Stabilization of a Coupled Multidimensional System

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ABSTRACT

We introduce a model of a vibrating multidimensional structure made of a n-dimensional body and a one-dimensional rod. We actually consider the anisotropic elastodynamic system in the n-dimensional body and the Euler-Bernouilli beam in the one-dimensional rod. These equations are coupled via their boundaries. Using appropriate feedbacks on a part of the boundary we show the exponential decay of the energy of the system.

Key words: multidimensional structures, stabilization, exponential decay.

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Introduction

Let Ω be a non empty bounded open subset of $\mathbb{R}^n, n \geq 1$, with a boundary Γ of class C^2. We denote by $\nu(x) = (\nu_1, \dots, \nu_n)^\top$ the unit outward normal vector at x along Γ. For a fixed $x_0 \in \mathbb{R}^n$ we define the function $m(x) = x - x_0$, $x \in \mathbb{R}^n$ and the following partition of the boundary Γ (see figures 1 and 2):

$$\Gamma_0 = \{\, x \in \Gamma \,:\, m(x) \cdot \nu(x) \leq 0 \,\},$$
$$\Gamma_N = \{\, x \in \Gamma \,:\, m(x) \cdot \nu(x) > 0 \,\}.$$

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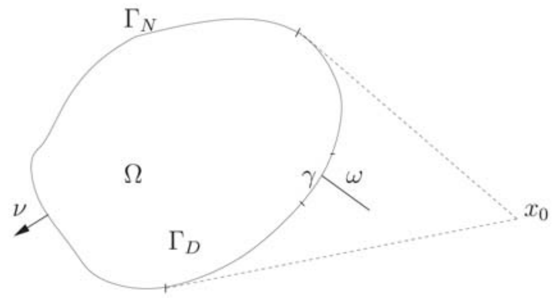

Figure 1: A pluridimensional structure for $n = 2$ — The case $\bar{\Gamma}_N \cap \bar{\Gamma}_D \neq \emptyset$

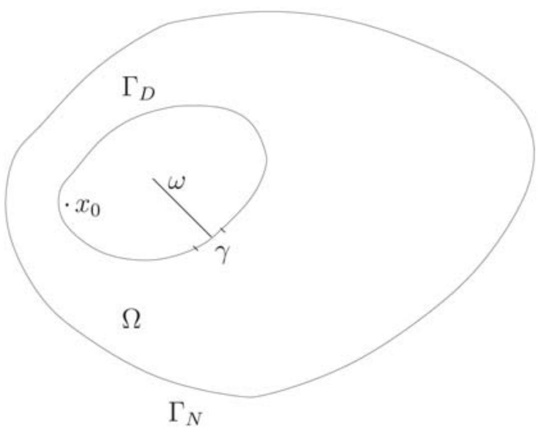

Figure 2: A pluridimensional structure for $n = 2$ — The case $\bar{\Gamma}_N \cap \bar{\Gamma}_D = \emptyset$

We also fix an open subset γ of Γ_0 such that

$$m(x)\cdot\nu(x)\le -\alpha_0<0,\quad \forall x\in\gamma,$$

and denote

$$\Gamma_D=\Gamma_0\setminus\gamma.$$

In the whole paper we suppose that $\operatorname{meas}\Gamma_D>0$, $\operatorname{meas}\Gamma_N>0$, $\operatorname{meas}\gamma>0$.

We further fix a 1-dimensional beam ω of length l attached to Ω at a point $a\in\gamma$ and orthogonal to Γ, in other words (see again figures 1 and 2),

$$\omega=\{\,a+s\nu(a)\,:\,0<s<l\,\}.$$

The derivation with respect to the parameter s will be denoted by ∂.

Finally let α be a non negative real number and θ be a function from γ to $\mathbb{R}^n$ of class C^1 with a compact support and such that $\theta\neq 0$.

We now consider the following problem:

$$\begin{cases} u''-\operatorname{div}\sigma(u)=0 & \text{in } \Omega\times\mathbb{R}^+,\\ v''+\rho\partial^4 v=0 & \text{in } \omega\times\mathbb{R}^+,\\ u=0 & \text{on } \Gamma_D\times\mathbb{R}^+,\\ \sigma(u)\cdot\nu+m\cdot\nu u'=0 & \text{on } \Gamma_N\times\mathbb{R}^+,\\ u(x,t)=v(0,t)\theta(x) & \text{on } \gamma\times\mathbb{R}^+,\\ \rho\partial^3 v(0,t)+\alpha v'(0,t)+\int_\gamma[\sigma(u)\cdot\nu]\cdot\theta(x)ds(x)=0 & \forall t\in\mathbb{R}^+,\\ \partial v(0,t)=\partial^2 v(l,t)=\partial^3 v(l,t)=0, \end{cases}\tag{1}$$

with initial conditions

$$\begin{cases} u(0)=u^0 & \text{in } \Omega,\\ u'(0)=u^1 & \text{in } \Omega,\\ v(0)=v^0 & \text{in } \omega,\\ v'(0)=v^1 & \text{in } \omega, \end{cases}$$

where, as usual, u' means $\frac{\partial u}{\partial t}$, $u=u(x,t)=(u_1,\dots,u_n)^\top$ denotes the displacement vector field in the domain Ω and $v=v(s,t)$ denotes the orthogonal displacement of the beam ω. The stress tensor σ is defined by $\sigma_{ij}(u)=a_{ijkl}\varepsilon_{kl}(u)$ (in the full paper we adopt the convention of repeated indices), where $\varepsilon(u)$ is the strain tensor given by (when $\partial_i=\frac{\partial}{\partial x_i}$)

$$\varepsilon_{ij}(u)=\frac{1}{2}(\partial_j u_i+\partial_i u_j),$$

the constant coefficients a_{ijkl} are such that

$$a_{ijkl}=a_{klij}=a_{jikl}$$

and satisfy the ellipticity condition

$$\exists \delta > 0 \, : \, a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \delta\varepsilon_{ij}\varepsilon_{ij}, \tag{2}$$

for all symmetric tensor ε_{ij}. Finally $\rho > 0$ corresponds to some mechanical properties of the beam ω.

The components of the vector field $\operatorname{div}\sigma(u)$ are given by

$$(\operatorname{div}\sigma(u))_i = \partial_j\sigma_{ij}, \ i = 1, \dots, n.$$

The system (1) is dissipative since its energy defined by

$$E(t) = \frac{1}{2}\int_\Omega \{|u'|^2 + \sigma(u) : \varepsilon(u)\}\, dx + \frac{1}{2}\int_\omega \{|v'|^2 + \rho|\partial^2 v|^2\}\, ds \tag{3}$$

is non increasing.

If $\bar\Gamma_N \cap \bar\Gamma_D \neq \emptyset$, we suppose that the elastodynamical system in Ω is reduced to the isotropic one, namely we assume that

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \operatorname{div} u I_n,$$

where $\lambda, \mu > 0$ are the Lamé coefficients and I_n is the identity matrix in $\mathbb{R}^n$. We further need to assume that (cf. [3]) $c := \bar\Gamma_N \cap \bar\Gamma_D$ is a $(n-2)$-dimensional submanifold of class C^3 such that there exists a neighborhood Ω' of c such that $\Gamma \cap \Omega'$ is a $(n-1)$-dimensional submanifold of class C^3. If $\tau(x)$ denotes the unit normal vector along c pointing outward of Γ_N, we assume that (see figure 1)

$$m(x) \cdot \tau(x) \leq 0, \quad \forall x \in c.$$

Note that the above system (1) is a coupled system between the anisotropic elastodynamical system in Ω and an Euler-Bernouilli beam equation in ω. The feedbacks correspond to the term $m \cdot \nu u'$ on Γ_N and the term $\alpha v'(0,t)$ on the junction γ. (Remark that α may be equal to zero.)

Simpler models were considered in [19, 30, 31], namely their system is a coupling between the wave equations in Ω and in ω. In [30, 31], the controllability of this system is considered using appropriate control on the boundary; while in [19] the stability of this system is considered with the help of a feedback only on Γ_N. As underlined in [31], the analysis of more realistic models should be made. Therefore our goal is to consider a simple but realistic model of the junction between the elasticity system and a beam. The junction between Ω and ω is made via the transmission conditions

$$\begin{aligned} &u(x,t) = v(0,t)\theta(x) && \text{on } \gamma \times \mathbb{R}^+, \\ &\rho\partial^3 v(0,t) + \alpha v'(0,t) + \int_\gamma [\sigma(u)\cdot\nu]\cdot\theta(x)ds(x) = 0 && \forall t \in \mathbb{R}^+. \end{aligned}$$

The first condition means that the displacement u on γ and v at its extremity a is prescribed via the profile θ, in a certain sense the beam is clamped at the domain Ω since we add the condition $\partial v(0) = 0$. The second condition is a (energy) balance law. The boundary conditions on the other extremity of the beam mean that the beam is free at that point. Note that the junction between Ω and ω is made through the profile θ, therefore the angle between ω and the boundary Γ of Ω could be different from $\pi/2$.

1. The main results

We define the following Hilbert spaces:

$$\begin{aligned}\mathcal{H} &= (L^2(\Omega))^n \times L^2(\omega),\\ H^1_{\Gamma_D}(\Omega) &= \{\, u \in H^1(\Omega) \,:\, u = 0 \text{ on } \Gamma_D \,\},\\ V &= \{\, (u, v) \in (H^1_{\Gamma_D}(\Omega))^n \times H^2(\omega) \,:\, u = \theta v(0) \text{ on } \gamma \text{ and } \partial v(0) = 0 \,\}.\end{aligned}$$

The space V is equipped with the natural norm

$$\|(u, v)\|_V^2 = \int_\Omega \sigma(u) : \varepsilon(u)\, dx + \int_\omega \rho(\partial^2 v)^2\, ds,$$

where $\sigma(u) : \varepsilon(u) = \sigma_{ij}(u)\varepsilon_{ij}(u)$.

Theorem 1.1. *For the initial data $((u_0, v_0), (u_1, v_1)) \in V \times \mathcal{H}$, the system* (1) *has a unique (weak) solution (u, v) satisfying*

$$(u, v) \in C^1([0, \infty); \mathcal{H}) \cap C([0, \infty); V).$$

The main result of our paper is the next theorem:

Theorem 1.2. *There exist positive constants M and δ such that the energy of any solution of* (1) *satisfies*

$$E(t) \le M e^{-\delta t}, \quad \forall t \ge 0.$$

Remark 1.3. In [19] the stability of the wave system is obtained under a geometric assumption between γ and the length of ω. Our paper shows that this assumption is unnecessary.

2. Well-posedness of the problem

In this section we prove Theorem 1.1 by reducing the system (1) to a first order evolution equation.

Let us define the operators

$$A : V \longmapsto V' \quad \text{and} \quad B : V \longmapsto V'$$

by

$$\langle A(u,v),(u^*,v^*)\rangle_{V',V} = \int_\Omega \sigma(u):\varepsilon(u^*)\,dx + \int_\omega \rho\partial^2 v\partial^2 v^*\,ds,$$
$$\langle B(u,v),(u^*,v^*)\rangle_{V',V} = \int_{\Gamma_N} m\cdot\nu u\cdot u^*\,d\Gamma + \alpha v(0)v^*(0).$$

Clearly the operators A and B are well defined. Now to obtain the abstract formulation of (1), we take an arbitrary element $(u^*,v^*)\in V$. We multiply the first identity of the system (1) by u^*, integrate by parts in Ω, and use the boundary conditions on Γ_D and Γ_N. This yields

$$\begin{aligned}0 &= \int_\Omega [u'' - \operatorname{div}(\sigma(u))]\cdot u^*\,dx\\ &= \int_\Omega u''\cdot u^* dx - \int_\Gamma (\sigma(u)\cdot\nu)\cdot u^*\,d\Gamma + \int_\Omega \sigma(u):\varepsilon(u^*)\,dx\\ &= \int_\Omega u''\cdot u^*\,dx + \int_\Omega \sigma(u):\varepsilon(u^*)\,dx + \int_{\Gamma_N} m\cdot\nu u'\cdot u^*\,d\Gamma - \int_\gamma [\sigma(u)\cdot\nu]\cdot u^*\,d\Gamma.\end{aligned}$$

In a similar manner, multiplying the second equation of (1) by v^*, and using integration by parts in ω and the boundary conditions, we obtain

$$\begin{aligned}0 &= \int_\omega [v'' + \rho\partial^4 v]v^*\,ds\\ &= \int_\omega v''v^*\,ds + \int_\omega \rho\partial^2 v\partial^2 v^*\,ds + [\rho\partial^3 v v^*]_0^l + [\rho\partial^2 v\partial v^*]_0^l\\ &= \int_\omega v''v^*\,ds + \int_\omega \rho\partial^2 v\partial^2 v^*\,ds - \rho\partial^3 v(0)v^*(0).\end{aligned}$$

Summing these two identities and taking into account the transmission condition on γ we arrive at the identity

$$(u,v)'' + A(u,v) + B(u',v') = (0,0) \text{ in } V'.$$

We now introduce the operators defined on $V\times V$ by

$$\mathbb{A}((u,v),(u^*,v^*)) = ((-u^*,-v^*),A(u,v)),$$
$$\mathbb{B}((u,v),(u^*,v^*)) = ((0,0),B(u^*,v^*)).$$

Setting

$$X = ((u,v),(u',v'))$$

and

$$\mathcal{A} = \mathbb{A} + \mathbb{B}, \tag{4}$$

the system (1) reduces to

$$\begin{cases} X' + \mathcal{A}X = 0, \\ X(0) = ((u_0, v_0), (u_1, v_1)). \end{cases}$$

Lemma 2.1. *Under the above hypotheses, the operator $\mathcal{A}$ defined on $\mathcal{H} \times \mathcal{H}$ by (4), with domain*

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \Big\{ ((u, v), (u^*, v^*)) \in V \times \mathcal{H} : (-\operatorname{div}(\sigma(u)), \partial^4 v) \in \mathcal{H}, \\ \sigma(u) \cdot \nu + m \cdot \nu u^* = 0 \quad \text{on } \Gamma_N, \\ \rho \partial^3 v(0) + \alpha v^*(0) + \int_\gamma [\sigma(u)\nu] \cdot \theta d\Gamma = 0, \\ \partial v(0) = \partial^2 v(l) = \partial^3 v(l) = 0 \Big\} \end{aligned}$$

is maximal dissipative. Moreover $D(\mathcal{A})$ is dense in $\mathcal{H} \times \mathcal{H}$.

The proof of this Lemma is quite standard (see for instance [12, section 2] or [17, Lemma 3.2]). The theory of linear semi-groups [29, 32] leads to Theorem 1.1. Note further that for initial data $((u_0, v_0), (u_1, v_1)) \in D(\mathcal{A})$, the system (1) has a unique strong solution (u, v) satisfying

$$(u, v) \in C^2([0, \infty); \mathcal{H}) \cap C^1([0, \infty); V) \cap C([0, \infty); D(\mathcal{A})).$$

3. Proof of Theorem 1.2

Deriving (3) in time and integrating by parts in space we readily see that

$$E'(t) = -\int_{\Gamma_N} m \cdot \nu |u'(t)|^2 d\Gamma - \alpha v'(0, t)^2$$

and consequently

$$E(S) - E(T) = \int_S^T \Big[\int_{\Gamma_N} m \cdot \nu |u'(t)|^2 dx + \alpha v'(0, t)^2 \Big] dt, \tag{5}$$

for all $0 \leq S \leq T < \infty$. This leads to the decay of the energy.

We will now obtain the exponential decay of this energy. For that purpose introduce the constant

$$R_0 = \max_{x \in \overline{\Omega}} \Big(\sum_{k=1}^n (x_k - x_{0k})^2 \Big)^{1/2}.$$

Let further μ be the smallest positive constant such that for all $u \in (H^1_{\Gamma_D}(\Omega))^n$

$$\int_{\Gamma_N} |u|^2 \, d\Gamma \leq \mu^2 \int_\Omega \sigma(u) : \varepsilon(u) \, dx.$$

We start with two technical Lemmas:

Lemma 3.1. *Let* (u, v) *be a strong solution of* (1). *Define*

$$M(u) = 2(m \cdot \nabla)u + (n-1)u$$

and

$$N(v) = 2(s-l)\partial v - v.$$

Then we have

$$\|M(u)(t)\|^2_{(L^2(\Omega))^n} \le CE(t), \quad \forall t \ge 0,$$
$$\|N(v)(t)\|^2_{L^2(\omega)} \le CE(t), \quad \forall t \ge 0,$$

where, here and below, $C > 0$ *means a positive constant independent of* (u, v).

Proof. By integration by parts we have

$$\begin{aligned}\|M(u)\|^2_{(L^2(\Omega))^n} &= \int_\Omega [|2(m\cdot\nabla)u|^2 + (n-1)^2|u|^2 + 4(n-1)u\cdot(m\cdot\nabla)u]\,dx\\ &= \int_\Omega [|2(m\cdot\nabla)u|^2 + (n-1)^2|u|^2 + 2(n-1)m\cdot\nabla(|u|^2)]\,dx\\ &= \int_\Omega [|2(m\cdot\nabla)u|^2 + (1-n^2)|u|^2]\,dx + 2(n-1)\int_\Gamma m\cdot\nu|u|^2\,d\Gamma\\ &\le 4R_0^2\int_\Omega |\nabla u|^2\,dx + 2(n-1)\int_\Gamma m\cdot\nu|u|^2\,d\Gamma.\end{aligned}$$

We conclude using Korn's inequality since Γ_D is not empty.

For the second estimate by integration by parts we have

$$\|N(v)(t)\|^2_{L^2(\omega)} \le 4\int_\omega (s-l)^2(\partial v(s,t))^2\,ds + 3\int_\omega v^2(s,t)\,ds - 2lv^2(0,t).$$

But Poincaré's inequality leads to

$$\int_\omega (\partial v(s,t))^2\,ds + \int_\omega v^2(s,t)\,ds \le C\Big(\int_\omega (\partial^2 v(s,t))^2\,ds + v^2(0,t)\Big).$$

These two inequalities yield

$$\|N(v)(t)\|^2_{L^2(\omega)} \le C(E(t) + v^2(0,t)). \tag{6}$$

Now the assumption $\theta \ne 0$ and the transmission condition $u = \theta v$ on γ lead to

$$v^2(0,t) \le \frac{1}{\int_\gamma \theta^2\,d\Gamma}\int_\gamma |u|^2\,d\Gamma,$$

and by Korn's inequality we obtain

$$v^2(0,t) \le C\int_\Omega \sigma(u) : \varepsilon(u)\,d\Gamma \le CE(t).$$

This estimate in (6) leads to the conclusion. □

For $0 \leq T \leq \infty$, we set

$$Q = \Omega \times (0,T), \qquad q = \omega \times (0,T)$$
$$\Sigma = \Gamma \times (0,T), \qquad \Sigma_D = \Gamma_D \times (0,T), \qquad \Sigma_N = \Gamma_N \times (0,T).$$

Lemma 3.2. *If $\alpha \geq 0$, there exists a constant $C \geq 0$ such that for all $\varepsilon \in (0,1)$ and $T \geq 0$, we have*

$$\int_0^T \int_{\Gamma_N} |u|^2 \, d\Gamma \, dt + \int_0^T |v(0,t)|^2 \, dt \leq \frac{C}{\varepsilon} E(0) + \varepsilon \int_0^T E(t) \, dt.$$

Proof. For $t \geq 0$, consider the solution $z = z(t)$ of (compare with [9, Lemma 5.2])

$$\begin{cases} \operatorname{div}(\sigma(z)) = 0 & \text{in } \Omega, \\ z = u & \text{on } \Gamma. \end{cases} \tag{7}$$

This solution is characterized by $z = \omega + u$ where $\omega \in (H_0^1(\Omega))^n$ is the unique solution of

$$\int_\Omega \sigma(\omega) : \varepsilon(v) \, dx = -\int_\Omega \sigma(u) : \varepsilon(v) \, dx \quad \forall v \in (H_0^1(\Omega))^n.$$

This identity means that

$$\int_\Omega \sigma(z) : \varepsilon(v) \, dx = 0 \quad \forall v \in (H_0^1(\Omega))^n.$$

Taking $v = z - u$ in this identity, we deduce that

$$\int_\Omega \sigma(z) : \varepsilon(u) \, dx = \int_\Omega \sigma(z) : \varepsilon(z) \, dx \geq 0. \tag{8}$$

One easily shows that z also satisfies (see [9, Lemma 5.2])

$$\int_\Omega f \cdot z \, dx = -\int_\Gamma z \cdot (\sigma(v_f)\nu) \, d\Gamma, \quad \forall f \in (L^2(\Omega))^n, \tag{9}$$

where $v_f \in (H_0^1(\Omega))^n$ is the unique solution of

$$\int_\Omega \sigma(v_f) : \varepsilon(w) \, dx = \int_\Omega f \cdot w \, dx, \forall w \in (H_0^1(\Omega))^n.$$

Taking $f = z$ in the identity (9), we may write

$$\int_\Omega |z|^2 \, dx = -\int_\Gamma z \cdot (\sigma(v_z)\nu) \, d\Gamma.$$

Since $z = u$ on Γ_N, $z = u = 0$ on Γ_D, and $z = u = \theta v$ on γ, by Cauchy-Schwarz's inequality we obtain

$$\int_\Omega |z|^2\,dx \le C(\|u\|_{(L^2(\Gamma_N))^n} + |v(0,t)|)\|\sigma(v_z)\nu\|_{(L^2(\Gamma))^n}. \tag{10}$$

As the boundary Γ is C^2, elliptic regularity results yield $v_z \in (H^2(\Omega))^n$ with the estimate

$$\|v_z\|_{(H^2(\Omega))^n} \le K\|z\|_{(L^2(\Omega))^n},$$

for some positive constant K. This estimate and a standard trace theorem lead to

$$\|\sigma(v_z)\nu\|_{(L^2(\Gamma))^n} \le K_1\|z\|_{(L^2(\Omega))^n},$$

for some positive constant K_1. Inserting this estimate in (10) we arrive at

$$\int_\Omega |z|^2\,dx \le C\Big(\int_{\Gamma_N} |u|^2\,d\Gamma + |v(0,t)|^2\Big). \tag{11}$$

Since z' is solution of problem (7) with u' instead of u, the above arguments yield

$$\int_\Omega |z'|^2\,dx \le C\Big(\int_{\Gamma_N} |u'|^2 d\Gamma + |v'(0,t)|^2\Big). \tag{12}$$

In the same manner for $t \ge 0$, consider the solution $w = w(t)$ of

$$\begin{cases} \partial^4 w = 0 & \text{in } \omega, \\ w(0) = v(0), \quad \partial w(0) = \partial v(0) = 0, & \partial^2 w(l) = \partial^3 w(l) = 0. \end{cases} \tag{13}$$

This solution w is characterized by $w = w_1 + v$ where $w_1 \in W$ is the unique solution of

$$\int_\omega \partial^2 w_1 \partial^2 k\,ds = -\int_\omega \partial^2 v \partial^2 k\,ds, \quad \forall k \in W,$$

the Hilbert space W (with the natural norm) being defined by

$$W = \{\, k \in H^2(\omega) : k(0) = \partial k(0) = \partial^2 k(l) = \partial^3 k(l) = 0 \,\}.$$

As before this identity means that

$$\int_\omega \partial^2 w \partial^2 k\,ds = 0 \quad \forall k \in W,$$

and taking $k = w - v$ in this identity, we deduce that

$$\int_\omega \partial^2 v \partial^2 w = \int_\omega (\partial^2 w)^2\,ds \ge 0. \tag{14}$$

Let us also notice that w satisfies

$$\int_\omega gw\,ds = -w(0)\partial^3 k_g(0), \quad \forall g \in L^2(\omega), \tag{15}$$

where $k_g \in W$ is the unique solution of

$$\int_\omega \partial^2 k_g \partial^2 k\,ds = \int_\omega gk\,ds, \forall k \in W.$$

Taking $g = w$ in the identity (15), we may write

$$\int_\omega |w|^2\,ds = -w(0)\partial^3 k_w(0),$$

and since $w(0) = v(0)$, we obtain

$$\int_\omega |w|^2\,ds \le |v(0,t)||\partial^3 k_w(0)|. \tag{16}$$

As $k_w \in H^4(\omega)$ with the estimate

$$\|k_w\|_{H^4(\omega)} \le K^* \|w\|_{L^2(\omega)},$$

for some positive constant K^*, by the Sobolev embedding theorem we obtain

$$|\partial^3 k_w(0)| \le K_1^* \|w\|_{L^2(\omega)},$$

for some positive constant K_1^*. Inserting this estimate in (16) we arrive at

$$\int_\omega |w|^2\,ds \le C|v(0,t)|. \tag{17}$$

Since w' is solution of problem (13) with v' instead of v, the above arguments yield

$$\int_\omega |w'|^2\,ds \le C|v'(0,t)|. \tag{18}$$

Now using a standard trace theorem and Korn's inequality (since $\Gamma_D \neq \emptyset$) we have

$$\int_{\Gamma_N \cup \gamma} |z|^2\,d\Gamma \le C \int_\Omega \sigma(z) : \varepsilon(z)\,dx.$$

Recalling that $z = u$ on Γ_N and $z = u = \theta v$ on γ, we get

$$\int_{\Gamma_N} |u|^2\,d\Gamma + |v(0,t)|^2 \le C \int_\Omega \sigma(z) : \varepsilon(z)\,dx.$$

This implies that

$$\int_{\Gamma_N} |u|^2 d\Gamma + |v(0,t)|^2 \leq C\Big(\int_\Omega \sigma(z) : \varepsilon(z)\, dx + \rho \int_\omega (\partial^2 w)^2\, ds\Big).$$

Using the identities (8) and (14) we get

$$\int_{\Gamma_N} |u|^2\, d\Gamma + |v(0,t)|^2 \leq C(\int_\Omega \sigma(z) : \varepsilon(u)\, dx + \rho \int_\omega \partial^2 v \partial^2 w).$$

Integrating this identity for $t \in (0,T)$, we find

$$\int_{\Sigma_N} |u|^2\, d\Gamma\, dt + \int_0^T |v(0,t)|^2\, dt \leq C\Big(\int_Q \sigma(u) : \varepsilon(z)\, dx\, dt + \rho \int_q \partial^2 v \partial^2 w\, ds\, dt\Big).$$

By integration by parts, we get

$$\int_{\Sigma_N} |u|^2\, d\Gamma\, dt + \int_0^T |v(0,t)|^2\, dt \leq C\Big(-\int_Q \operatorname{div} \sigma(u) \cdot z\, dx\, dt + \rho \int_q \partial^4 v w\, ds\, dt + \int_\Sigma \sigma(u)\nu \cdot z\, d\Gamma\, dt + \rho \int_0^T w(0,t)\partial^3 v(0,t)\, dt\Big).$$

As $z = u$ on Σ_N, $z = 0$ on Σ_D, $z = \theta v(0,t)$ on $\gamma \times (0,T)$, and $w(0) = v(0,t)$, using the boundary conditions on Σ_N and on $\gamma \times (0,T)$ for u we may write

$$\int_\Sigma \sigma(u)\nu \cdot z\, d\Gamma dt = \int_{\Sigma_N} m \cdot \nu u' u\, d\Gamma dt - \int_0^T v(0,t)(\rho \partial^3 v(0,t) + \alpha v'(0,t))\, dt.$$

Inserting this identity in the last one and using the first and second identities of (1), we arrive at

$$\int_{\Sigma_N} |u|^2\, d\Gamma + \int_0^T |v(0,t)|^2\, dt \leq C\Big(-\int_Q u'' \cdot z\, dx\, dt - \int_q v'' w\, ds\, dt + \int_{\Sigma_N} m \cdot \nu u' u\, d\Gamma\, dt - \alpha \int_0^T v(0,t) v'(0,t)\, dt\Big).$$

Now integrating by parts in time, we obtain

$$\int_{\Sigma_N} |u|^2\, d\Gamma + \int_0^T |v(0,t)|^2\, dt \leq C\Big(\int_Q u' \cdot z'\, dx\, dt + \int_q v' w'\, ds\, dt - \int_\Omega z u' |_0^T - \int_\omega w v' |_0^T + \int_{\Sigma_N} m \cdot \nu u' u d\Gamma\, dt - \alpha \int_0^T v(0,t) v'(0,t)\, dt\Big). \quad (19)$$

Fix an arbitrary $\varepsilon_0 \geq 0$. Using several times (5), (11), (12), (17), (18), and Young's inequality, we can estimate the different integrals of the right-hand side of the above inequality as follows:

$$\begin{aligned}\int_{\Sigma_N} m\cdot\nu u u' d\Sigma &\leq \varepsilon_0 \int_{\Sigma_N} |u|^2\, d\Sigma + \frac{1}{4\varepsilon_0}\int_{\Sigma_N} m\cdot\nu |u'|^2\, d\Sigma\\ &\leq 2\varepsilon_0\mu^2 \int_0^T E(t)\, dt + \frac{1}{4\varepsilon_0} E(0),\end{aligned}$$

$$\begin{aligned}\int_Q z'u' dxdt &\leq \varepsilon_0 \int_Q |u'|^2\, dx\, dt + \frac{1}{4\varepsilon_0}\int_Q |z'|^2\, dx\, dt\\ &\leq 2\varepsilon_0 \int_0^T E(t) dt + \frac{C}{4\varepsilon_0} E(0),\end{aligned}$$

$$\int_q w'v' dxdt \leq 2\varepsilon_0 \int_0^T E(t)\, dt + \frac{C}{4\varepsilon_0} E(0),$$

$$-\int_\Omega zu'|_0^T \leq 4(1 + C\mu^2)E(0),$$

$$-\int_\omega wv'|_0^T \leq CE(0),$$

$$\begin{aligned}-\alpha \int_0^T v(0,t)v'(0,t)\, dt &\leq \frac{1}{\varepsilon_0}\alpha \int_0^T |v'(0,t)|^2\, dt + \varepsilon_0 \int_0^T |v(0,t)|^2\, dt\\ &\leq \frac{1}{\varepsilon_0} E(0) + \varepsilon_0 \int_0^T |v(0,t)|^2\, dt.\end{aligned}$$

Using these different estimates in (19), we arrive at the requested estimate by choosing ε_0 appropriately. □

Proof of Theorem 1.2. Without loss of generality we can assume that

$$\Big(\frac{l}{2} + \int_\gamma m\cdot\nu |\theta(x)|\, d\Gamma\Big) \leq 0. \tag{20}$$

Indeed if (20) is not satisfied, we can use the following scaling argument: For a parameter $\beta > 0$ fixed later on, let us set

$$\hat{v}(\hat{s},t) = v(\beta\hat{s},t) \text{ on } \hat{\omega},$$

where

$$\hat{\omega} = \{\, a + \hat{s}\nu(a) \,:\, 0 < \hat{s} < \hat{l}\,\},$$

$\hat{l} = l/\beta$ being the length of $\hat{\omega}$. We then see that the pair $(u, \hat{v})$ is solution of (1) with ω (resp. ρ) replaced by $\hat{\omega}$ (resp. $\hat{\rho} = \beta^{-4}\rho$). For this new system, the condition (20) is equivalent to

$$\Big(\frac{l}{2\beta} + \int_\gamma m \cdot \nu |\theta(x)|\, d\Gamma\Big) \leq 0,$$

which holds if β is chosen sufficiently large, namely if

$$\beta \geq -\frac{l}{2\int_\gamma m \cdot \nu |\theta(x)|\, d\Gamma}. \tag{21}$$

For a fixed β, we further notice that

$$\min\{1, \beta\}\hat{E}(t) \leq E(t) \leq \max\{1, \beta\}\hat{E}(t),$$

where $\hat{E}(t)$ is the energy of the new system:

$$\hat{E}(t) = \frac{1}{2}\int_\Omega \{|u'|^2 + \sigma(u) : \varepsilon(u)\}\, dx + \frac{1}{2}\int_{\hat{\omega}} \{|\hat{v}'|^2 + \hat{\rho}|\hat{\partial}^2 \hat{v}|^2\}\, d\hat{s}.$$

Consequently the exponential stability of the energy E is equivalent to the exponential stability of the energy $\hat{E}$. Therefore if (20) does not hold, it suffices to consider the new system for $(u, \hat{v})$ for a fixed β satisfying (21) and the exponential stability of this new system (proved below) will imply the exponential stability of the original system

Assume first that (u, v) is a strong solution of (1).

If $\bar{\Gamma}_D \cap \bar{\Gamma}_N = \emptyset$, then multiplying the first identity of (1) by

$$M(u) = 2(m \cdot \nabla)u + (n-1)u$$

and integrating by parts on Q we obtain

$$\begin{aligned} 0 = (u', M(u))|_0^T &+ \int_Q |u'|^2\, dx\, dt - \int_{\Sigma_N} m \cdot \nu |u'|^2 d\Sigma \\ &- \int_{\gamma \times [0,T]} m \cdot \nu |u'|^2\, ds(x)\, dt + \int_Q \sigma(u) : \varepsilon(u)\, dx\, dt \\ &- \int_\Sigma [(\sigma(u)\nu) \cdot M(u) - (m \cdot \nu)\sigma(u) : \varepsilon(u)]\, d\Sigma. \end{aligned}$$

If $\Gamma_D \cap \Gamma_N \neq \emptyset$, then applying [3, Theorem 4.1], we have

$$\begin{aligned} 0 \geq (u', M(u))|_0^T &+ \int_Q |u'|^2\, dx\, dt - \int_{\Sigma_N} m \cdot \nu |u'|^2\, d\Sigma \\ &- \int_{\gamma \times [0,T]} m \cdot \nu |u'|^2\, ds(x)\, dt + \int_Q \sigma(u) : \varepsilon(u)\, dx\, dt \\ &- \int_\Sigma [(\sigma(u)\nu) \cdot M(u) - (m \cdot \nu)\sigma(u) : \varepsilon(u)]\, d\Sigma. \end{aligned}$$

Similarly multiplying the second identity of (1) by $N(v)$ and integrating by parts on q we obtain

$$0 = 2\int_0^T \int_\omega |v'|^2 - l\int_0^T |v'(0,t)|^2\,dt$$
$$+ \int_\omega v'N(v)|_0^T + 2\rho\int_0^T \int_0^l (\partial^2 v)^2\,dt + 2l\rho\int_0^T \partial^3 v(0,t)\partial v(0,t)$$
$$+ \rho\int_0^T \partial^3 v(0,t)v(0,t)\,dt.$$

These two identities (or inequalities if $\bar\Gamma_D \cap \bar\Gamma_N \neq \emptyset$) allow to obtain

$$\int_0^T E(t)\,dt \le \int_{\Sigma_N} m\cdot\nu|u'|^2 + \sum_{i=1}^4 I_i$$

where

$$I_1 = -\int_\Omega (u', M(u))|_0^T - \frac{1}{2}\int_\omega N(v)v'|_0^T,$$
$$I_2 = \int_{\Sigma_N\cup\Sigma_D} [(\sigma(u)\nu)\cdot M(u) - (m\cdot\nu)\sigma(u):\varepsilon(u)]d\Sigma,$$
$$I_3 = \int_{\gamma\times(0,T)} m\cdot\nu|u'|^2\,d\Gamma\,dt + \frac{l}{2}\int_0^T v'^2(0,t)\,dt,$$
$$I_4 = \int_{\gamma\times(0,T)} [(\sigma(u)\nu)\cdot M(u) - (m\cdot\nu)\sigma(u):\varepsilon(u)]d\Sigma - \frac{1}{2}\rho\int_0^T \partial^3 v(0,t)v(0,t)\,dt.$$

Lemma 3.1 yields

$$I_1 \le CE(0).$$

As in [1, 9] using local coordinates systems we obtain the estimate

$$I_2 \le C(E(0) + \int_{\Sigma_N} (|u|^2 + |u'|^2)\,d\Sigma).$$

Using the boundary condition $u = \theta v$ on $\gamma \times (0,T)$ in system (1) and the condition (20), we get

$$I_3 = \Big(\frac{l}{2} + \int_\gamma m\cdot\nu|\theta(x)|^2\Big)\int_0^T v'^2(0,t)dt \le 0.$$

Again using the boundary condition on $\gamma \times (0,T)$ in system (1) we may write

$$I_4 = \int_{\gamma\times(0,T)} [(\sigma(u)\nu)\cdot M(u) - (m\cdot\nu)\sigma(u) : \varepsilon(u)]\, d\Sigma \\ + \frac{1}{2}\int_{\gamma\times(0,T)} (\sigma(u)\nu)\cdot\theta v(0,t)\, d\Sigma + \frac{\alpha}{2}\int_0^T v'(0,t)v(0,t)\, dt. \quad (22)$$

The estimation of I_4 is also based on the use of local coordinates systems (cf. [1]). Namely for all $x\in\Gamma$, we denote by $\pi(x)$ the orthogonal projection on the tangent hyperplane $T_x(\Gamma)$. Any vector field $v : \bar{\Omega} \to \mathbb{R}^n$ will be split up as follows:

$$v(x) = v_T(x) + v_\nu(x)\nu(x),$$

where $v_T(x) = \pi(x)v(x)$ is the tangential component of v and $v_\nu(x) = v(x)\cdot\nu(x)$. We further denote by $\partial_\nu v = \nu\cdot\nabla v$, the normal derivative of v and by $\nabla_T v = \nabla v - \partial_n v$ the tangential gradient of v. For further uses, we set $\partial_T v = \overline{\nabla_T v}$, the tangential derivation of v, where $\overline{\tau}$ means the transposed matrix of the matrix τ. Similarly for a vector v, $\overline{v}$ will mean its transposed vector.

Following [15] or [33], the strain tensor is written as follows:

$$\varepsilon(v) = \varepsilon_T(v) + \nu\overline{\varepsilon_S(v)} + \varepsilon_S(v)\overline{\nu} + \varepsilon_\nu(v)\nu\overline{\nu} \qquad \text{on} \quad \Gamma,$$

with

$$\begin{aligned} 2\varepsilon_T(v) &= \pi(\partial_T v_T)\pi + \pi\overline{\partial_T v_T}\pi + 2v_\nu\partial_T\nu, \\ 2\varepsilon_S(v) &= \partial_\nu v_T + \nabla_T v_\nu - (\partial_T\nu)v_T, \\ \varepsilon_\nu(v) &= \partial_\nu v_\nu, \end{aligned}$$

where $(\partial_T\nu)$ is the curvature operator of Γ. Similarly the stress tensor may be written

$$\sigma(v) = \sigma_T(v) + \nu\overline{\sigma_S(v)} + \sigma_S(v)\overline{\nu} + \sigma_\nu(v)\nu\overline{\nu} \qquad \text{on } \Gamma,$$

where $\sigma_T(v)$ is an endomorphism on the tangent hyperplane, $\sigma_S(v)$ is a tangent vector field and $\sigma_\nu(v)$ is a scalar field.

These splittings allow to write

$$\begin{aligned} \varepsilon(v) : \varepsilon(v) &= \varepsilon_T(v) : \varepsilon_T(v) + 2|\varepsilon_S(v)|^2 + |\varepsilon_\nu(v)|^2 && \text{on } \Gamma, \\ \sigma(v) : \varepsilon(v) &= \sigma_T(v) : \varepsilon_T(v) + 2\overline{\sigma_S(v)}\varepsilon_S(v) + \sigma_\nu(v)\varepsilon_\nu(v) && \text{on } \Gamma. \end{aligned}$$

Using these local coordinates systems and the boundary condition on $\gamma\times(0,T)$ in system (1) we obtain

$$\begin{aligned} \sigma(u)\nu &= \sigma_S(u) + \sigma_\nu(u)\nu && \text{on } \gamma\times(0,T), \\ M(u) &= 2(m\cdot\nu)\partial_\nu u + v_1(\theta)v(0,t) && \text{on } \gamma\times(0,T), \end{aligned}$$

for some vector valued function $v_1(\theta)$ (depending on θ and its tangential gradient). This yields

$$\begin{aligned}\sigma(u)\nu \cdot M(u) &= \overline{\sigma(u)\nu} M(u) \\ &= 2(m\cdot\nu)(\bar{\sigma}_S(u) + \sigma_\nu(u)\bar{\nu})\partial_\nu u + \sigma(u)\nu \cdot v_1(\theta) v(0,t) \\ &= 2(m\cdot\nu)(\bar{\sigma}_S(u)\partial_\nu u_T + \sigma_\nu(u)\bar{\nu}\partial_\nu u_\nu) \\ &\quad + \sigma(u) : C_1(\theta) v(0,t) \text{ on } \gamma \times (0,T),\end{aligned}$$

for some matrix valued function $C_1(\theta)$ (depending on θ and its tangential gradient).

On the other hand, we recall that

$$\sigma(u) : \varepsilon(u) = \sigma_T(u) : \varepsilon_T(u) + 2\overline{\sigma_S(u)}\varepsilon_S(u) + \sigma_\nu(u)\varepsilon_\nu(u) \qquad \text{on } \gamma \times (0,T),$$

and again using the boundary condition, we obtain

$$\sigma(u) : \varepsilon(u) = (\bar{\sigma}_S(u)\partial_\nu u_T + \sigma_\nu(u)\bar{\nu}\partial_\nu u_\nu) + \sigma(u) : C_2(\theta) v(0,t) \qquad \text{on } \gamma \times (0,T).$$

All together we arrive at

$$\begin{aligned}(\sigma(u)\nu)\cdot M(u) - (m\cdot\nu)\sigma(u) : \varepsilon(u) = (m\cdot\nu)&\sigma(u) : \varepsilon(u) \\ &+ \sigma(u) : C_3(\theta) v(0,t) \qquad \text{on } \gamma \times (0,T).\end{aligned}$$

Inserting this identity into (22), we obtain

$$\begin{aligned}I_4 &= \int_{\gamma\times(0,T)} (m\cdot\nu)\sigma(u) : \varepsilon(u)\, d\Sigma \\ &\quad + \int_{\gamma\times(0,T)} [\sigma(u) : C_3(\theta) + \frac{1}{2}(\sigma(u)\nu)\cdot\theta] v(0,t)\, d\Sigma + \frac{\alpha}{2}\int_0^T v'(0,t) v(0,t)\, dt.\end{aligned}$$

By Young's inequality we obtain

$$\begin{aligned}I_4 \leq \int_{\gamma\times(0,T)} &(m\cdot\nu)\sigma(u) : \varepsilon(u)\, d\Sigma\, dt \\ &+ \epsilon \int_{\gamma\times(0,T)} |\sigma(u)|^2\, d\Sigma\, dt + \frac{C}{\epsilon}\int_0^T |v(0,t)|^2\, dt \\ &+ \alpha \int_0^T |v'(0,t)|^2 dt, \forall \epsilon \in (0,1).\end{aligned}$$

Now using the assumption (2), we may write

$$|\sigma(u)|^2 \leq C|\varepsilon(u)|^2 \leq C\sigma(u) : \varepsilon(u).$$

Therefore reminding that $m \cdot \nu < -\alpha_0 < 0$ on γ, by fixing $\epsilon < \frac{C\alpha_0}{2}$, we obtain

$$I_4 \leq \int_{\gamma\times(0,T)} \frac{m\cdot\nu}{2}\sigma(u) : \varepsilon(u)\, d\Sigma\, dt + C\int_0^T |v(0,t)|^2\, dt + \alpha\int_0^T |v'(0,t)|^2\, dt.$$

Since $m \cdot \nu \leq 0$ on γ, we conclude that

$$I_4 \leq C\int_0^T |v(0,t)|^2\, dt + \alpha\int_0^T |v'(0,t)|^2\, dt.$$

The estimates on $I_i, i = 1, \dots, 4$ yield

$$2\int_0^T E(t)\, dt \leq C(E(0) + \int_{\Sigma_N} m\cdot\nu|u'|^2 d\Sigma + \alpha\int_0^T |v'(0,t)|^2 dt) \\ + C\int_{\Sigma_N} |u|^2\, d\Sigma + C(\theta)\int_0^T v^2(0,t)dt. \quad (23)$$

By Lemma 3.2 the above estimate (23) becomes

$$2\int_0^T E(t)\, dt \leq C(E(0) + \int_{\Sigma_N} m\cdot\nu|u'|^2 d\Sigma + \alpha\int_0^T |v'(0,t)|^2 dt) \\ + \frac{C}{\varepsilon}E(0) + \varepsilon\int_0^T E(t)\, dt,$$

for any $\varepsilon > 0$. By choosing ε small enough, we arrive at the observability estimate

$$\int_0^T E(t)\, dt \leq C(E(0) + \int_{\Sigma_N} m\cdot\nu|u'|^2\, d\Sigma + \alpha\int_0^T |v'(0,t)|^2\, dt).$$

This estimate remains valid for weak solutions by a density argument. The conclusion now follows from this estimate as shown in [26, Theorem 3.3]. □

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