Fundamental Groups of Some Special Quadric Arrangements

Meirav Amram [1] and Mina Teicher

Einstein Mathematics Institute
The Hebrew University
Jerusalem — Israel
ameirav@math.huji.ac.il

Department of Mathematics
Bar-Ilan University
Ramat-Gan — Israel
teicher@macs.biu.ac.il

Received: June 2, 1005
Accepted: January 1, 2006

ABSTRACT

Continuing our work on the fundamental groups of conic-line arrangements [3], we obtain presentations of fundamental groups of the complements of three families of quadric arrangements in $\mathbb{P}^2$. The first arrangement is a union of n conics, which are tangent to each other at two common points. The second arrangement is composed of n quadrics which are tangent to each other at one common point. The third arrangement is composed of n quadrics, $n-1$ of them are tangent to the nth one and each one of the $n-1$ quadrics is transversal to the other $n-2$ ones.

Key words: fundamental groups, complement of curve, conic arrangement.

2000 Mathematics Subject Classification: 14H20, 14H30, 14Q05.

Introduction

The aim of the present article is the computation of the fundamental groups to complements of some quadric arrangements in $\mathbb{P}^2$. Recall that given a quadric-line arrangement in $\mathbb{P}^2$, we are interested in computing the fundamental group of its complement. In [3], general presentations for some families of quadric-line arrangement

[1] Partially supported by EU-network HPRN-CT-2009-00099 (EAGER), the Emmy Noether Research Institute for Mathematics, the Israel Science Foundation grant #8008/02-3 (Excellency Center "Group Theoretic Methods in the Study of Algebraic Varieties"), the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

ISSN: 1139-1138
http://dx.doi.org/10.5209/rev_REMA.2006.v19.n2.16577

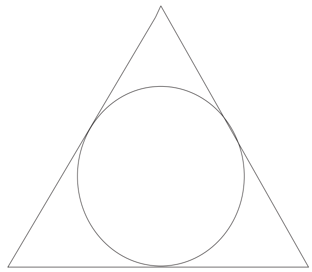

Figure 1: The arrangement $\mathcal{A}_3$

were computed. The present paper is devoted to the computation of the fundamental groups related to three infinite families of quadric arrangements. The three types of interesting families of quadric curves in this paper are as follows. The first arrangement is a union of n quadrics, which are tangent to each other at two common points (figure 2). The second arrangement is composed of n quadrics which are tangent to each other at one common point (figure 3). The third arrangement is composed again of n quadrics, $n-1$ of them are tangent to the nth one and each one of the $n-1$ quadrics is transversal to the other $n-2$ ones (figure 4).

Some work has been done concerning line arrangements (see e.g. [7, 11, 12]), and other progress has been done also concerning quadric-line arrangements (see [1–4]).

Let $C \subset \mathbb{P}^2$ be a plane curve and $* \in \mathbb{P}^2 \setminus C$ a base point. By abuse of notation, we will call the group $\pi_1(\mathbb{P}^2 \setminus C, *)$ the *fundamental group of* C, and we shall frequently omit base points and write $\pi_1(\mathbb{P}^2 \setminus C)$. One is interested in the group $\pi_1(\mathbb{P}^2 \setminus C)$ mainly for two reasons. First, when the curve appears to be a branch curve, then $\pi_1(\mathbb{P}^2 \setminus C)$ is an important invariant, concerning either the branch curve or the surface itself. Secondly, it contributes to the study of the Galois coverings $X \to \mathbb{P}^2$ branched along C. Many interesting surfaces have been constructed as branched Galois coverings of the plane. An example has been already given in [3]. It involves the arrangement $\mathcal{A}_3$ (shown in figure 1 above), which has Galois coverings $X \to \mathbb{P}^2$ branched along it, $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, or X is either an abelian surface, a $K3$ surface, or a quotient of the two-ball $\mathbb{B}_2$ (see [8, 14, 16]). Moreover, some line arrangements defined by unitary reflection groups studied in [10] are related to $\mathcal{A}_3$ via orbifold coverings. For example, if $\mathcal{L}$ is the line arrangement given by the equation

$$xyz(x+y+z)(x+y-z)(x-y+z)(x-y-z) = 0,$$

then the image of $\mathcal{L}$ under the branched covering map

$$[x:y:z] \in \mathbb{P}^2 \to [x^2:y^2:z^2] \in \mathbb{P}^2$$

is the arrangement $\mathcal{A}_3$, see [14] for details.

The standard tool for fundamental group computations is the Zariski-van Kampen algorithm [15, 17] (see [5] for a modern approach and [3] for a detailed explanation). We use a variation of this algorithm developed in [13] for computing the fundamental groups of quadrics arrangements and avoid lengthy monodromy computations. This approach has the advantage that it permits to capture the local fundamental groups around special and complicated singularities of these arrangements. The local fundamental groups are needed for the study of the singularities of $\mathbb{P}^2$ branched along these arrangements.

This paper is divided into four parts. In section 1, we quote basic definitions and an alternative way for computing quadric arrangements, by defining them as birational to line arrangements or as their coverings. In section 2, we quote the results of this work (Theorems 2.2, 2.5, 2.8), using the techniques from [3] and [13]. In sections 3, 4, and 5, we prove these theorems.

1. Quadric arrangements related to line arrangements

1.1. Meridians

Let $C \subset X$ be a curve in a smooth complex surface X and $p \in C$. A *meridian* μ of C at p, based on a point $* \in \mathbb{P}^2 \setminus C$ is a loop in $\mathbb{P}^2 \setminus C$ obtained by following a path ω with $\omega(0) = *$ and $\omega(1)$ belonging to a small neighborhood of p, turning around C in the positive sense along the boundary of a small disc Δ, having with C a single intersection at p, and then turning back to $*$ along ω. If $B \subset C$ is an irreducible component, a meridian μ_p of C at a point $p \in B \setminus \mathrm{Sing}(C)$ will be called a *meridian* of B. It is well-known that (homotopy classes of) any two meridians of B are conjugate elements in $\pi_1(X \setminus C, *)$ (see e.g. [9, section 7.5]). When p is a singular point of C, we have the following result.

Lemma 1.1. *Let $p \in C$ be a singular point, μ_p a meridian of C at p, and let $\sigma : Y \to \mathbb{P}^2$ be the blow-up of X at p. Denote by C the proper transform of C and by P the exceptional divisor. Then $\sigma(\mu_p)$ is a meridian of P. In particular, any two meridians of C at p are conjugate elements of $\pi_1(X \setminus C) \simeq \pi_1(Y \setminus (C \cup P))$.*

Proof. The spaces $Y \setminus (C \cup P)$ and $X \setminus C$ are homeomorphic. By definition, $\mu_p = \omega \cdot \partial\Delta \cdot \omega^{-1}$, where Δ is a disc, having an intersection with C at p, implying that the disc $\sigma(\Delta)$ intersects P transversally and away from C. In other words, the loop $\sigma(\mu)$ is a meridian of P. □

The group $\pi_1(X \setminus C)$ is an invariant of the pair $(\mathbb{P}^2, C)$. Since meridians are well-defined up to a conjugacy class, they can be considered as supplementary invariants of $\pi_1(X \setminus C)$. What follows is a description of how to capture the meridians at singular points of C, during the computation of the group $\pi_1(X \setminus C)$ by Zariski-van Kampen.

Lemma 1.2. *Let $C \subset \mathbb{P}^2$ be a curve, L_0 a line in general position with respect to C and let $* \in L_0 \setminus C$ be a base point. Let p be a singular point of C. We assume that*

L_0 passes through a sufficiently small ball V around p, so that the disc $\Delta := L_0 \cap V$ meets all branches of C meeting at p. Take a path ω in L_0 connecting $*$ to a boundary point q of Δ. Then $\mu_0 := \omega \cdot \partial\Delta \cdot \omega^{-1}$ is a homotopic to a meridian of C at p.

Proof. Let L_1 be the line through $*$ and p. Consider the projection $\phi : \mathbb{P}^2 \to \mathbb{P}^1$ from the point $*$ and with $\phi(L_0) = a$, $\phi(L_1) = b$ and take a path $\gamma \subset \phi(V)$ with $\gamma(0) = a$ and $\gamma(1) = b$. Put L_t for the fiber above $\gamma(t)$. Let $\sigma \subset \partial V$ be a lift of the path γ with $\sigma(0) = q$. Assume that σ_τ is the loop σ until τ, i.e. with $\sigma_\tau(t) := \sigma(\tau t)$ $(t \in [0, 1])$. Put $\omega_\tau := \sigma_\tau \cdot \omega$ and define $\mu_\tau := \omega_\tau \cdot \partial\Delta_\tau \cdot \omega_\tau^{-1}$, where $\Delta_\tau := L_\tau \cap V$. Then $M(t, \tau) := \mu_\tau(t)$ gives a homotopy between μ_0 and μ_1, and this latter loop is obviously a meridian of C at p. □

1.2. Quadric arrangements birational to line arrangements

Assume that $\mathcal{A}$ is a line arrangement, and let ψ be the involution

$$\psi : [x : y : z] \in \mathbb{P}^2 \to [1/x : 1/y : 1/z] \in \mathbb{P}^2.$$

Suppose that the lines X, Y, Z are respectively given by the equations $x = 0$, $y = 0$, and $z = 0$. If $\mathcal{A}$ is in general position with respect to $X \cup Y \cup Z$, then $\psi(\mathcal{A})$ is an arrangement of smooth quadrics. In addition to those of $\mathcal{A}$, this arrangement has three more singular points where all the irreducible components of $\psi(\mathcal{A})$ meet transversally.

In this case, the group $\pi_1(\mathbb{P}^2 \setminus \psi(\mathcal{A}))$ can easily be found in terms of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ as follows: Assuming $\mathcal{A} = \cup_{i=1}^n L_i$, let

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \simeq \langle\, \mu_1, \dots, \mu_n \mid w_1 = \cdots = w_m = \mu_1 \cdots \mu_n = 1 \,\rangle \tag{1}$$

be a presentation obtained by an application of Zariski-van Kampen, where μ_i is a meridian of L_i. Put $\mathcal{A}' := \mathcal{A} \cup X \cup Y \cup Z$. Since $\mathcal{A}$ is in general position with respect to $X \cup Y \cup Z$, one has by [7]

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}') \simeq \left\langle \begin{array}{c|c} \mu_1, \dots, \mu_n, & [\mu_i, \sigma_j] = [\sigma_j, \sigma_k] = 1 \quad (i \in [1, n],\ j, k \in [1, 3]) \\ \sigma_1, \sigma_2, \sigma_3 & w_1 = \cdots = w_m = \mu_1 \cdots \mu_n \sigma_1 \sigma_2 \sigma_3 = 1 \end{array} \right\rangle \tag{2}$$

where σ_1, σ_2, σ_3 are, respectively, meridians of X, Y, and Z. Let $p := X \cap Y$, $q = Y \cap Z$, and $r := Z \cap X$. Then $\sigma_1\sigma_2$ (respectively, $\sigma_2\sigma_3$, $\sigma_3\sigma_1$) is a meridian of $\mathcal{A}'$ at p (respectively, q, r). Hence, the group $\pi_1(\mathbb{P}^2 \setminus \psi(\mathcal{A}))$ can be obtained by setting $\sigma_1\sigma_2 = \sigma_2\sigma_3 = \sigma_3\sigma_1 = 1$ in the presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$. But these relations imply $\sigma := \sigma_1 = \sigma_2 = \sigma_3$ and $\sigma^2 = 1$ and since by the projective relation one has $\mu_1 \dots \mu_n \sigma_1 \sigma_2 \sigma_3 = 1$, it suffices to replace this latter relation by $(\mu_1 \dots \mu_n)^2 = 1$. Hence

$$\pi_1(\mathbb{P}^2 \setminus \psi(\mathcal{A})) \simeq \left\langle \mu_1, \dots, \mu_n \;\middle|\; \begin{array}{l} [\mu_i, \mu_1 \dots \mu_n] = 1 \quad (i \in [1, n]), \\ w_1 = \cdots = w_m = (\mu_1 \cdots \mu_n)^2 = 1 \end{array} \right\rangle.$$

Since σ is a central element of this group, this proves the following result.

Theorem 1.3. *For any arrangement of n lines $\mathcal{A}$, there is an arrangement of n smooth quadrics $\mathcal{B}$ with a central extension*

$$0 \to \mathbb{Z}/(2) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \to 0.$$

1.3. Quadric arrangements as coverings of line arrangements

Assume that $\mathcal{A}$ is a line arrangement, and let ϕ be the branched covering

$$\phi : [x : y : z] \in \mathbb{P}^2 \to [x^2 : y^2 : z^2] \in \mathbb{P}^2.$$

Suppose that the lines X, Y, Z, are respectively given by the equations $x = 0$, $y = 0$, and $z = 0$. If $\mathcal{A}$ is in general position to $X \cup Y \cup Z$, then $\phi^{-1}(\mathcal{A})$ is an arrangement of smooth quadrics. Above any singular point of $\mathcal{A}$ lie four singular points of $\phi^{-1}(\mathcal{A})$ of the same type. In this case, the group $\pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}))$ can easily be found in terms of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ as follows: Assuming $\mathcal{A} = \bigcup_{i=1}^n L_i$, one has a presentation (1). For the arrangement $\mathcal{A}' := \mathcal{A} \cup X \cup Y \cup Z$, the presentation (2) is valid. There is an exact sequence

$$0 \to \pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}')) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}') \to \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \to 0.$$

The group $\pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}))$ is the quotient $\pi_1(\mathbb{P}^2 \setminus \phi^{-1}(\mathcal{A}'))$ by the subgroup generated by the meridians of $\phi^{-1}(X)$, $\phi^{-1}(Y)$ and $\phi^{-1}(Z)$.

2. Statements of results

In this section we give in Theorems 2.2, 2.5, and 2.8 the presentations of the fundamental groups of the three quadric arrangements in $\mathbb{P}^2$. We prove them in the forthcoming sections. For our computations, we need the following definition.

Definition 2.1. A group G is said to be *big* if it contains a non-abelian free subgroup, and *small* if G is almost solvable.

2.1. The quadric arrangement $\mathcal{A}_n$

Let $\mathcal{A}_n := Q_1 \cup \cdots \cup Q_n$ be a quadric arrangement, which is a union of n quadrics tangent to each other at two common points, see figure 2.

Theorem 2.2. *The fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ of $\mathcal{A}_n$ in $\mathbb{P}^2$ admits the presentation*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n) \simeq \left\langle a_1, a_2, \ldots, a_n \;\middle|\; \begin{array}{l} (a_n \cdots a_2 a_1)^2 = (a_1 a_n \cdots a_2)^2 = \cdots = (a_{n-1} a_{n-2} \cdots a_n)^2 \\ a_n a_{n-1} \cdots a_2 a_1^2 a_2 \cdots a_{n-1} a_n = e \end{array} \right\rangle, \quad (3)$$

where $a_1, \ldots, a_n$ are meridians of $Q_1, \ldots, Q_n$, respectively.

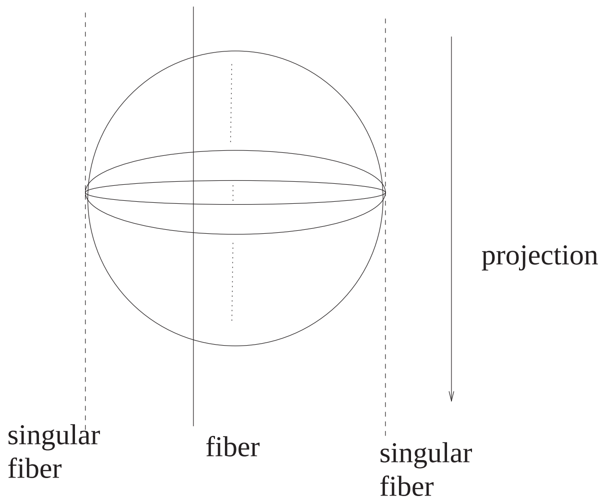

Figure 2: The arrangement $\mathcal{A}_n$

Proposition 2.3.

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_1) = \mathbb{Z}_2, \tag{4}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2) \simeq \left\langle a, b \;\middle|\; \begin{array}{l} (ab)^2 = (ba)^2 \\ b^2a^2 = 1 \end{array} \right\rangle \quad \text{is infinite and solvable,} \tag{5}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \simeq \left\langle a, b, c \;\middle|\; \begin{array}{l} (cba)^2 = (acb)^2 = (bac)^2 \\ cba^2bc = 1 \end{array} \right\rangle \quad \text{is big.} \tag{6}$$

Corollary 2.4. *The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ is big for $n \geq 3$.*

2.2. The quadric arrangement $\mathcal{B}_n$

Let $\mathcal{B}_n := Q_1 \cup \cdots \cup Q_n$ be a quadric arrangement, composed of n quadrics tangent to each other at one common point, see Figure 3.

Theorem 2.5. *The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ admits the presentation*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n) \simeq \left\langle a_1, a_2, \dots, a_n \;\middle|\; \begin{array}{l} (a_1 \cdots a_n)^4 = (a_n a_1 \cdots a_{n-1})^4 = \cdots = (a_2 \cdots a_n a_1)^4 \\ a_n^2 \cdots a_1^2 = 1 \end{array} \right\rangle, \tag{7}$$

where $a_1, \dots, a_n$ are meridians of $Q_1, \dots, Q_n$, respectively.

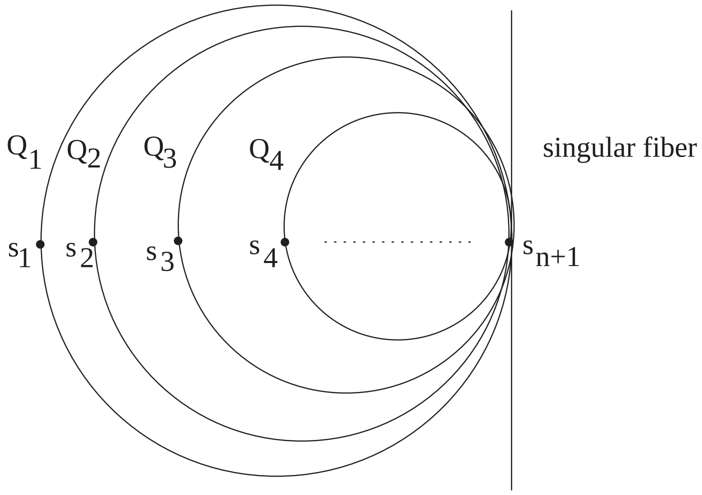

Figure 3: The arrangement $\mathcal{B}_n$

Proposition 2.6.

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_1) = \mathbb{Z}_2, \tag{8}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2) \simeq \left\langle a, b \;\middle|\; \begin{array}{l} (ab)^4 = (ba)^4 \\ b^2a^2 = 1 \end{array} \right\rangle \quad \textit{is infinite and solvable,} \tag{9}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3) \simeq \left\langle a, b, c \;\middle|\; \begin{array}{l} (cba)^4 = (acb)^4 = (bac)^4 \\ c^2b^2a^2 = 1 \end{array} \right\rangle \quad \textit{is big.} \tag{10}$$

Corollary 2.7. *The group* $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ *is big for* $n \geq 3$.

2.3. The quadric arrangement $\mathcal{C}_n$

Let $\mathcal{C}_n := Q_1 \cup \cdots \cup Q_n$ be a quadric arrangement, which is a union of n quadrics, $n-1$ of them are tangent to the nth one and each one of these $n-1$ quadrics is transversal to the other $n-2$, see figure 4.

Theorem 2.8. *The group* $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n)$ *admits the presentation*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n) \simeq \left\langle a_1, \ldots, a_n \;\middle|\; \begin{array}{ll} [a_i, a_j] = 1, & 2 \leq i, j \leq n, \quad i \neq j \\ (a_1a_k)^4 = (a_ka_1)^4, & 2 \leq k \leq n \\ a_n^2 \cdots a_1^2 = 1 & \end{array} \right\rangle, \tag{11}$$

where $a_1, \ldots, a_n$ *are meridians of* $Q_1, \ldots, Q_n$, *respectively.*

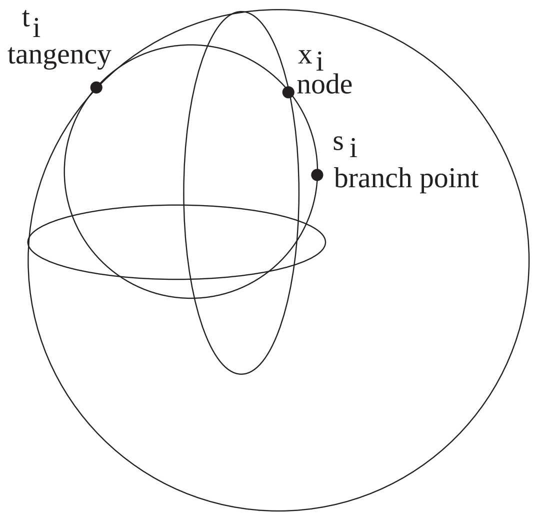

Figure 4: The arrangement $\mathcal{C}_n$

Proposition 2.9.

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1) = \mathbb{Z}_2, \tag{12}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_2) \simeq \left\langle a_1, a_2 \;\middle|\; \begin{array}{l} (a_1a_2)^4 = (a_2a_1)^4 \\ a_2^2a_1^2 = 1 \end{array} \right\rangle \quad \textit{is infinite and solvable,} \tag{13}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_3) \simeq \left\langle a_1, a_2, a_3 \;\middle|\; \begin{array}{l} [a_2, a_3] = 1 \\ (a_1a_2)^4 = (a_2a_1)^4 \\ (a_1a_3)^4 = (a_3a_1)^4 \\ a_3^2a_2^2a_1^2 = 1 \end{array} \right\rangle. \tag{14}$$

3. Proof of Theorem 2.2

Take the following affine quadric arrangement $\mathcal{A}_n$, which is composed of n quadrics tangent to each other at two points, see figure 2. Say that these points are $(1, 0)$ and $(-1, 0)$. Let $p_1 : \mathbb{C}^2 \setminus \mathcal{A}_n \to \mathbb{C}$ be the first projection. The base of this projection will be denoted by B. Identify the base B of the projection p_1 with the line $y = -2 \subset \mathbb{C}^2$. The projections of $(1, 0)$ and $(-1, 0)$ are $(1, -2)$ and $(-1, -2)$, respectively. Take $* := (M, -2)$ to be the base point. Put $F_x := p_1^{-1}(x)$, and denote by S the set of singular fibers of p_1. It is clear that if $F_x \in S$, then $x \in [-1, 1]$. Therefore, the only singular fibers are F_1 and F_{-1}, both corresponding to $(1, 0)$ and $(-1, 0)$.

In order to compute the fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$, we have to study first the

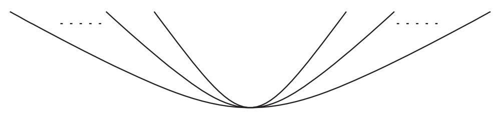

Figure 5: A tangency of n quadrics

local monodromy around the points $(1, 0)$ and $(-1, 0)$. This is done in the following proposition.

Proposition 3.1. *Let* 0 *be a point (figure 5) which is a tangency of* n *quadrics defined locally by*

$$(y - x^2)(y - 2x^2)(y - 3x^2) \cdots (y - nx^2).$$

Then the local monodromy around 0 *is a fulltwist* H^2 *of the* n *points (*H *is a counterclockwise halftwist).*

Proof. Take a loop $x = e^{2\pi it}$ in $y = 0$, starting (and ending) at the point $*$ and encircling the point 0, $0 \leq t \leq 1$. Take a typical fiber F_* next to the fiber F_0. We have n points on F_*, say $1, 2, \ldots, n$ (the intersection points of the arrangement with the fiber).

When $t = 0$, the points $1, 2, \ldots, n$ are still in their initial positions. When t is proceeding from 0 to 1, there is an induced motion of the points. The point 1 is rotating around the other points with a fulltwist $e^{2\pi i}$, the point 2 is rotating along a closed curve which bounds a disk, containing the trajectory of the point 1, and so on. This gives the required monodromy. □

Proof of Theorem 2.2. Let us denote the loops around the points $1, \ldots, n$ in Proposition 3.1 as $a_1, \ldots, a_n$, respectively. These loops get the forms $\tilde{a_1}, \ldots, \tilde{a_n}$, as depicted in figure 6.

By the Zariski Theorem [17],

$$\begin{aligned}
\tilde{a_1} &= a_n \cdots a_2 a_1 a_n \cdots a_2 a_1 {a_2}^{-1} \cdots {a_n}^{-1} {a_1}^{-1} {a_2}^{-1} \cdots {a_n}^{-1} \\
\tilde{a_2} &= a_n \cdots a_2 a_1 a_n \cdots a_2 a_1 a_2 {a_1}^{-1} {a_2}^{-1} \cdots {a_n}^{-1} {a_1}^{-1} {a_2}^{-1} \cdots {a_n}^{-1} \\
&\vdots \\
\tilde{a_n} &= a_n \cdots a_1 a_n \cdots a_1 a_n {a_1}^{-1} \cdots {a_n}^{-1} {a_1}^{-1} \cdots {a_n}^{-1}.
\end{aligned}$$

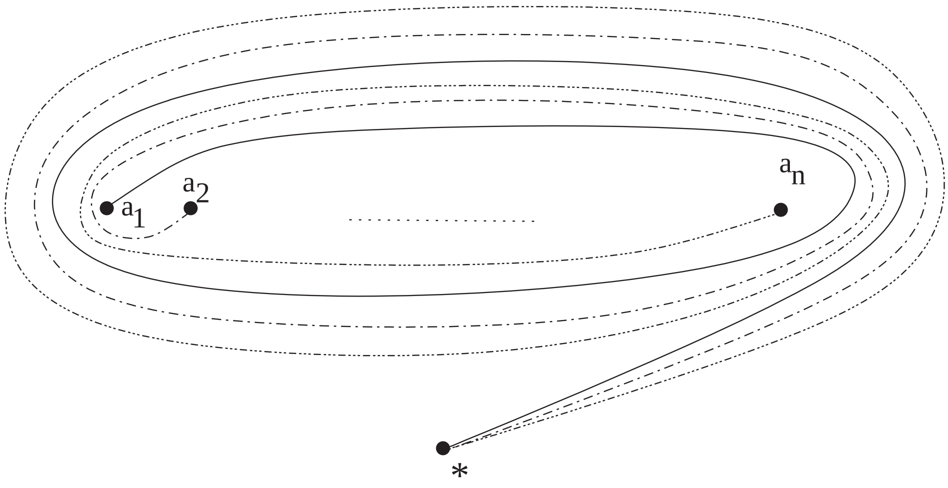

Figure 6: The resulting loops

These relations get the forms

$$\begin{aligned}(a_n \cdots a_1)^2 &= (a_1 a_n \cdots a_2)^2 \\ (a_n \cdots a_1)^2 &= (a_2 a_1 a_n \cdots a_3)^2 \\ &\vdots \\ (a_n \cdots a_1)^2 &= (a_{n-1} a_{n-2} \cdots a_1 a_n)^2.\end{aligned}$$

Since the points $(1,0)$ and $(-1,0)$ are intersections of branch points, we have

$$a_i = a_i' \quad \text{for } 1 = 1, \dots, n.$$

The projective relation

$$a_1' a_2' \cdots a_n' a_n \cdots a_2 a_1 = 1$$

gets the form

$$a_1 a_2 \cdots {a_n}^2 \cdots a_2 a_1 = 1.$$

Therefore, the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ admits (3). □

Proof of Proposition 2.3. It is easy to see that the arrangement $\mathcal{A}_1$ consists of a smooth quadric, and the group is $\mathbb{Z}_2$, see (4).

For the case $n = 2$, we depict figure 7. We substitute $n = 2$ in (3) and get the presentation (5). Now we prove that $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ is an infinite solvable group. Apply

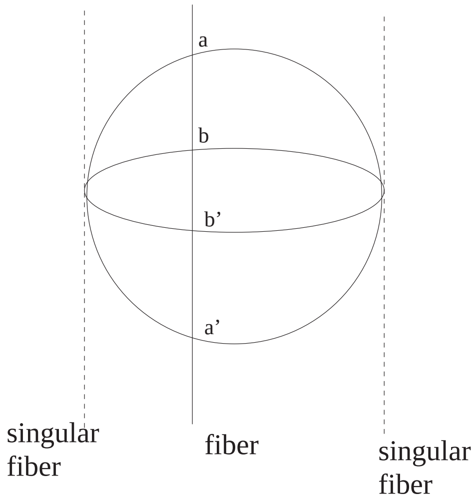

Figure 7: The arrangement A_2

a change of generators in (5): $\alpha := ba$, $\beta := b$, and get

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2) \simeq \left\langle \alpha, \beta \;\middle|\; \begin{array}{l} \beta\alpha^2\beta^{-1} = \alpha^2 \\ \alpha^{-1}\beta = \beta\alpha \end{array} \right\rangle.$$

The first relation gets the form (by the second one) $\alpha^{-1}\beta\alpha\beta^{-1} = \alpha^2$. This yields $\beta\alpha\beta^{-1} = \alpha^3$. We repeat this procedure and obtain $\alpha^4 = 1$. Therefore the subgroup generated by α is a finite one. Since $\beta\alpha\beta^{-1} = \alpha^3$, $\beta\alpha^3\beta^{-1} = \alpha$, and $\beta\alpha^2\beta^{-1} = \alpha^2$, this subgroup is normal. Sending $\alpha \to 1$, the image is generated by β and no relations remain. Thus $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ is an extension of $\mathbb{Z}_4$ by $\mathbb{Z}$, and in particular an infinite solvable one.

Now, when we substitute $n = 3$ in (3) we get (6). We prove that $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ is big. We apply a change of generators in (6), $\alpha := cba$, $\beta := cb$, $\gamma := b$, which gives

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \simeq \left\langle \alpha, \beta, \gamma \;\middle|\; \begin{array}{l} [\alpha^2, \beta] = [\alpha^2, \gamma] = 1 \\ \gamma\beta\gamma^{-1} = \alpha^{-1}\beta\alpha^{-1} \end{array} \right\rangle.$$

There is a surjection of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ onto its quotient, by adding $\alpha^2 = 1$

$$\left\langle \alpha, \beta, \gamma \;\middle|\; \begin{array}{l} \alpha^2 = 1 \\ \alpha\gamma\beta = \beta\alpha\gamma \end{array} \right\rangle. \tag{15}$$

An isomorphism $\alpha, \beta, \gamma \to \alpha, \beta, \alpha^{-1}\gamma$, respectively, gives

$$\left\langle \alpha, \beta, \gamma \;\middle|\; \begin{array}{l} \alpha^2 = 1 \\ \gamma\beta = \beta\gamma \end{array} \right\rangle.$$

Mapping $\beta \to 1$ and fixing $\gamma^3 = 1$, we get $\mathbb{Z}_2 * \mathbb{Z}_3$. This group has a free subgroup [6], therefore $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ is big. □

Proof of Corollary 2.4. $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ is a quotient group of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ ($n \geq 3$), and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ is big, therefore $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ is a big group. □

4. Proof of Theorem 2.5

The arrangement $\mathcal{B}_n$ is defined by

$$(x^2 + y^2 - 1)\Big(\Big(x - \frac{1}{2}\Big)^2 + y^2 - \frac{1}{4}\Big)\Big(\Big(x - \frac{3}{4}\Big)^2 + y^2 - \frac{1}{16}\Big) \cdots .$$

The quadrics $Q_1, \ldots, Q_n$ are tangent to each other at one common tangency $(1, 0)$, see figure 3. As in the case of the arrangement $\mathcal{A}_n$, it is readily seen that arrangements of type $\mathcal{B}_n$ are all isotopic to each other for fixed n.

The projection to the line $y = -2$ has two types of singular fibers:

(i) the fibers F_{-1}, F_0, $F_{\frac{1}{2}}$, ...corresponding to branch points of the n quadrics,

(ii) the fiber F_1 corresponding to the tangency $(1, 0)$.

In order to find the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$, we shall apply the same procedure as in section 3. Let $*$ be the base point and F_* a typical fiber. Denote the branch points of the quadrics as $s_1, \ldots, s_n$ and $(1, 0)$ as s_{n+1}. Say that the typical fiber intersects Q_1 at a_1, a_1', Q_2 at a_2, a_2', and so on.

First we study the local monodromy around the point s_{n+1}.

Proposition 4.1. *Define the unique tangency of n quadrics (e.g. figure 5) locally by*

$$(y - x^4)(y - 2x^4)(y - 3x^4) \cdots (y - nx^4).$$

Let $a_1, \ldots, a_n$ be the intersection points of the quadrics with a typical fiber. Then the local monodromy around this tangency is a double fulltwist H^4 of the points $a_1, \ldots, a_n$.

Proof. Take a loop $x = e^{2\pi i t}$ in $y = 0$, starting (and ending) at the point $*$ and encircling the point 0, $0 \leq t \leq 1$.

When $t = \frac{1}{2}$, the resulting motion of the points is: the point a_1 is turning around all the points in a twist of $e^{4\pi i}$, the point a_2 is turning around in a bigger twist of $2e^{4\pi i}$, and so on. □

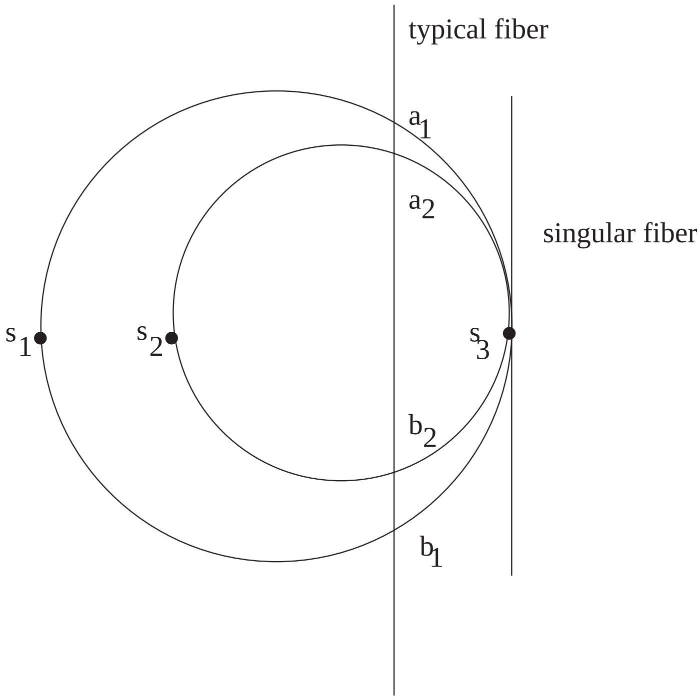

Figure 8: The arrangement $\mathcal{B}_2$

Proof of Theorem 2.5. We compute the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$. Take a g-base $a_1, \dots, a_n$ for $F_* \setminus \{a_1, \dots, a_n\}$.

By Proposition 4.1, for $t = 1$ the motion of the n points is a $e^{8\pi i}$ twist. The branch points $s_1, \dots, s_n$ contribute $a_i = a'_i$ for $i = 1, \dots, n$, and together with the projective relation, lengthy computations yield the presentation (7). □

Proof of Proposition 2.6. It is easy to see that the arrangement $\mathcal{B}_1$ consists of a smooth quadric, and the group is $\mathbb{Z}_2$, see (8).

The arrangement $\mathcal{B}_2$ is shown in figure 8. Substituting $n = 2$ in (7) yields (9).

Now we show another way to find $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$. We understand first the local monodromy around the point $(1, 0)$. A tangency of two quadrics is homotopic equivalent to a tangency of a line with a quadric. Therefore, we can use figure 9 as a local model defined by $y(y - x^4) = 0$.

Lemma 4.2. *The local monodromy around the point* 0 *is a twist* H^4 *of* b *around* a.

Proof. Since we explained already the construction of a g-base for a fundamental group in [3], we can directly compute the local monodromy around the point 0. Take

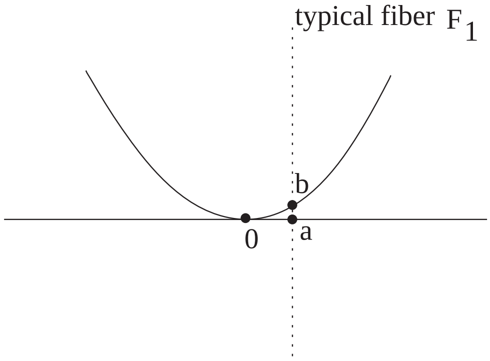

Figure 9: The local model

a loop $x = e^{2\pi it}$ in $y = 0$, starting (and ending) at the point $*$ and encircling the point 0, $0 \leq t \leq 1$. Take, for example, the fiber F_1 in figure 9. The points on this fiber are $a = (1,0)$ and $b = (1,1)$. They are encircled by the loops (liftings of $x = e^{2\pi it}$) a and b of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$. It is easy to see that when $t = \frac{1}{2}$, the induced motion is a double fulltwist of b around a. □

When $t = 1$, the point $y = 0$ is left in its place, while the second point is now $y = e^{8\pi i}$. That means that the second point was encircling 0 in four counterclockwise fulltwists. The resulting loops $\tilde{a}$ and $\tilde{b}$ are shown in figure 10 and formulated as follows:

$$\tilde{a} = babababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$$
$$\tilde{b} = bababababa^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}.$$

These relations get the form

$$(ab)^4 = (ba)^4.$$

s_1 and s_2 are branch points (figure 8), therefore $a_1 = b_1$ and $a_2 = b_2$. The projective relation

$$b_1 b_2 a_2 a_1 = 1$$

is transformed to

$$b^2 a^2 = 1.$$

Therefore we obtain (9).

We prove that $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$ is an infinite solvable group. Apply a change of generators in (9), $\alpha := ba$, $\beta := b$, and get

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2) \simeq \left\langle \alpha, \beta \;\middle|\; \begin{array}{l} \beta\alpha^4\beta^{-1} = \alpha^4 \\ \beta\alpha\beta^{-1} = \alpha^{-1} \end{array} \right\rangle.$$

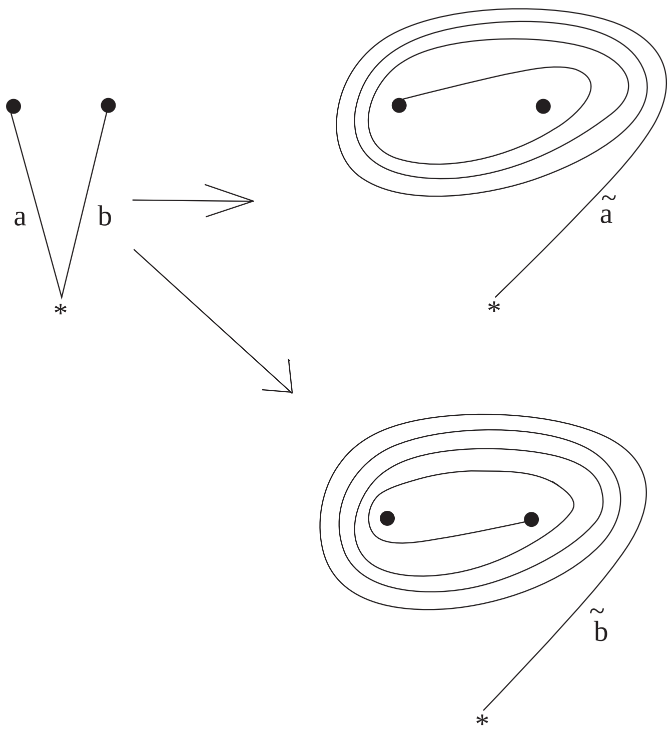

Figure 10: Resulting generators $\tilde{a}, \tilde{b}$ in $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$

The first relation is transformed by the second relation to $\alpha^{-1}\beta\alpha^3\beta^{-1} = \alpha^4$. This yields $\beta\alpha^3\beta^{-1} = \alpha^5$. We repeat this procedure and obtain $\alpha^8 = 1$. Therefore the subgroup generated by α is a finite one. One can prove that it is also normal. Mapping $\alpha \to 1$, the image is generated by β and no relations remain. Thus $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$ is an extension of $\mathbb{Z}_8$ by $\mathbb{Z}$, and in particular infinite and solvable.

Now, substituting $n = 3$ in (7), we obtain (10). We prove that it is a big group. By a change of generators $\kappa := cba$, $\tau := cb$, $m := c$ in (10), we obtain

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3) \simeq \left\langle \kappa, \tau, m \;\middle|\; \begin{array}{l} [\kappa^4, \tau] = [\kappa^4, m] = 1 \\ \kappa m \tau = \tau \kappa^{-1} m \end{array} \right\rangle.$$

Adding the relation $\kappa^2 = 1$, we have a surjection of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ onto

$$\left\langle \kappa, \tau, m \;\middle|\; \begin{array}{l} [\kappa m, \tau] = 1 \\ \kappa^2 = 1 \end{array} \right\rangle,$$

which is equal to (15). At this point, we have to repeat the proof which we did for $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$, and then we get $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ is also big. □

Proof of Corollary 2.7. The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ is a quotient of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ $(n \geq 3)$, and $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_3)$ is big, therefore $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ is big. □

5. Proof of Theorem 2.8

Let $\mathcal{C}_n$ be a quadric arrangement, composed of n quadrics as shown in Figure 4. Each one of the quadrics $Q_2, \ldots, Q_n$ is tangent to the quadric Q_1 in one tangency and intersects each one of the other quadrics at two points.

The projection to the line $y = -2$ has three types of singular fibers:

(i) the fibers F_{s_i}, where s_i is a branch point of a quadric Q_i, $1 \leq i \leq n$,

(ii) the fibers F_{x_i} corresponding to the nodes,

(iii) the fibers F_{t_i} corresponding to the $n-1$ tangencies.

In order to find the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n)$, take $* \in \mathbb{R}$ a base point and F_* a typical fiber. Next to each branch point one can find a fiber which intersects the same quadric in two real points. Denote this pair of points as a_i, b_i for $1 \leq i \leq n$.

Proof of Theorem 2.8. Since we are familiar with the types of the singularities, we can compute easily the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n)$. The nodes and tangencies give $[a_i, a_j] = 1$ for $i \neq j$, $2 \leq i, j \leq n$, and $(a_1 a_k)^4 = (a_k a_1)^4$ for $2 \leq k \leq n$. The branch points $s_1, \ldots, s_n$ give $a_i = b_i$. The projective relation is ${a_n}^2 \cdots {a_1}^2 = 1$. Therefore we get 2.8. □

Proof of Proposition 2.9. The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1)$ is again $\mathbb{Z}_2$, see (12).

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_2)$ admits the presentation (13), hence is isomorphic to the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$. Therefore it is an infinite solvable group.

The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_3)$ admits (14). We apply a change of generators in (14), $\kappa := a_2a_1$, $\tau := a_2$, $m := a_3$, which yields a better look at the group:

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_3) \simeq \left\langle \kappa, \tau, m \;\middle|\; \begin{array}{l} [m,\tau] = 1 \\ [\kappa^4, \tau] = 1 \\ (\tau^{-1}\kappa m)^4 = (m\tau^{-1}\kappa)^4 \\ \tau\kappa\tau^{-1}\kappa m^2 = 1 \end{array} \right\rangle. \qquad \square$$

Acknowledgements. The first author thanks the Einstein Institute, Jerusalem, and especially Professor Ruth Lawrence-Neumark for her hospitality. This work began while the first author was at the Mathematics Institute, Erlangen-Nürnberg University, Germany, and the assistance of Professor Wolf Barth is gratefully acknowledged.

The authors are grateful to Muhammed Uludağ for his ideas and suggestions concerning the types of the arrangements studied in this paper.

References

[1] M. Amram, *Braid group and braid monodromy*, M. Sc. Thesis, Bar-Ilan University, 1995.

[2] M. Amram and M. Teicher, *Braid monodromy of special curves*, J. Knot Theory Ramifications **10** (2001), no. 2, 171–212.

[3] M. Amram, M. Teicher, and A. M. Uludag, *Fundamental groups of some quadric-line arrangements*, Topology Appl. **130** (2003), no. 2, 159–173.

[4] M. Amram, D. Garber, and M. Teicher, *Fundamental groups of tangented conic-line arrangements with singularities up to order* 6, Math. Z., to appear.

[5] D. Chéniot, *Topologie du complémentaire d'un ensemble algébrique projectif*, Enseign. Math. (2) **37** (1991), no. 3-4, 293–402.

[6] A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer-Verlag, New York, 1992.

[7] D. Garber and M. Teicher, *The fundamental group's structure of the complement of some configurations of real line arrangements*, Complex analysis and algebraic geometry, A volume in memory of Michael Schneider, de Gruyter, Berlin, 2000, pp. 173–223.

[8] R.-P. Holzapfel and N. Vladov, *Quadric-line configurations degenerating plane Picard Einstein metrics. I, II*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, Berliner Math. Gesellschaft, Berlin, 2001, pp. 79–141.

[9] K. Lamotke, *The topology of complex projective varieties after S. Lefschetz*, Topology **20** (1981), no. 1, 15–51.

[10] P. Orlik and L. Solomon, *Arrangements defined by unitary reflection groups*, Math. Ann. **261** (1982), no. 3, 339–357.

[11] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, vol. 300, Springer-Verlag, Berlin, 1992.

[12] A. I. Suciu, *Fundamental groups of line arrangements: enumerative aspects*, Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math., vol. 276, Amer. Math. Soc., Providence, RI, 2001, pp. 43–79.

[13] A. M. Uludağ, *On finite smooth uniformizations of the plane along line arrangements*, in preparation.

[14] ______, *Covering relations between ball-quotient orbifolds*, Math. Ann. **328** (2004), no. 3, 503–523.

[15] E. R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), no. 1-4, 255–267.

[16] J. Kaneko, S. Tokunaga, and M. Yoshida, *Complex crystallographic groups. II*, J. Math. Soc. Japan **34** (1982), no. 4, 595–605.

[17] O. Zariski, *On the Poincaré group of rational plane curves*, Amer. J. Math. **58** (1936), no. 3, 607–619.