The Homotopy Type of the Space of Degree 0 — Immersed Plane Curves

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ABSTRACT

The space $B_i^0 = \operatorname{Imm}^0(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ of all immersions of rotation degree 0 in the plane modulo reparameterizations has homotopy groups $\pi_1(B_i^0) = \mathbb{Z}$, $\pi_2(B_i^0) = \mathbb{Z}$, and $\pi_k(B_i^0) = 0$ for $k \geq 3$.

Key words: homotopy groups, immersed plane curves.

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Introduction

For an immersion $c : S^1 \to \mathbb{R}^2$ the (rotation) degree is the winding number of $c' : S^1 \to \mathbb{R}^2$ around 0. Let $\operatorname{Imm}^k(S^1,\mathbb{R}^2)$ denote the connected smooth Fréchet manifold of all immersions of degree k. It was shown in [4] that for $k \neq 0$ the space $\operatorname{Imm}^k(S^1,\mathbb{R}^2)$ contains a copy of S^1 as a smooth strong deformation retract and that the infinite dimensional orbifold $\operatorname{Imm}^k(S^1,\mathbb{R}^2)/\operatorname{Diff}^+(S^1)$ is contractible, where $\operatorname{Diff}^+(S^1)$ denotes the regular Fréchet Lie group of orientation preserving diffeomorphisms. The proof in [4] consists in expanding the classical proof of the theorem of Whitney and Graustein (see [1, 2, 8]) into the construction of an S^1-equivariant smooth deformation retraction. For $k = 0$ this did not work.

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In this paper we treat the case $k = 0$. In section 1 we first give simple argument which shows that $\pi_1(B^0(S^1,\mathbb{R}^2))$ contains $\mathbb{Z}$. In section 2 we give a more involved proof that $\mathrm{Imm}^0(S^1,\mathbb{R}^2)$ is homotopy equivalent to S^1. In section 3 we show that factoring out $\mathrm{Diff}^+(S^1)$ gives a fibration with homotopically trivially embedded fiber, and then the homotopy sequence shows that $\pi_1(B^{0,+}(S^1,\mathbb{R}^2)) = \mathbb{Z}$, $\pi_2(B^{0,+}(S^1,\mathbb{R}^2)) = \mathbb{Z}$, and $\pi_k(B^{0,+}(S^1,\mathbb{R}^2)) = 0$ for $k > 2$. Factoring out the larger group $\mathrm{Diff}(S^1)$ gives a two-sheeted covering and the final result.

1. A simple proof that $\mathbb{Z} \subseteq \pi_1(B^0(S^1,\mathbb{R}^2))$

Proposition 1.1. $\mathrm{Imm}^0(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ *is not contractible.*

Proof. We shall view a curve $c \in \mathrm{Imm}(S^1,\mathbb{R}^2)$ as a 2π-periodic plane valued function. A smooth function $a = a(c,\quad) : \mathbb{R} \to \mathbb{R}$ is called an argument of a curve c if

$$\frac{c'(\theta)}{|c'(\theta)|} = \exp(i\,a(\theta));$$

it is unique up to addition of an integer multiple of 2π. If the curve c has degree k then $a(\theta + 2\pi) - a(\theta) = 2k\pi$. Thus, a curve c is in $\mathrm{Imm}^0(S^1,\mathbb{R}^2)$ if and only if some (any) argument of c is 2π-periodic. For a curve $c \in \mathrm{Imm}^0(S^1,\mathbb{R}^2)$, we define the *average argument* $\alpha(c) \in S^1$ by

$$\alpha(c) = \exp\biggl(\frac{i}{l(c)}\int_0^{2\pi} a(c,\theta)|c'(\theta)|\,d\theta\biggr),$$

which does not depend on the choice of $a(c,\quad)$ and defines a well-defined smooth mapping $\alpha : \mathrm{Imm}^0(S^1,\mathbb{R}^2) \to S^1$. Also, since any argument a of a degree 0 curve is 2π-periodic, $\alpha(c)$ is invariant under the action of $\mathrm{Diff}^+(S^1)$. So we can view α as a map

$$\alpha : B^{0,+}(S^1,\mathbb{R}^2) = \mathrm{Imm}^0(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1) \to S^1.$$

For $\varphi \in S^1 \subset \mathbb{C} = \mathbb{R}^2$, the rotation map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ act on $B^{0,+}(S^1,\mathbb{R}^2)$ and obviously

$$\alpha(\varphi \cdot c) = \varphi \cdot \alpha(c).$$

So choosing a free orbit $S^1.C$ for the rotation action of S^1 on $B^{0,+}(S^1,\mathbb{R}^2)$, the composition

$$S^1 \cdot C \hookrightarrow \mathrm{Imm}^0(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1) \xrightarrow{\alpha} S^1$$

equals the identity on S^1, thus $\pi_1(S^1) = \mathbb{Z} \subset \pi_1(B^{0,+}(S^1,\mathbb{R}^2))$.

Moreover, $\alpha(c(-\quad)) = -\alpha(c)$ implies that α factors as follows, where the vertical arrows are 2-sheeted coverings:

$$\begin{array}{ccccc} B^{0,+}(S^1,\mathbb{R}^2) & = & \operatorname{Imm}^0(S^1,\mathbb{R}^2)/\operatorname{Diff}^+(S^1) & \xrightarrow{\alpha} & S^1 \\ & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\ B^0(S^1,\mathbb{R}^2) & = & \operatorname{Imm}^0(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1) & \xrightarrow{\bar\alpha} & S^1 \end{array}$$

Thus we also get in a similar way $\pi_0(S^1) = \mathbb{Z} \subset \pi_0(B^0(S^1,\mathbb{R}^2))$. □

2. The homotopy type of $\operatorname{Imm}^0(S^1,\mathbb{R}^2)$

Proposition 2.1. *The space* $\operatorname{Imm}^0(S^1,\mathbb{R}^2)$ *of degree* 0 *immersions in the plane is homotopy equivalent to* S^1.

Proof. This will follow from 2.2–2.5 below. □

2.2. Let $\operatorname{Imm}^{0,*}(S^1,\mathbb{R}^2) := \{ c \in \operatorname{Imm}^0(S^1,\mathbb{R}^2) : c(0) = 0 \}$. Clearly we have $\operatorname{Imm}^0(S^1,\mathbb{R}^2) \cong \operatorname{Imm}^{0,*}(S^1,\mathbb{R}^2) \times \mathbb{R}^2$ and $\operatorname{Imm}^0(S^1,\mathbb{R}^2) \sim \operatorname{Imm}^{0,*}(S^1,\mathbb{R}^2)$, where $\cong$ denotes homeomorphic and $\sim$ homotopy equivalent. Let us define a map

$$\Phi\colon \operatorname{Imm}^{0,*} \to C^\infty(S^1,\mathbb{R}_+) \times C^\infty(S^1,S^1)$$
$$\Phi(c)(\theta) = \left(|c_\theta(\theta)|, \frac{c_\theta(\theta)}{|c_\theta(\theta)|} \right) =: (v(\theta), e(\theta)).$$

The map Φ is injective. For $(v,e) = \Phi(c)$, the winding number of e equals the degree 0 of c and thus $\int_0^{2\pi} v \cdot e \, d\theta = 0$.

Lemma. *The length of the image of e is greater than π.*

Proof. If not, there exists a number $r \in \mathbb{R}$ such that

$$\exp(ir) \in \operatorname{Im}(e) \subset \exp(i[r - \pi/2, r + \pi/2]).$$

Then, $\langle \exp(ir), e(\theta) \rangle$ is nonnegative for any θ and strictly positive for some θ. Therefore $\int_0^{2\pi} \langle \exp(ir), v \cdot e \rangle \, d\theta > 0$. This contradicts

$$\int_0^{2\pi} \langle \exp(ir), v \cdot e \rangle \, d\theta = \left\langle \exp(ir), \int_0^{2\pi} v \cdot e \, d\theta \right\rangle = \langle \exp(ir), 0 \rangle = 0. \qquad \square$$

2.3. Let us define the set

$$C^{\infty,0}_{>\pi}(S^1,S^1) = \{ e \in C^\infty(S^1,S^1) : \deg(e) = 0, \operatorname{length}(\operatorname{Im}(e)) > \pi \}$$

and consider the map

$$\operatorname{pr}_2 \circ \Phi\colon \operatorname{Imm}^{0,*}(S^1,S^1) \to C^{\infty,0}_{>\pi}(S^1,S^1),$$

where pr_2 denotes the second projection.

Lemma. *The map* $\mathrm{pr}_2 \circ \Phi \colon \mathrm{Imm}^{0,*}(S^1, S^1) \to C^{\infty,0}_{>\pi}(S^1, S^1)$, *is surjective, has contractible fibers, admits a global smooth section, and is a homotopy equivalence.*

Proof. For a map $e \in C^{\infty,0}_{>\pi}(S^1, S^1)$, there exist points θ_1, θ_2, θ_3 such that $0 \in \mathrm{int}([e(\theta_1), e(\theta_2), e(\theta_3)])$, where $[\,\cdot,\cdot,\cdot\,]$ denotes the convex hull of three points. Let $v_1 \in C^\infty(S^1, \mathbb{R}_{>0})$ be a map such that $\int_0^{2\pi} v_1\, d\theta = 1$ and $v_1(\theta)$ is close to 0 if θ is not close to θ_1. Then $\int_0^{2\pi} v_1 \cdot e\, d\theta$ is close to $e(\theta_1)$. We also define v_2 and v_3 similarly, so that

$$0 \in \mathrm{int}\Big(\Big[\int_0^{2\pi} v_1 \cdot e\, d\theta, \int_0^{2\pi} v_2 \cdot e\, d\theta, \int_0^{2\pi} v_3 \cdot e\, d\theta\Big]\Big).$$

Therefore there exist positive numbers a_1, a_2, a_3 with

$$a_1 \int_0^{2\pi} v_1 \cdot e\, d\theta + a_2 \int_0^{2\pi} v_2 \cdot e\, d\theta + a_3 \int_0^{2\pi} v_3 \cdot e\, d\theta = 0.$$

Define c by

$$c(\theta) = \int_0^\theta (a_1 v_1(u) + a_2 v_2(u) + a_3 v_3(u)) e(u)\, du.$$

Then c is in $\mathrm{Imm}^{0,*}$ and $(\mathrm{pr}_2 \circ \Phi)(c) = e$, which means that $\mathrm{pr}_2 \circ \Phi$ is surjective.

We next show that for any $e \in C^{\infty,0}_{>\pi}(S^1, S^1)$, the inverse image $(\mathrm{pr}_2 \circ \Phi)^{-1}(e)$ is contractible. Namely, let $V(e) \subset C^\infty(S^1, \mathbb{R}_+)$ be given by

$$V(e) = \Big\{ v \in C^\infty(S^1, \mathbb{R}_+) \;:\; \int_0^{2\pi} v \cdot e\, d\theta = 0 \Big\},$$

an open convex subset of the linear subspace $\{\, v \in C^\infty(S^1, \mathbb{R}) : \int_0^{2\pi} v \cdot e\, d\theta = 0 \,\} \subset C^\infty(S^1, \mathbb{R})$. Thus $V(e)$ is contractible for each e. Moreover, $V(e)$ is homeomorphic to $(\mathrm{pr}_2 \circ \Phi)^{-1}(e)$ by the map $\mathrm{pr}_1 \circ \Phi \colon (p_2 \circ \Phi)^{-1}(e) \to V(e)$.

For fixed $\theta_1, \theta_2, \theta_3$ the construction above works for each $e \in C^{\infty,0}_{>\pi}(S^1, S^1)$ for which 0 is contained in the interior of the convex hull of $e(\theta_1), e(\theta_2), e(\theta_3)$; these e form an open set in $C^{\infty,0}_{>\pi}(S^1, S^1)$ on which we get a continuous (even smooth) section of $p_2 \circ \Phi$. Open sets like that cover $C^{\infty,0}_{>\pi}(S^1, S^1)$. So we get smooth local sections whose domains cover the base. Since the base is open in a nuclear Fréchet space, it is smoothly paracompact (see [3, 16.10]) we can use convexity of all fibers and a smooth partition of unity on the base $C^{\infty,0}_{>\pi}(S^1, S^1)$ to construct a global smooth section s.

Finally, since all fibers are convex, there is a smooth strong fiber preserving deformation retraction of $\mathrm{Imm}^{0,*}(S^1, S^1)$ onto the image the global section s. □

2.4. To study the topology of $C^{\infty,0}_{>\pi}(S^1, S^1)$, we introduce the set of 2π-periodic functions

$$C^{\infty,p}(\mathbb{R}, \mathbb{R}) = \{\, c \in C^\infty(\mathbb{R}, \mathbb{R}) : c(\theta + 2\pi) = c(\theta) \,\}.$$

For $c \in C^{\infty,p}(\mathbb{R},\mathbb{R})$ let $\operatorname{Var}(c) = \max c - \min c$ and let $\operatorname{Ave}(c) = \frac{1}{2\pi}\int_0^{2\pi} c\,d\theta$. For $k \geq 0$, $C^{\infty,p}_{>k}(\mathbb{R},\mathbb{R}) = \{\, c \in C^{\infty,p}(\mathbb{R},\mathbb{R}) : \operatorname{Var}(c) > k \,\}$. Define a diffeomorphism $g : C^{\infty,p}_{>0}(\mathbb{R},\mathbb{R}) \to C^{\infty,p}_{>\pi}(\mathbb{R},\mathbb{R})$ by

$$g(c) = \frac{\operatorname{Var}(c) + \pi}{\operatorname{Var}(c)}(c - \operatorname{Ave}(c)) + \operatorname{Ave}(c).$$

The diffeomorphism g satisfies $g(c(\quad +2n\pi)) = g(c)(\quad +2n\pi)$, thus induces the diffeomorphism

$$\tilde{g} : C^{\infty,0}_{>0}(S^1,S^1) \to C^{\infty,0}_{>\pi}(S^1,S^1),$$

where $C^{\infty,0}_{>0}(S^1,S^1)$ denotes the set of nonconstant smooth maps of degree 0 in $C^\infty(S^1,S^1)$.

2.5. We consider now the evaluation ev_1 at $1 \in S^1$ whose fiber at $1 \in S^1$ is the smooth manifold of based smooth loops of degree 0 in S^1, with the constant loop 1 deleted:

$$C^{\infty,0}((S^1,1),(S^1,1)) \setminus \{1\} \hookrightarrow C^{\infty,0}_{>0}(S^1,S^1) \xrightarrow{\operatorname{ev}_1} S^1$$

Lemma. *The map $\operatorname{ev}_1 : C^{\infty,0}_{>0}(S^1,S^1) \to S^1$ is a smooth trivial fibration with a global section and smoothly contractible fibers. Moreover, it is a homotopy equivalence.*

Proof. A smooth section $s : S^1 \to C^{\infty,0}_{>0}(S^1,S^1)$ of ev_1 is given by $s(\varphi)(\theta) = \varphi \cdot \exp(i\operatorname{Im}(\theta))$. The fiber of ev_1 over φ is the space $C^{\infty,0}((S^1,1),(S^1,\varphi))\setminus\{\varphi\}$ consisting of all non-constant smooth loops of degree 0 mapping 1 to φ, which is diffeomorphic to the fiber $C^{\infty,0}((S^1,1),(S^1,1)) \setminus \{1\}$ via multiplication by φ.

It remains to show that the fiber $C^{\infty,0}((S^1,1),(S^1,1)) \setminus \{1\}$ is contractible. Via lifting to the universal cover, $C^{\infty,0}((S^1,1),(S^1,1))$ is diffeomorphic to the space $\{\, f \in C^{\infty,p}(\mathbb{R},\mathbb{R}) : f(0) = 0 \,\}$ of periodic functions mapping 0 to 0. Via Fourier expansion $f(t) = \sum_{n\in\mathbb{Z}} a_n \exp(int)$ this is isomorphic to the space of all rapidly decreasing complex sequences $(a_k)_{k\in\mathbb{Z}}$ with $\overline{a_k} = a_{-k}$ and $\sum_k a_k = 0$. This space is isomorphic to the space $\mathfrak{s}$ of rapidly decreasing sequences $(b_n)_{n\geq 1}$ by $a_n = b_n$ for $n \geq 1$, $a_{-n} = \overline{b}_n$, and $a_0 = 2\operatorname{Re}(\sum_{n\geq 1} b_n)$.

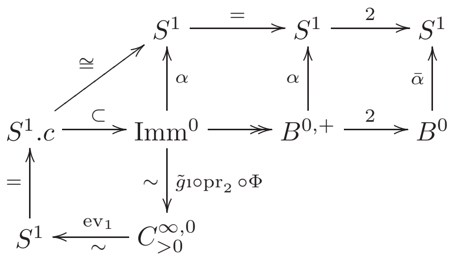
Now we have to show that this is still contractible if we remove the constant sequence 0. Then it is homotopy equivalent to its intersection with the sphere in ℓ^2, i.e., to the space $S := \{\, b \in \mathfrak{s} : \sum_{n\geq 1} b_n^2 = 1 \,\}$. But this is contractible by a standard argument which is explained on page 513 of [3] for the space of finite sequences. Namely, consider the homotopy $A : \mathfrak{s} \times [0,1] \to \mathfrak{s}$ through isometries which is given by $A_0 = \operatorname{Id}$ and by

$$\begin{aligned} A_t(b_1,b_2,b_3,\dots) = (&b_1,\dots,b_{n-2}, b_{n-1}\cos\theta_n(t), b_{n-1}\sin\theta_n(t),\\ &b_n\cos\theta_n(t), b_n\sin\theta_n(t), b_{n+1}\cos\theta_n(t), b_{n+1}\sin\theta_n(t),\dots) \end{aligned}$$

for $\frac{1}{n+1} \leq t \leq \frac{1}{n}$, where $\theta_n(t) = \varphi(n((n+1)t-1))\frac{\pi}{2}$ for a fixed smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ which is 0 on $(-\infty,0]$, grows monotonely to 1 in $[0,1]$, and equals 1 on

$[1, \infty)$. The mapping A is Lipschitz continuous for each seminorm $\|b\|_k = \sup\{|b_n|n^k : n \geq 1\}$ of $\mathfrak{s}$ with constant 2^k, and is isometric for ℓ^2. Then $A_{1/2}(b_1, b_2, \dots) = (b_1, 0, b_2, 0, \dots)$ is in $\mathfrak{s}_{\text{odd}}$, and on the other hand $A_1(b_1, b_2, \dots) = (0, b_1, 0, b_2, 0, \dots)$ is in $\mathfrak{s}_{\text{even}}$. This is a variant of a homotopy constructed by [6]. Now $A_t|S$ for $0 \leq t \leq 1/2$ is a homotopy on S between the identity and $A_{1/2}(S) \subset \mathfrak{s}_{\text{odd}}$. The latter set is contractible, for example in a stereographic chart. □

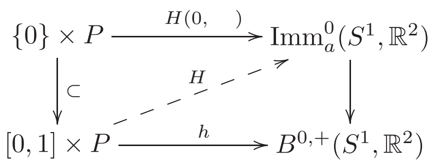
2.6. If we put together all mappings constructed above we get the following commutative diagram where we indicate isomorphism $\cong$, homotopy equivalence $\sim$, or 2-sheeted covering 2, and a free orbit $S^1 \cdot c$ for the rotation action on Imm^0:



3. The homotopy type of $B^0(S^1, \mathbb{R}^2)$

Proposition 3.1. *The mapping* $\mathrm{Imm}^0(S^1, \mathbb{R}^2) \to B^{0,+}(S^1, \mathbb{R}^2)$ *is a (Serre) fibration.*

Proof. First we replace $\mathrm{Imm}^0(S^1, \mathbb{R}^2)$ by the subset $\mathrm{Imm}^0_a(S^1, \mathbb{R}^2)$ consisting of all immersions which are parametrized by scaled arc-length which is a strong deformation retract, see [4, 2.6]. The normalizer of the $\mathrm{Diff}^+(S^1)$-action on it is just the action of S^1 which shifts the initial point. We have to show that for any compactly generated space P and a homotopy $h : [0, 1] \times P \to B^{0,+}$ whose initial value $h(0, \quad)$ admits a continuous lift there exists a continuous lift of the whole homotopy:



To get the lift H we just have to specify the initial point coherently from $H(0, p)(1)$ over $[0, 1] \ni t \mapsto h(t, p)$.

For that we need a description of the elements in $B^{0,+}(S^1, \mathbb{R}^2)$. A point C in it can be described by the following data:

For some n and $i = 1, \dots, n$, there are open sets $U_i = U_i(C) \subseteq \mathbb{R}^2$, smooth functions $f_i = f_i(C) : U_i \to \mathbb{R}$ such that $f_i^{-1}(0) =: C_i$ is a component C_i of C with

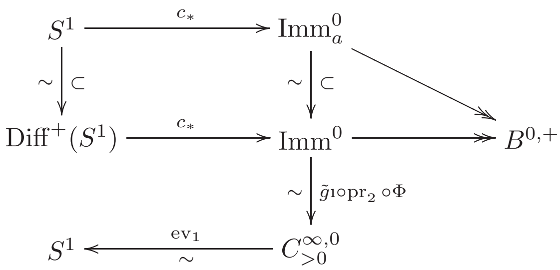
$\operatorname{grad}(f_i)$ is a unit vector field with flow lines unit speed straight lines passing orthogonally through C_i in such a way that for $x \in C_i$ the frame consisting of $\operatorname{grad}(f_i)(x)$ and the unit tangent to C_i at x is positively oriented. The unparameterized smooth oriented 1-manifolds $C_1, C_2, \dots, C_n$ (in that order) describe C. Note that there is a choice for the U_i and their cyclic order, but then the f_i are unique.

For every $p \in P$ the initial point $H(0,p)(1)$ lies in some component $h(0,p)_i$ of $h(0,p)$, and we may move it orthogonally along $\operatorname{grad}(f_i(h(t,p)))$ to get a coherent choice of initial points. This takes care of the lift H. □

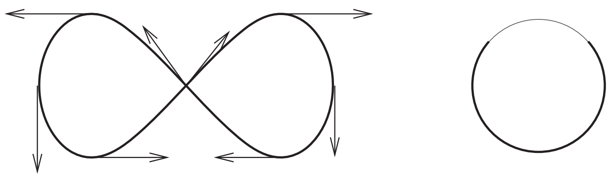
Lemma 3.2. *The fiber* $\operatorname{Diff}^+(S^1)$ *maps homotopically trivial into the fibration*

$$\operatorname{Imm}^0(S^1, \mathbb{R}^2) \to B^{0,+}(S^1, \mathbb{R}^2).$$

Proof. As in the proof of 3.1 we consider the space $\operatorname{Imm}^0_a(S^1, \mathbb{R}^2)$ of degree 0 immersions with constant speed parameterizations. Let c be the unit speed parameterized horizontal figure eight, and consider the diagram where $c^*(f) = c \circ f$:



We have to show that the mapping from the upper left S^1 to the lower left S^1 is nullhomotopic. It is essentially (suppressing $\tilde{g}^{-1}$) given by $\beta \mapsto \frac{c'(\beta)}{|c'(\beta)|}$. From the figure



we see that this mapping covers everything below the northern polar region twice and avoids the northern polar region, so it is nullhomotopic. □

Corollary 3.3. *We have the following homotopy groups:*

$$\begin{aligned} &\pi_1(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}, && \pi_1(B^{0}(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\ &\pi_2(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}, && \pi_2(B^{0}(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\ &\pi_k(B^{0,+}(S^1, \mathbb{R}^2)) = 0, && \pi_k(B^{0}(S^1, \mathbb{R}^2)) = 0 \quad \textit{for } k > 2. \end{aligned}$$

Proof. By 3.1 we have the long exact homotopy sequence

$$\cdots \to \pi_k(S^1) \xrightarrow{0} \pi_k(\mathrm{Imm}^0_a) \to \pi_k(B^{0,+}) \to \pi_{k-1}(S^1) \to \cdots$$

and by section 2 the space Imm^0_a is homotopy equivalent to S^1. This gives the homotopy groups of $B^{0,+}(S^1,\mathbb{R}^2))$. Since $B^{0,+}(S^1,\mathbb{R}^2)) \to B^0(S^1,\mathbb{R}^2))$ is a two-sheeted covering, we can also read of the homotopy groups of $B^{0,+}(S^1,\mathbb{R}^2))$. ☐

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