(\tilde{m})) \le \frac{lD_{N-1}(l)}{l-1}\bar{\mu}_N(B_R(\hat{m})). \tag{15}$$

Together with the previous inequality this will yield

$$\bar{\mu}_N(B_{R_2}(\hat{m})) - \bar{\mu}_N(B_{R_1}(\hat{m})) \le \frac{2lD_{N-1}(l)}{l-1}\cdot\frac{\bar{\mu}_N(B_{R_2}(\hat{m}))}{R_2}(R_2 - R_1).$$

Finally choose here $l = 1 + \frac{1}{N-1}$ and use Lemma 2.1. This will give the required inequality.

Hence, it remains to establish (15). By the definition of $D_{N-1}(l)$ we have for $l > 1$ using the previous lemma

$$\bar{\mu}_N(B_{lR}(\hat{m})) = 2l\int_0^R \bar{\mu}_{N-1}(B_{ls}(\tilde{m}))\,ds \le 4l\int_{R/2}^R \bar{\mu}_{N-1}(B_{ls}(\tilde{m}))\,ds$$
$$\le 4lD_{N-1}(l)\int_{R/2}^R \bar{\mu}_{N-1}(B_s(\tilde{m}))\,ds \le 2lD_{N-1}(l)\bar{\mu}_N(B_R(\hat{m})).$$

On the other hand, replacing $[0,R]$ by $[l^{-1}R,R]$ we also have

$$\bar{\mu}_N(B_{lR}(\hat{m})) \ge 2l\bar{\mu}_{N-1}(B_R(\tilde{m}))(R - l^{-1}R) = 2(l-1)R\bar{\mu}_{N-1}(B_R(\tilde{m})).$$

Combining the last two inequalities we get (15). □

Extension operator. We define the required simultaneous extension

$$E : \mathrm{Lip}(M_N^o, X) \to \mathrm{Lip}(M_N, X)$$

using the standard average operator *Ave* defined on continuous and locally bounded functions $g : M_N^o \to X$ by

$$\mathrm{Ave}(g;\hat{m},R) := \frac{1}{\bar{\mu}_N(B_R(\hat{m}))}\int_{B_R(\hat{m})} g\,d\bar{\mu}_N.$$

To be well-defined the domain of integration $B_R(\hat{m}) \cap M_N^o$ should be of strictly positive $\bar{\mu}_N$-measure (i.e., μ_N^o-measure). This condition is fulfilled in the case presented now. Namely, we define the simultaneous extension E on functions $f \in \mathrm{Lip}(M_N^o, X)$ by

$$(Ef)(\hat{m}) := \begin{cases} f(\hat{m}) & \text{if} \quad \hat{m} \in M_N^o, \\ \mathrm{Ave}(f;m,R(\hat{m})) & \text{if} \quad \hat{m} \notin M_N^o \end{cases} \tag{16}$$

where we set

$$R(\hat{m}) := 8d_N(\hat{m}, M_N^o).$$

The required estimate of $\|E\|$ is presented below. To formulate the result we set

$$K_N(l) := A_N D_N(l)(4l+1) \tag{17}$$

where the first of two factors are defined by (14) and (5).

Proposition 2.3. *The following inequality,*

$$\|E\| \le 20A_N + \max\Big(\frac{4l+1}{2(l-1)}, K_N(l)\Big),$$

is true provided $l := 1 + 1/N$.

Before we begin the proof let us derive from here the desired result. Namely, choose

$$N := [3\log_2 D] + 6$$

and use Lemma 2.1 and (14) to estimate $D_N(1 + 1/N)$ and A_N. Then we get

$$\|E\| \le C(\log_2 D + 2)$$

with some numerical constant C. This clearly gives (2).

Proof. We have to show that for every $\hat{m}_1, \hat{m}_2 \in M_N$

$$\|(Ef)(\hat{m}_1) - (Ef)(\hat{m}_2)\|_X \le K\|f\|_{\mathrm{Lip}(M_N,X)} d_N(\hat{m}_1, \hat{m}_2) \tag{18}$$

where K is the constant in the inequality of the proposition.

It suffices to consider only two cases:

(i) $\hat{m}_1 \in M_N^o$ and $\hat{m}_2 \notin M_N^o$,

(ii) $\hat{m}_1, \hat{m}_2 \notin M_N^o$.

We assume without loss of generality that

$$\|f\|_{\mathrm{Lip}(M_N^o,X)} = 1$$

and simplify the computations by introducing the following notations:

$$R_i := d_N(\hat{m}_i, M_N^o), \quad B_{ij} := B_{8R_j}(\hat{m}_i), \quad v_{ij} := \bar{\mu}_N(B_{ij}), \quad 1 \le i, j \le 2. \tag{19}$$

We assume also for definiteness that

$$0 < R_1 \le R_2.$$

By the triangle inequality we then have

$$0 \le R_2 - R_1 \le d_N(\hat{m}_1, \hat{m}_2). \tag{20}$$

Further, by Lemma 2.2 the quantities introduced satisfy the following inequality:

$$v_{i2} - v_{i1} \le \frac{A_N v_{i2}}{R_2}(R_2 - R_1). \tag{21}$$

Let now $\hat{m}^*$ be such that $d_N(\hat{m}_1, \hat{m}^*) < 2R_1$. Set

$$\hat{f}(\hat{m}) := f(\hat{m}) - f(\hat{m}^*). \tag{22}$$

From the triangle inequality we then obtain

$$\max\{\, \|\hat{f}(\hat{m})\|_X \;:\; \hat{m} \in B_{i2} \cap M_N^o \,\} \le 10R_2 + (i-1)d_N(\hat{m}_1, \hat{m}_2). \tag{23}$$

(Here $i = 1, 2$.)

We now prove (18) for $\hat{m}_1 \in M_N^o$ and $\hat{m}_2 \notin M_N^o$. We begin with the evident inequality

$$\|(Ef)(\hat{m}_2) - (Ef)(\hat{m}_1)\|_X = \frac{1}{v_{22}} \left\| \int_{B_{22}} \hat{f}(\hat{m})\, d\bar{\mu}_N \right\|_X \le \max_{B_{22} \cap M_N^o} \|\hat{f}\|_X,$$

see (19) and (22). Applying (23) with $i = 2$ we then bound this maximum by $10R_2 + d_N(\hat{m}_1, \hat{m}_2)$. But $\hat{m}_1 \in M_N^o$ and so

$$R_2 = d_N(\hat{m}_2, M_N^o) \le d_N(\hat{m}_1, \hat{m}_2);$$

therefore (18) holds in this case with $K = 11$.

The remaining case $\hat{m}_1, \hat{m}_2 \notin M_N^o$ requires some additional auxiliary results. For their formulations we first write

$$(Ef)(\hat{m}_1) - (Ef)(\hat{m}_2) := D_1 + D_2 \tag{24}$$

where

$$\begin{aligned} D_1 &:= \mathrm{Ave}(\hat{f}; \hat{m}_1, 8R_1) - \mathrm{Ave}(\hat{f}; \hat{m}_1, 8R_2), \\ D_2 &:= \mathrm{Ave}(\hat{f}; \hat{m}_1, 8R_2) - \mathrm{Ave}(\hat{f}; \hat{m}_2, 8R_2), \end{aligned} \tag{25}$$

see (16) and (22).

Lemma 2.4. *We have*

$$\|D_1\|_X \le 20 A_N d_N(\hat{m}_1, \hat{m}_2).$$

Recall that A_N is the constant defined by (14).

Proof. By (25), (22), and (19),

$$\begin{aligned} D_1 &= \frac{1}{v_{11}} \int_{B_{11}} \hat{f}\, d\bar{\mu}_N - \frac{1}{v_{12}} \int_{B_{12}} \hat{f}\, d\bar{\mu}_N \\ &= \Big(\frac{1}{v_{11}} - \frac{1}{v_{12}}\Big) \int_{B_{11}} \hat{f}\, d\bar{\mu}_N - \frac{1}{v_{12}} \int_{B_{12} \setminus B_{11}} \hat{f}\, d\bar{\mu}_N. \end{aligned}$$

This immediately implies that

$$\|D_1\|_X \le 2 \cdot \frac{v_{12} - v_{11}}{v_{12}} \cdot \max_{B_{12} \cap M_N^o} \|\hat{f}\|_X.$$

Applying now (21) and (20), and then (23) with $i = 1$ we get the desired estimate. □

To obtain a similar estimate for D_2 we will use the following two facts.

Lemma 2.5. *Assume that for a given $l > 1$*

$$d_N(\hat{m}_1, \hat{m}_2) \le 8(l-1)R_2. \tag{26}$$

Let for definiteness

$$v_{22} \le v_{12}. \tag{27}$$

Then we have

$$\bar{\mu}_N(B_{12} \Delta B_{22}) \le A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \tag{28}$$

(Here Δ denotes symmetric difference of sets.)

Proof. Set

$$R := 8R_2 + d_N(\hat{m}_1, \hat{m}_2).$$

Then $B_{12} \cup B_{22} \subset B_R(\hat{m}_1) \cap B_R(\hat{m}_2)$, and

$$\bar{\mu}_N(B_{12} \Delta B_{22}) \le (\bar{\mu}_N(B_R(\hat{m}_1)) - \bar{\mu}_N(B_{8R_2}(\hat{m}_1))) + (\bar{\mu}_N(B_R(\hat{m}_2)) - \bar{\mu}_N(B_{8R_2}(\hat{m}_2))).$$

Estimating the terms on the right-hand side by Lemma 2.2 we bound them by

$$A_N \frac{\bar{\mu}_N(B_R(\hat{m}_1))}{R} (R - 8R_2) + A_N \frac{\bar{\mu}_N(B_R(\hat{m}_2))}{R} (R - 8R_2).$$

Moreover, $8R_2 \le R \le 8lR_2$ and $R - 8R_2 := d_N(\hat{m}_1, \hat{m}_2)$, see (26); taking into account (5), (19), and (27) we therefore have

$$\bar{\mu}_N(B_{12} \Delta B_{22}) \le A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \qquad \square$$

Lemma 2.6. *Under the assumptions of the previous lemma we have*

$$v_{12} - v_{22} \le A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \tag{29}$$

Proof. By (19) the left-hand side is bounded by $\bar{\mu}_N(B_{12} \Delta B_{22})$. $\square$

We now estimate D_2 from (25) beginning with

Lemma 2.7. *Under the conditions of Lemma 2.5 we have*

$$\|D_2\|_X \le K_N(l) d_N(\hat{m}_1, \hat{m}_2)$$

where $K_N(l) := A_N D_N(l)(4l+1)$.

Proof. By the definition of D_2 and our notation, see (25), (22), and (19),

$$\begin{aligned}\|D_2\|_X &:= \left\| \frac{1}{v_{12}} \int_{B_{12}} \hat{f}\, d\bar{\mu}_N - \frac{1}{v_{22}} \int_{B_{22}} \hat{f}\, d\bar{\mu}_N \right\|_X \\ &\le \frac{1}{v_{12}} \int_{B_{12}\Delta B_{22}} \|\hat{f}\|_X\, d\bar{\mu}_N + \left| \frac{1}{v_{12}} - \frac{1}{v_{22}} \right| \int_{B_{22}} \|\hat{f}\|_X\, d\bar{\mu}_N := J_1 + J_2 \ .\end{aligned}$$

By (28), (26), and (23)

$$\begin{aligned}J_1 &\le \frac{1}{v_{12}} \bar{\mu}_N(B_{12}\Delta B_{22}) \sup_{(B_{12}\Delta B_{22})\cap M_N^o} \|\hat{f}\|_X \\ &\le \frac{A_N D_N(l)}{4R_2} d_N(\hat{m}_1, \hat{m}_2)(d_N(\hat{m}_1, \hat{m}_2) + 10R_2) \\ &\le A_N D_N(l)(2l + 1/2) d_N(\hat{m}_1, \hat{m}_2).\end{aligned}$$

Also, (29), (23), and (26) yield

$$J_2 \le A_N D_N(l)(2l + 1/2) d_N(\hat{m}_1, \hat{m}_2).$$

Combining these we get the required estimate. □

It remains to consider the case of $\hat{m}_1, \hat{m}_2 \notin M_N$ satisfying the inequality

$$d_N(\hat{m}_1, \hat{m}_2) > 8(l-1)R_2$$

converse to (26). Now the definition (25) of D_2 and (23) imply that

$$\|D_2\|_X \le 2 \sup_{(B_{12}\cup B_{22})\cap M_N^o} \|\hat{f}\|_X \le 2(10R_2 + d_N(\hat{m}_1, \hat{m}_2)) \le \frac{4l+1}{2(l-1)} d_N(\hat{m}_1, \hat{m}_2).$$

Combining this with the inequalities of Lemmas 2.4 and 2.7 and equality (24) we obtain the required estimate of the Lipschitz norm of the extension operator E:

$$\|E\| \le 20A_N + \max\left(\frac{4l+1}{2(l-1)}, K_N(l) \right)$$

where $K_N(l)$ is the constant in (17). □

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