

Schottky Uniformizations of \mathbb{Z}_2^2 Actions on Riemann Surfaces

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ABSTRACT

Given a closed Riemann surface S together a group of its conformal automorphisms $H \cong \mathbb{Z}_2^2$, it is known that there are Schottky uniformizations of S realizing H . In this note we proceed to give an explicit Schottky uniformizations for each of all different topological actions of \mathbb{Z}_2^2 as group of conformal automorphisms on a closed Riemann surface.

Key words: Schottky groups, Riemann surfaces, automorphisms, uniformization.

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Introduction

A Schottky group of genus zero is just the trivial group. A Schottky group of positive genus is defined as follows. Assume we have a collection of $2p$ ($p > 0$) pairwise disjoint simple loops, say $\alpha_1, \alpha'_1, \dots, \alpha_p, \alpha'_p$, in the Riemann sphere bounding a common region \mathcal{D} of connectivity $2p$, and that there are loxodromic transformations L_1, \dots, L_p so that

- (i) $L_j(\alpha_j) = \alpha'_j$, for $j = 1, \dots, p$, and
- (ii) $L_j(\mathcal{D}) \cap \mathcal{D} = \emptyset$, for each $j = 1, 2, \dots, p$.

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The group G , generated by L_1, \dots, L_p , is called a Schottky group of genus $p > 0$. The collection of loops $\alpha_1, \alpha'_1, \dots, \alpha_p$ and α'_p , is called a fundamental system of loops of G respect to the Schottky generators L_1, \dots, L_p . In [3] was obtained that any set of p generators of a Schottky group of genus p is in fact a set of Schottky generators. A Schottky group of genus p is a free group of rank p , purely loxodromic and Kleinian. That these properties define a Schottky group of genus p was obtained in [18]. If we denote by Ω the region of discontinuity of a Schottky group G of genus p , then the quotient $S = \Omega/G$ turns out to be a closed Riemann surface of genus p . The reciprocal is valid by the retrosection theorem [12] (see [1] for a modern proof using quasiconformal deformation theory). A triple $(\Omega, G, P : \Omega \rightarrow S)$ is called a Schottky uniformization of a closed Riemann surface S if G is a Schottky group with Ω as its region of discontinuity and $P : \Omega \rightarrow S$ is a holomorphic regular covering with G as covering group. A Schottky group G is called a hyperelliptic Schottky group if it uniformizes a hyperelliptic Riemann surface and the hyperelliptic involution lifts to such a covering. A set of Schottky generators L_1, \dots, L_p of a Schottky group G is called a set of hyperelliptic Schottky generators if there is an elliptic transformation E of order 2 satisfying the equality $E \circ L_j \circ E = L_j^{-1}$, for all $j = 1, \dots, p$. It was proved by L. Keen [10] that every hyperelliptic Schottky group has a set of hyperelliptic Schottky generators and that hyperelliptic Riemann surface may be uniformized by a suitable hyperelliptic Schottky group. Given a closed Riemann surface S together a group of conformal automorphisms $H < \text{Aut}(S)$ isomorphic to \mathbb{Z}_2^2 we know from the results in [8] the existence of a Schottky uniformization $(\Omega, G, P : \Omega \rightarrow S)$ of S for which H lifts, that is, for each $h \in H$ there is a Möbius transformation k that keeps Ω invariant and so that $P \circ k = h \circ P$. In this note, we provide explicitly a Schottky uniformization as above for each possible topological action of \mathbb{Z}_2^2 . There are exactly two different topological actions of \mathbb{Z}_2^2 acting free fixed points. We observe that one of these actions cannot be produced by hyperelliptic Riemann surfaces.

This note is organized as follows. In section 1 we recall some basic tools from quasiconformal deformation of Kleinian group we will need in the last section. In section 2 we recall a description of the different topological actions of \mathbb{Z}_2^2 and we observe that in the free fixed point action one of the two topological cases cannot be produced by hyperelliptic Riemann surfaces. In section 3 we describe explicit Schottky uniformizations for each of the two possible topological free actions of \mathbb{Z}_2^2 and in section 4 we provide explicit Schottky uniformizations for each of the possible topological actions of \mathbb{Z}_2^2 .

1. Quasiconformal deformations

Details of the following definitions and results concerning quasiconformal homeomorphisms can be found, for instance, in [2, 15].

1.1. Quasiconformal homeomorphisms

Let us consider an orientation-preserving homeomorphism $W : \Omega_1 \rightarrow \Omega_2$, where Ω_j is a region on the Riemann sphere, for $j = 1, 2$. We say that W is a quasiconformal homeomorphism if:

- (i) W has distributional partial derivatives ∂W and $\bar{\partial}W$ in $L^2_{loc}(\Omega_1)$, and
- (ii) there is a measurable function $\mu \in L^\infty(\Omega_1)$ such that $\|\mu\|_\infty < 1$ and

$$\bar{\partial}W(z) = \mu(z)\partial W(z) \quad \text{a.e. } \Omega_1.$$

The measurable function μ is called a Beltrami coefficient of W . If there is some $K \geq 1$ so that

$$\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \leq K,$$

then we say that W is a K -quasiconformal homeomorphism.

An orientation-preserving homeomorphism between Riemann surfaces S and R is said to be K -quasiconformal if, for every local coordinates of both S and R , we have a K -quasiconformal homeomorphism. If the surfaces are compact, then W is K -quasiconformal (some $1 \leq K < \infty$) if and only if is quasiconformal in local coordinates. We will use the following fact.

Theorem 1.1 ([15]). *Let S and R be closed Riemann surfaces and $f : S \rightarrow R$ be an orientation-preserving homeomorphism. Then there is a quasiconformal homeomorphism $w : S \rightarrow R$ which is homotopic to f .*

The above result permits us to assume all orientation-preserving homeomorphisms between closed Riemann surfaces to be quasiconformal.

Theorem 1.2 (Ahlfors-Bers’ theorem [1]). *Let μ be a function of $L^\infty(\hat{\mathbb{C}})$ of norm strictly less than 1. Then, there is a unique quasiconformal homeomorphism $W : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ that satisfies the following two properties:*

- (i) $\bar{\partial}W(z) = \mu(z)\partial W(z)$ a.e. Ω_1 ,
- (ii) $W(0) = 0, W(1) = 1$, and $W(\infty) = \infty$.

1.2. Quasiconformal deformations of Kleinian groups

A Kleinian group is a group K of Möbius transformations acting discontinuously at some point on the Riemann sphere. The region of discontinuity of a Kleinian group K is the open set of points on which it acts discontinuously, which is denoted by $\Omega(K)$. Generalities on Kleinian groups can be found in [17].

Let G be a Kleinian group, with region of discontinuity $\Omega(G)$, and μ be a measurable function defined on $\hat{\mathbb{C}}$ such that:

- (i) $\|\mu\|_\infty \leq k < 1$, and
- (ii) $\mu(g(z))\overline{g'(z)} = \mu(z)g'(z)$, a.e. $\widehat{\mathbb{C}}$ for all $g \in G$.

We say that μ is a Beltrami coefficient for G . Ahlfors-Bers' theorem asserts that there is a quasiconformal homeomorphism W of the Riemann sphere with Beltrami coefficient μ , which is unique if we require it to fix three different points. The group $W \circ G \circ W^{-1}$ turns out to be a Kleinian group with region of discontinuity $\Omega(W \circ G \circ W^{-1}) = W(\Omega(G))$ and is called a quasiconformal deformation of G .

If G is a Schottky group, then any quasiconformal deformation of it is again a Schottky group of the same genus.

2. Topological classification

In [22] is considered the problem of classifying the (topological) effective actions of the Abelian group $\mathbb{Z}_p^r = \mathbb{Z}_p \oplus \dots \oplus \mathbb{Z}_p$ (where the number p is a prime and the number of summands is r) on a closed orientable surface. In there, it is determined the equivalent classes in the sense of putting them into one-to-one correspondence with the equivalence classes of some algebraic system. In some simple instances it is also computed the number of different classes. In the particular case $p = 2$ and $r = 2$ there is no mention of an explicit formula which permit to count such a number of equivalence classes. On the other hand, in [4, 21] there are descriptions of the topological actions of \mathbb{Z}_2^2 . Formula for the number of different type actions of \mathbb{Z}_2^2 may be obtained from these descriptions using similar methods to the ones in the works of Alexander Mednykh [20] or Jin Ho Kwak [13, 14]. In this section we recall, as a matter of completeness, (1) the description of all possible topological actions of \mathbb{Z}_2^2 , as a group of orientation preserving homeomorphisms of a closed orientable surface, and (2) a simple formula to count the different topological classes.

Remark 2.1. As a consequence of the Nielsen's realization theorem, proved in its generality by Kerckhoff in [11], such a topological classification is equivalent to the topological classification of the actions of \mathbb{Z}_2^2 as a group of conformal automorphisms of a closed Riemann surface. For that reason, we only need to deal with pairs (S, H) , where S is a closed Riemann surface and $H \cong \mathbb{Z}_2^2$ is a group of conformal automorphisms on S .

We denote by \mathfrak{F}_g the collection of pairs (S, H) , where S is a closed Riemann surface of genus g and $H \cong \mathbb{Z}_2^2$ is a group of conformal automorphisms of S . Two pairs $(S_1, H_1), (S_2, H_2) \in \mathfrak{F}_g$ are said to be topologically equivalent (some authors call this weak equivalence) if there is an orientation preserving homeomorphism $f : S_1 \rightarrow S_2$ so that $f \circ H_1 \circ f^{-1} = H_2$.

Assume we have a pair $(S, H) \in \mathfrak{F}_g$. We denote by g and γ the genus of S and S/H , respectively, and by $\pi : S \rightarrow S/H$ the branched holomorphic regular covering induced by H . In H there are three different non-trivial elements, say a, b , and

$a \circ b$, each one an involution. As a conformal involution has an even number of fixed points, a consequence of the Riemann-Hurwitz formula [6], and the stabilizer of any point in S by H is cyclic, we may assume that the number of fixed points of a , b , and $a \circ b$ are, respectively, $2r$, $2s$, and $2t$. The branch locus $\mathcal{B}_\pi \subset S/H$ of π can be decomposed into three pairwise disjoint sets, say $\mathcal{B}_\pi = \mathcal{R} \dot{\cup} \mathcal{S} \dot{\cup} \mathcal{T}$, where \mathcal{R} , \mathcal{S} , and \mathcal{T} contain, respectively, the projections under π of the fixed points of a , b , and $a \circ b$. The cardinalities of \mathcal{R} , \mathcal{S} , and \mathcal{T} are then, respectively, r , s , and t . The Riemann-Hurwitz formula asserts the equality

$$g = 4(\gamma - 1) + 1 + r + s + t,$$

in particular

$$\begin{cases} r + s + t \leq g + 3, \\ r + s + t \equiv g + 3 \pmod{4}. \end{cases}$$

Let us denote by $\mathfrak{F}_{g,r,s,t}$ the collection of pairs $(S, H) \in \mathfrak{F}_g$ so that the number of fixed points of the three different non-trivial elements of H are $2r$, $2s$ and $2t$. Our definition asserts that $\mathfrak{F}_{g,r,s,t} = \mathfrak{F}_{g,s,t,r} = \mathfrak{F}_{g,t,r,s} = \mathfrak{F}_{g,s,r,t} = \mathfrak{F}_{g,r,t,s} = \mathfrak{F}_{g,t,s,r}$. In this way, we have that \mathfrak{F}_g can be decomposed into a disjoint union of subfamilies $\mathfrak{F}_{g,r,s,t}$ (maybe some of them are empty ones), where

$$\begin{cases} 0 \leq r \leq s \leq t, \\ 0 \leq r + s + t \leq g + 3, \\ r + s + t \equiv g + 3 \pmod{4}. \end{cases} \tag{1}$$

Observe that if we have two pairs, say $(S_1, H_1), (S_2, H_2) \in \mathfrak{F}_g$, which are topologically equivalent, then both pairs should belong to the same subfamily $\mathfrak{F}_{g,r,s,t}$.

The following two theorems can be obtained from the descriptions obtained in [4, 21].

Theorem 2.2. *If two pairs in \mathfrak{F}_g are topologically equivalent, then they belong to the same subfamily $\mathfrak{F}_{g,r,s,t}$. Moreover, if $(S_1, H_1), (S_2, H_2) \in \mathfrak{F}_{g,r,s,t}$, where $r + s + t > 0$, then the two pairs are topologically equivalent.*

In particular, second part of theorem 2.2 tells us that the number of topologically non-equivalent actions of \mathbb{Z}_2^2 as group of orientation-preserving homeomorphisms on closed orientable surfaces, acting with fixed points, is at most the number of solutions of equation (1) for which $r + s + t \neq 0$. Respect to the actions without fixed points (freely acting), we have the following.

Theorem 2.3. (i) *The family $\mathfrak{F}_{1,0,0,0}$ consists exactly of one point.*

(ii) *If $g \not\equiv 1 \pmod{4}$, then $\mathfrak{F}_{g,0,0,0} = \emptyset$.*

(iii) *If $g \geq 2$ and $g \equiv 1 \pmod{4}$, then the family $\mathfrak{F}_{g,0,0,0}$ consists of exactly two disjoint subfamilies, say $\mathfrak{F}_{g,0,0,0}^1$ and $\mathfrak{F}_{g,0,0,0}^2$, so that:*

- (a) two pairs in the same subfamily are topologically equivalent, and
- (b) pairs belonging to different subfamilies are topologically non-equivalent.

It is well known that on genus $g = 1$ there is exactly one topological free action of the group \mathbb{Z}_2^2 . The third equation of (1) asserts part (ii).

Proposition 2.4. *Let S be a closed Riemann surface of genus g admitting the Klein group H as group of conformal automorphisms. Let, as before, denote by $r, s,$ and t the cardinalities of the subsets $\mathcal{R}, \mathcal{S},$ and \mathcal{T} of S/H . Then r, s, t are simultaneously even or simultaneously odd.*

Proof. We only first need to observe that any simple closed loop on S/H bounding a closed disc that contains all branching on its interior must lift to a simple loop, that is, corresponds to the trivial element of H . This is consequence of the fact that such a loop is homotopic to the product of commutators and the covering $\pi : S \rightarrow S/H$ is Abelian. Secondly, if we have a simple loop bounding a close disc on S/H that contains on its interior an even number of branch points, all of them belonging to the same set \mathcal{R} or \mathcal{S} of \mathcal{T} , then it should correspond to the trivial element of H , that is, it lifts to a loop. Third, a simple loop bounding a closed disc on S/H containing on its interior only at most one point from each set \mathcal{R} or \mathcal{S} of \mathcal{T} will lift to a loop if there is exactly one point from each such set. The above three observations imply the result. \square

We have decomposed the family \mathfrak{F}_g into a disjoint union of subfamilies $\mathfrak{F}_{g,r,s,t}$, where (r, s, t) are restricted to the conditions given in (1). As mentioned before, some of them maybe are empty ones. We proceed to see which ones are non-empty, so we may count the number of different topological actions. An extra restriction is given by the Riemann-Hurwitz formula, which asserts that a conformal involution $j : S \rightarrow S$, where S is a closed Riemann surface of genus $g \geq 2$, has at most $2(g + 1)$ fixed points, that is, $t \leq g + 1$. Moreover, if the involution j has $2(g + 1)$ fixed points, then S is hyperelliptic, j is the hyperelliptic involution and any other automorphism of S has at most 4 fixed points [6]. It follows that if $t = g + 1$, then we should have $s \leq 2$. As a consequence of these facts and proposition 2.4, we need to replace the restrictions (1) by the following ones:

$$\begin{cases} 0 \leq r \leq s \leq t \leq (g + 1), \\ r + s + t \equiv (g + 3) \pmod{4}, \\ 0 \leq r + s + t \leq g + 3, \\ r, s, t \text{ all have the same parity.} \end{cases} \tag{2}$$

Let us denote by N_g the collection of the solutions (r, s, t) of the system (2). We have, as a consequence of the Schottky constructions we provide later in section 4, the following non-emptiness situation.

Proposition 2.5. *Let us fix a value of g . If $(r, s, t) \in N_g$, then $\mathfrak{F}_{g,r,s,t} \neq \emptyset$. If $(r, s, t) \notin N_g$, then $\mathfrak{F}_{g,r,s,t} = \emptyset$.*

Clearly, the second part of proposition 2.5 is consequence of the above arguments.

Let us denote by Ψ_g the cardinality of the set N_g formed by the triples $(r, s, t) \in \mathbb{N}_0^3$ satisfying the conditions of (2). We observe first that as r, s, t all have the same parity and since $g + 3$ is congruent to $r + s + t$ module 4, both $r + s + t$ and $g + 3$ also have the same parity as for r .

CASE $g \equiv 0 \pmod 4$

In this situation we must have r, s, t odd integers and $r + s + t \equiv 3 \pmod 4$. It follows we need to consider the values $r + s + t = 3 + 4l$, $l = 0, 1, \dots, g/4$. Now, for each value of l as before, we may write $t = 3 + 4l - 2q$, where q a non-negative integer. Since $t \leq g + 1$, we also must have $3 + 4l - 2q \leq g + 1$, that is, $q \geq q_{0,l} = \max\{0, (4l + 2 - g)/2\}$. We observe that $q_{0,l} = 0$ for $l = 0, 1, \dots, g/4 - 1$ and $q_{0,g/4} = 1$. As $r \leq s \leq t$, we have that $3 + 4l = r + s + t \leq 3t = 9 + 12l - 6q$. It follows that the maximum possible value of q is given by $q_l = 1 + l$, which is obtained by the triple $(r = 1, s = 2l + 1, t = 2l + 1)$. Now, for each value of $q \in \{q_{0,l}, \dots, q_l\}$, the possible triples we have are of the form

$$(r = 1 + 2u, s = 2q - (2u + 1), t = 3 + 4l - 2q),$$

where $u \geq 0$ and $0 \leq 1 + 2u \leq 2q - 2u - 1 \leq 3 + 4l - 2q$. The last inequality is trivial as $q \leq q_l$. It follows that the above is equivalent to have $u \in \{0, 1, \dots, [(q - 1)/2]\}$.

CASE $g \equiv 1 \pmod 4$

In this situation we must have r, s, t even integers and $r + s + t \equiv 0 \pmod 4$. It follows we need to consider the values $r + s + t = 4l$, $l = 0, 1, \dots, (g + 3)/4$. Now, for each value of l as before, we may write $t = 4l - 2q$, where q a non-negative integer. Since $t \leq g + 1$, we also must have $4l - 2q \leq g + 1$, that is, $q \geq \max\{0, (4l - g - 1)/2\}$. As $r \leq s \leq t$, we have that the maximum possible value of q is $q_l = l$, which is given by the triple $(r = 0, s = 2l, t = 2l)$, where $q_l = l$. Now, for each value of $q \in \{0, 1, \dots, q_l\}$, the possible triples we have are of the form $(r = 2u, s = 2q - 2u, t = 4l - 2q)$, where $0 \leq 2u \leq 2q - 2u \leq 4l - 2q$. The last inequality is trivial since $q \leq q_l$. It follows that the above is equivalent to have $u \in \{0, 1, \dots, [q/2]\}$.

CASE $g \equiv 2 \pmod 4$

In this situation we must have r, s, t odd integers and $r + s + t \equiv 1 \pmod 4$. It follows we need to consider the values $r + s + t = 1 + 4l$, $l = 0, 1, \dots, (g + 2)/4$. Now, for value of l as before, we may write $t = 1 + 4l - 2q$, where q a non-negative integer. Since $t \leq g + 1$, we also must have $1 + 4l - 2q \leq g + 1$, that is, $q \geq \max\{0, (4l - g)/2\}$. As $r \leq s \leq t$, we have that the maximum possible value of q is $q_l = [(1 + 2l)/2]$, which is given by the

triple $(r = 1, s = 2q_l - 1, t = 1 + 4l - 2q_l)$. Now, for each value of $q \in \{0, 1, \dots, q_l\}$, the possible triples we have are of the form $(r = 1 + 2u, s = 2q - (2u + 1), t = 1 + 4l - 2q)$, where $0 \leq 1 + 2u \leq 2q - 2u - 1 \leq 1 + 4l - 2q$. The last inequality is trivial since $q \leq q_l$. It follows that the above is equivalent to have $u \in \{0, 1, \dots, [(q - 1)/2]\}$.

CASE $g \equiv 3 \pmod 4$

In this situation we must have r, s, t even integers and $r + s + t \equiv 2 \pmod 4$. It follows we need to consider the values $r + s + t = 2 + 4l, l = 0, 1, \dots, (g + 1)/4$. Now, for each value of l as before, we may write $t = 4l - 2q$, where q a non-negative integer. Since $t \leq g + 1$, we also must have $4l - 2q \leq g + 1$, that is, $q \geq \max\{0, (4l - 1 - g)/2\}$. As $r \leq s \leq t$, we have that the maximum possible value of q is $q_l = l$, obtained by the triple $(r = 0, s = 2l, t = 2l)$. Now, for each value of $q \in \{0, 1, \dots, q_l\}$, the possible triples we have are of the form $(r = 2u, s = 2q - 2u, t = 4l - 2q)$, where $0 \leq 2u \leq 2q - 2u \leq 4l - 2q$. The last inequality is trivial since $q \leq q_l$. It follows that the above is equivalent to have $u \in \{0, 1, \dots, [q/2]\}$.

It follows from the above

$$\Psi_g = \begin{cases} \sum_{l=0}^{\frac{g}{4}} \sum_{q=\max\{0, \frac{4l+2-g}{2}\}}^{1+l} \left(\left[\frac{q-1}{2}\right] + 1\right), & g \equiv 0 \pmod 4, \\ \sum_{l=0}^{\frac{g+3}{4}} \sum_{q=\max\{0, \frac{4l-g-1}{2}\}}^l \left(\left[\frac{q}{2}\right] + 1\right), & g \equiv 1 \pmod 4, \\ \sum_{l=0}^{\frac{g+2}{4}} \sum_{q=\max\{0, \frac{4l-g}{2}\}}^{\lceil \frac{1+2l}{2} \rceil} \left(\left[\frac{q-1}{2}\right] + 1\right), & g \equiv 2 \pmod 4, \\ \sum_{l=0}^{\frac{g+1}{4}} \sum_{q=\max\{0, \frac{4l-1-g}{2}\}}^l \left(\left[\frac{q}{2}\right] + 1\right), & g \equiv 3 \pmod 4, \end{cases}$$

where $[x]$ denotes the integral part of the real number x and $\max(a, b)$ denotes the maximum between a and b . As a consequence of theorems 2.2, 2.3 and proposition 2.5, we have the following.

Theorem 2.6. *For each non-negative integer g there are exactly*

- (i) Ψ_g , for either case $g \not\equiv 1 \pmod 4$ or $g = 1$,
- (ii) $\Psi_g + 1$, for $g \equiv 1 \pmod 4$ and $g \neq 1$,

topologically non-equivalent pairs $(S, H) \in \mathfrak{F}_g$.

The first values of Ψ_g are given by

$$\Psi_0 = 1, \quad \Psi_1 = 2, \quad \Psi_2 = 1, \quad \Psi_3 = 3, \quad \Psi_4 = 3.$$

In order to produce Schottky uniformizations as desired, we need a result that permits to describe the different topological actions of \mathbb{Z}_2^2 . Let S be a closed Riemann surface of genus g and H be a group of conformal automorphisms of S isomorphic

to \mathbb{Z}_2^2 . We denote by \tilde{S} the quotient Riemann (orbifold) surface S/H , by γ its genus and by $P : S \rightarrow \tilde{S}$ the natural holomorphic (branched) covering induced by the action of H on S . If we fix a and b any two different elements of order two in H , then a presentation of H is given by

$$H = \langle a, b : a^2 = b^2 = (a \circ b)^2 = 1 \rangle.$$

Let η be any loop on \tilde{S} disjoint from the branch locus of P . Choose any point $z \in \eta$ and choose a point $x \in S$ such that $P(x) = z$. Lift the loop η starting at the x . Let y be the end point of such a lifting. Since $P(y) = z$ and our covering (branched) is regular, there exists a unique element $h \in H$ such that $y = h(x)$. We say that the loop η corresponds to the automorphism h or reciprocally. As the covering group H is Abelian, this correspondence is well defined, that is, does not depend on the choice of either x nor z . We denote this correspondence as

$$\eta \rightarrow h.$$

The following lemma permits to construct a special family of loops on \tilde{S} respect to the above chosen generators a and b for H . This special loops gives an alternative proof of theorems 2.2 and 2.3. This also permits to construct canonical homology basis on S adapted to the action of H and may help in the computation of Riemann period matrices of S (see also section 5).

Lemma 2.7 ([7]). *If $\gamma > 0$, then there is a set of simple loops on \tilde{S} , say*

$$\alpha_1, \dots, \alpha_\gamma, \beta_1, \dots, \beta_\gamma,$$

satisfying the following properties:

- (i) $\alpha_i \cap \alpha_j = \emptyset$, for $i \neq j$,
- (ii) $\beta_i \cap \beta_j = \emptyset$, for $i \neq j$,
- (iii) $\alpha_i \cap \beta_j = \emptyset$, for $i \neq j$,
- (iv) $\alpha_i \cap \beta_i$ consists of exactly one point,
- (v) $\tilde{S} - \bigcup_{i=1}^\gamma (\alpha_i \cup \beta_i)$ is topologically a sphere with γ pairwise disjoint deleted discs,
- (vi) *If some non-trivial element of H acts with fixed points, then*

$$\begin{cases} \alpha_1 \rightarrow a, \\ \beta_1 \rightarrow b, \\ \alpha_j, \beta_j \rightarrow 1, \quad j \geq 2. \end{cases}$$

(vii) If H acts freely, then either

$$\begin{cases} \alpha_1 \rightarrow a, \\ \beta_1 \rightarrow b, \\ \alpha_j, \beta_j \rightarrow 1, \quad j \geq 2. \end{cases} \tag{3a}$$

or

$$\begin{cases} \alpha_1 \rightarrow a, \\ \alpha_2 \rightarrow b, \\ \alpha_j \rightarrow 1, \quad j \geq 3, \\ \beta_j \rightarrow 1, \quad j \geq 1. \end{cases} \tag{3b}$$

We have two different topological actions of \mathbb{Z}_2^2 acting free fixed points; these two actions are described below in an explicit manner.

2.0.1. FIRST ACTION

Let us consider the set \mathcal{G} formed of the edges of the cube $\mathcal{Q} \subset \mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. Let $V(\mathcal{G})$ be a regular tubular neighborhood of \mathcal{G} , invariant under the Euclidean isometry group

$$H_0 = \langle A(x, y, z) = (x, -y, -z), B(x, y, z) = (-x, y, -z) \rangle \cong \mathbb{Z}_2^2.$$

Let $S_{\mathcal{G}}$ be the boundary of $V(\mathcal{G})$. We have that $S_{\mathcal{G}}$ is a closed orientable surface of genus $g = 5$ with H_0 as group of orientation preserving homeomorphisms. Let us consider a closed disc $D_S \subset S$ so that $h(D_S) \cap D_S = \emptyset$, for all $h \in H$. Set S^* to be S minus the interior of D_S and its H -translates. Let R be a closed Riemann surface of genus $(g - 5)/4$ and $D_R \subset R$ a closed disc. Set R^* to be R minus the interior of D_R . We proceed to glue R^* to S^* along the boundary of D_R and D_S . Then we use the action of H to glue three other copies of R^* to S^* along the other three boundary components. We end with a closed orientable surface S' , of genus g admitting H as a freely acting group of orientation preserving homeomorphisms.

2.0.2. SECOND ACTION

Let us consider $\tau \in \mathbb{H}^2$, the upper-half-plane. Let $A(z) = z + 1/2$, $B_{\tau}(z) = z + \tau/2$ and $\Lambda_{\tau} = \langle A^2, B_{\tau}^2 \rangle$. The torus

$$T_{\tau} = \mathbb{C}/\Lambda_{\tau}$$

admits the group $H_{\tau} = \langle A, B_{\tau} \rangle/\Lambda_{\tau} \cong \mathbb{Z}_2^2$ as group of conformal automorphisms acting freely. Let us consider a small closed Euclidean disc Δ centered at the origin so that it is disjoint from all its translates under the Euclidean group $\langle A, B_{\tau} \rangle$. If C

is the boundary circle of Δ and σ the reflection on C , then we consider the group Γ generated by the transformations

$$\begin{aligned} &A^2, \quad B_\tau^2, \quad \sigma \circ A^2 \circ \sigma, \quad \sigma \circ B_\tau^2 \circ \sigma, \\ &A \circ \sigma \circ A^2 \circ \sigma \circ A, \quad A \circ \sigma \circ B_\tau^2 \circ \sigma \circ A, \\ &B \circ \sigma \circ A^2 \circ \sigma \circ B, \quad B \circ \sigma \circ B_\tau^2 \circ \sigma \circ B, \\ &B \circ A \circ \sigma \circ A^2 \circ \sigma \circ A \circ B, \quad B \circ A \circ \sigma \circ B_\tau^2 \circ \sigma \circ A \circ B. \end{aligned}$$

We have that, if Ω denotes the region of discontinuity of Γ , then $S = \Omega/\Gamma$ is a closed Riemann surface of genus $g = 5$ with the group $H = \langle A, B_\tau \rangle/\Gamma \cong \mathbb{Z}_2^2$ as group of conformal automorphisms acting freely.

Now we may proceed as in the previous case. Let us consider a closed disc $D_S \subset S$ so that $h(D_S) \cap D_S = \emptyset$, for all $h \in H$. Set S^* to be S minus the interior of D_S and its H -translates. Let R be a closed Riemann surface of genus $(g - 5)/4$ and $D_R \subset R$ a closed disc. Set R^* to be R minus the interior of D_R . We proceed to glue R^* to S^* along the boundary of D_R and D_S . Then we use the action of H to glue three other copies of R^* to S^* along the other three boundary components. We end with a closed orientable surface S' , of genus g admitting H as a freely acting group of orientation preserving homeomorphisms.

Remark 2.8. A description of the two free actions of \mathbb{Z}_2^2 at the level of Schottky groups is given in section 3.

The next result asserts that one of these actions cannot be produced using hyperelliptic Riemann surfaces.

Theorem 2.9. *In Theorem 2.3, part (iii), one of the two subfamilies consists only of non-hyperelliptic Riemann surfaces. The other family has the property that if (S, H) is one of its members so that S/H is hyperelliptic, then S is necessarily hyperelliptic.*

Proof. Let us denote by $\mathfrak{F}_{g,0,0,0}^1$ the subfamily of $\mathfrak{F}_{g,0,0,0}$ that corresponds to the topological class described in section 2.0.1 and by $\mathfrak{F}_{g,0,0,0}^2$ the subfamily corresponding to the one described in section 2.0.2.

Let $(S, H) \in \mathfrak{F}_{g,0,0,0}$ so that S is hyperelliptic. We have by the previous descriptions that the hyperelliptic involution $j_S : S \rightarrow S$ is not in H . We consider the natural 2-fold branched covering $\pi : S \rightarrow \hat{\mathbb{C}}$ induced by j_S . As j_S belongs to the center of $\text{Aut}(S)$ and $j_S \notin H$, we have that H descends to a subgroup $K \cong \mathbb{Z}_2^2$ of Möbius transformations that keep invariant the $2(g + 1)$ branch values of π . As none of the non-trivial element in H has fixed points, we have also that no non-trivial element of K fixes one of the branch values of π . In fact, assume we have some non-trivial element of $\hat{h} \in H/\langle j_S \rangle$ that fixes a branch value q of π . Let $h \in H$ be so that $\pi \circ h = \hat{h} \circ \pi$. In this way, if we denote by $p \in S$ the fixed point of j_S so that $\pi(p) = q$, then we have $\pi(h(p)) = \hat{h}(\pi(p)) = \hat{h}(q) = q = \pi(p)$, that is, $h(p) \in \{p, j_S(p)\}$. But as p is fixed point of j_S , we have that $h(p) = p$, a contradiction to the fact that H is acting free fixed points.

Up to conjugation in the Möbius group, we may assume K is generated by $A(z) = -z$ and $B(z) = 1/z$. It follows that S can be then represented by an algebraic curve of the form

$$y^2 = \prod_{j=1}^{(g+1)/2} \left(x^4 - \left(\lambda_j^2 + \frac{1}{\lambda_j^2} \right) x^2 + 1 \right),$$

where $\lambda_j \in \mathbb{C} - \{0, \pm 1, \pm i\}$, are so that

$$\lambda_j^2 + \frac{1}{\lambda_j^2} \neq \lambda_k^2 + \frac{1}{\lambda_k^2}, \quad \text{for } j \neq k.$$

It follows that the topological action of H is uniquely determined in the hyperelliptic case. (see Remark 2.10 below). In this way, we have that one of the two families $\mathfrak{F}_{g,0,0,0}^1$ or $\mathfrak{F}_{g,0,0,0}^2$ cannot have hyperelliptic Riemann surfaces. By looking at the action in the algebraic curve, we may see that the hyperelliptic case belongs to the family $\mathfrak{F}_{g,0,0,0}^2$ (see also theorem 3.6). This gives half part of theorem 2.9; the other half will be consequence of theorem 3.6. \square

Remark 2.10. Let us consider a pair $(S, H) \in \mathcal{F}_{g,0,0,0}^2$ so that S is hyperelliptic. As previously seen, S is represented by an algebraic curve of the form

$$y^2 = \prod_{j=1}^{(g+1)/2} \left(x^4 - \left(\lambda_j^2 + \frac{1}{\lambda_j^2} \right) x^2 + 1 \right),$$

where $\lambda_j \in \mathbb{C} - \{0, \pm 1, \pm i\}$, are so that

$$\lambda_j^2 + \frac{1}{\lambda_j^2} \neq \lambda_k^2 + \frac{1}{\lambda_k^2}, \quad \text{for } j \neq k.$$

The quotient surface S/H corresponds to the algebraic curve

$$w^2 = (u^2 - 1) \prod_{j=1}^{(g+1)/2} \left(u - \frac{1}{2} \left(\lambda_j^2 + \frac{1}{\lambda_j^2} \right) \right),$$

the Weierstrass values are given by the points

$$\pm \lambda_1, \dots, \pm \lambda_{(g+1)/2}, \pm \lambda_1^{-1}, \dots, \pm \lambda_{(g+1)/2}^{-1}$$

and $\pi : S \rightarrow S/H$ corresponds to the map

$$w = Q(z) = \frac{1}{2} \left(z^2 + \frac{1}{z^2} \right).$$

Let us assume we have two tuples of points as above, say $(\lambda_1, \dots, \lambda_{(g+1)/2})$ and $(\lambda_1^*, \dots, \lambda_{(g+1)/2}^*)$. Let S and S^* be the corresponding hyperelliptic Riemann surfaces

and let $H < \text{Aut}(S)$ and $H^* < \text{Aut}(S^*)$ the corresponding groups of automorphisms induced by A and B , where $A(z) = -z$, $B(z) = 1/z$. We have that any homeomorphism $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ so that:

- (i) $f \circ A \circ f^{-1} = A$ and $f \circ B \circ f^{-1} = B$, and
- (ii) f sends $\lambda_j, \lambda_j^{-1}, -\lambda_j, -\lambda_j^{-1}$ onto $\lambda_j^*, \lambda_j^{*-1}, -\lambda_j^*, -\lambda_j^{*-1}$, respectively,

then lifts to an homeomorphism $\widehat{f} : S \rightarrow S^*$ that conjugate H into H^* .

3. Constructions of Schottky uniformizations for the free fixed point case

In this section we will construct Schottky uniformizations of each pair $(S, H) \in \mathcal{F}_{g,0,0,0}$, where $g > 1$. These Schottky uniformizations will have the property that the H -action is reflected on a set of generators of the corresponding Schottky group. Recall that for $\mathcal{F}_{g,0,0,0} \neq \emptyset$, we necessary need to have $g \equiv 1 \pmod{4}$, which will be assumed in the rest of this section. As a consequence of our construction, we will have that such a condition is also sufficient, proving proposition 2.5 at least for this case. We also obtain the last part of theorem 2.9. In theorem 2.3 we have that there are at most two possible topological actions of \mathbb{Z}_2^2 in this case. They give us a decomposition of $\mathfrak{F}_{g,0,0,0}$ into two subfamilies $\mathfrak{F}_{g,0,0,0}^1$ and $\mathfrak{F}_{g,0,0,0}^2$. The family $\mathfrak{F}_{g,0,0,0}^1$ has only non-hyperelliptic Riemann surfaces.

3.1. The family $\mathfrak{F}_{g,0,0,0}^1$

As $g \equiv 1 \pmod{4}$ and $g > 1$, we have that $\gamma = (g + 3)/4 \in \{2, 3, \dots\}$. Let us consider a Schottky group K of genus γ , and a set of Schottky generators of K , say L_1, \dots, L_γ . We may now consider the biggest normal subgroup G_K of K containing the elements $L_1^2, L_2^2, (L_2 \circ L_1)^2, L_3, \dots, L_\gamma$. It follows that $K/G_K \cong \mathbb{Z}_2^2$. As G_K is a subgroup of index 4 of a free group of rank γ , we have that G_K is itself a free group of rank g [16], that is a Schottky group of genus g , which is a normal subgroup of the Schottky group K . If we denote by Ω_K the region of discontinuity of K , then $(S_K = \Omega_K/G_K, H_K = K/G_K) \in \mathcal{F}_{g,0,0,0}^1$. As a consequence of quasiconformal deformation theory and theorem 2.3, we have the following.

Theorem 3.1. *If $g \equiv 1 \pmod{4}$ and $g > 1$, then $\mathcal{F}_{g,0,0,0}^1 \neq \emptyset$. Moreover, if $(S, H) \in \mathcal{F}_{g,0,0,0}^1$, then there is a Schottky group K of genus $(g + 3)/4$ so that (S, H) is biholomorphically equivalent to (S_K, H_K) .*

3.1.1. THE PARTICULAR CASE: $\mathfrak{F}_{5,0,0,0}^1$

In the genus $g = 5$ situation we can be more precise.

Proposition 3.2. *Let K be a Schottky group of genus 2 and set $G = \langle U^2 : U \in K \rangle$. Then:*

- (i) $G \triangleleft K$, $K/G \cong \mathbb{Z}_2^2$.
- (ii) G is a Schottky group of genus 5.
- (iii) If $K = \langle A, B \rangle$, then

$$G = \langle C_1 = A^2, C_2 = B^2, C_3 = B^{-1} \circ A^2 \circ B, \\ C_4 = B^{-1} \circ A^{-1} \circ B \circ A, C_5 = A^{-1} \circ B \circ A \circ B \rangle.$$

- (iv) If $\alpha_1, \alpha'_1, \alpha_2, \alpha'_2$ is a fundamental set of loops for K , respect to the generators A and B , then a fundamental set of loops for G , respect to the generators C_1, C_2, C_3, C_4, C_5 is given by

$$\begin{cases} \delta_1 = A^{-1}(\alpha_1), & \delta'_1 = \alpha'_1, \\ \delta_2 = B^{-1}(\alpha_2), & \delta'_2 = \alpha'_2, \\ \delta_3 = B^{-1}(\delta_1), & \delta'_3 = B^{-1}(\alpha'_1), \\ \delta_4 = A^{-1}(\alpha_2), & \delta'_4 = B^{-1}(A^{-1}(\alpha'_2)), \\ \delta_5 = B^{-1}(A^{-1}(\alpha_2)), & \delta'_5 = A^{-1}(\alpha'_2). \end{cases}$$

- (v) If Ω is the region of discontinuity of K , then it is also the region of discontinuity of G and $(\Omega/G, K/G) \in \mathcal{F}_{5,0,0,0}^1$.
- (vi) Each pair $(S, H) \in \mathcal{F}_{5,0,0,0}^1$ can be obtained as above.

Proof. Part (i) is clear. Part (ii) is consequence of the fact that a finite index 4 subgroup of a free group of rank 2 is a free group of rank 5 [16]. To obtain part (iii), we only need to observe that the subgroup L , generated by C_1, C_2, C_3, C_4, C_5 , is a normal subgroup of K so that $K/L \cong \mathbb{Z}_2^2$ and $L < G$. Then apply (i). Part (iv) is easy to see from figure 1. Part (v) is due to the facts that $S = \Omega/G$ is a genus 5 closed Riemann surface with $H = K/G$ as group of automorphisms so that S/H is the genus two surface Ω/K , and that the topological action coincides with the cube description. Part (vi) is just a consequence of Theorem 2.3. \square

Remark 3.3. We also have the following equalities which we will not use in the rest of this paper, but which may be used in order to construct Riemann period matrices of the corresponding uniformized Riemann surfaces (see section 5):

$$\begin{aligned} A \circ C_1 \circ A^{-1} &= C_1, & B \circ C_1 \circ B^{-1} &= C_2 \circ C_3 \circ C_2^{-1}, \\ A \circ C_2 \circ A^{-1} &= C_1 \circ C_5 \circ C_4 \circ C_1^{-1}, & B \circ C_2 \circ B^{-1} &= C_2, \\ A \circ C_3 \circ A^{-1} &= C_1 \circ C_4^{-1} \circ C_3 \circ C_4 \circ C_1^{-1}, & B \circ C_3 \circ B^{-1} &= C_1, \\ A \circ C_4 \circ A^{-1} &= C_1 \circ C_4^{-1} \circ C_3^{-1}, & B \circ C_4 \circ B^{-1} &= C_5 \circ C_2^{-1}, \\ A \circ C_5 \circ A^{-1} &= C_2 \circ C_3 \circ C_4 \circ C_1^{-1}, & B \circ C_5 \circ B^{-1} &= C_2 \circ C_4. \end{aligned}$$

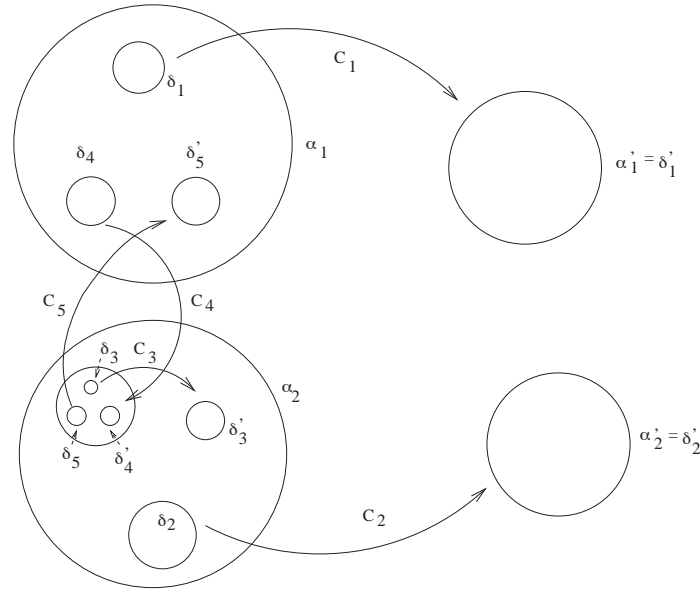


Figure 1

Remark 3.4. The above, in particular, asserts that the family $\mathcal{F}_{5,0,0,0}^1$ can be uniformized by the Schottky space of genus 2.

Let us consider a Schottky group $K = \langle A, B \rangle$ of genus 2. If we denote by $E = A \circ B - B \circ A$, then we have that [10]

$$\begin{aligned} E^2 &= I, \\ E \circ A \circ E &= A^{-1}, \\ E \circ B \circ E &= B^{-1}. \end{aligned}$$

The involution E defines the hyperelliptic involution on the surface of genus 2 given by $R = \Omega/K$, where Ω is the region of discontinuity of K . If we consider the Schottky group of genus 5

$$\begin{aligned} G = \langle x^2 : x \in K \rangle &= \langle C_1 = A^2, C_2 = B^2, C_3 = B^{-1} \circ A^2 \circ B, \\ &C_4 = B^{-1} \circ A^{-1} \circ B \circ A, C_5 = A^{-1} \circ B \circ A \circ B \rangle, \end{aligned}$$

we observe that $E \circ G \circ E = G$. It follows that E also induces a conformal involution e on $S = \Omega/G$, a lifting of the hyperelliptic involution on R . If we denote by a and b the conformal involutions in $H = K/G$ induced by A and B , respectively, equality (1) asserts that $\langle a, b, e \rangle \cong \oplus^3 \mathbb{Z}_2$. This observation and Proposition 3.2 ensure the following.

Corollary 3.5. *Let $(S, H) \in \mathcal{F}_{5,0,0,0}^1$. Then the hyperelliptic involution on S/H lifts to a conformal involution on S , which together H generate a group of conformal automorphisms of S isomorphic to \mathbb{Z}_2^3 . The lifted involution has exactly 8 fixed points on S .*

3.2. The family $\mathfrak{F}_{g,0,0,0}^2$

As $g \equiv 1 \pmod 4$ and $g > 1$, we have that $\gamma = (g + 3)/4 \in \{2, 3, \dots\}$. Let us consider the circles (see figure 2 for $\gamma = 2$)

$$\begin{aligned} \Sigma_1 &= \{z \in \mathbb{C} : |z| = 1\}, \\ \Sigma_2 &= \{z \in \mathbb{C} : |z| = 5\}, \\ \Sigma_j, \quad j &= 3, 4, \dots, \gamma + 1, \end{aligned}$$

so that

- (i) each circle Σ_j , for $j = 3, \dots, \gamma + 1$, is orthogonal to the imaginary axis,
- (ii) each circle is contained in the upper-half plane $\{\text{Im}(z) > 0\}$,
- (iii) each circle is contained in the annulus $\{1 < |z| < 5\}$,
- (iv) all circles are pairwise disjoint, and
- (v) no circle Σ_j , for $j \geq 3$, separates any of them from Σ_1 .

Let σ_j be the reflection on Σ_j , for $j = 1, \dots, \gamma + 1$. If $\sigma_{00}(z) = \bar{z}$ and $\sigma_{01}(z) = -\bar{z}$, then we set

$$\begin{aligned} J(z) &= -z, \\ T(z) &= \frac{-1}{z}, \\ L(z) &= \sigma_{01} \circ \sigma_2(z), \\ N_j(z) &= \sigma_{01} \circ \sigma_j(z), \quad j = 3, \dots, \gamma + 1. \end{aligned}$$

If we denote by K_0 the group generated by $J, T, L, N_3, \dots, N_{\gamma+1}$, then we have the presentation

$$\begin{aligned} K_0 &= \langle J, T, L, N_3, \dots, N_{\gamma+1} : \\ & \quad J^2 = T^2 = L^2 = N_3^2 = \dots = N_{\gamma+1}^2 = (J \circ T)^2 = (J \circ L)^2 = I \rangle. \end{aligned}$$

In figure 2 there is shown (the dashed part) a fundamental domain for K_0 in the case $\gamma = 2$. If Ω_0 denotes the region of discontinuity of K_0 , then Ω_0 is connected and Ω_0/K_0 has signature $(0; \underbrace{2, \dots, 2}_{2\gamma+2})$. If we denote by K_1 the group generated by J ,

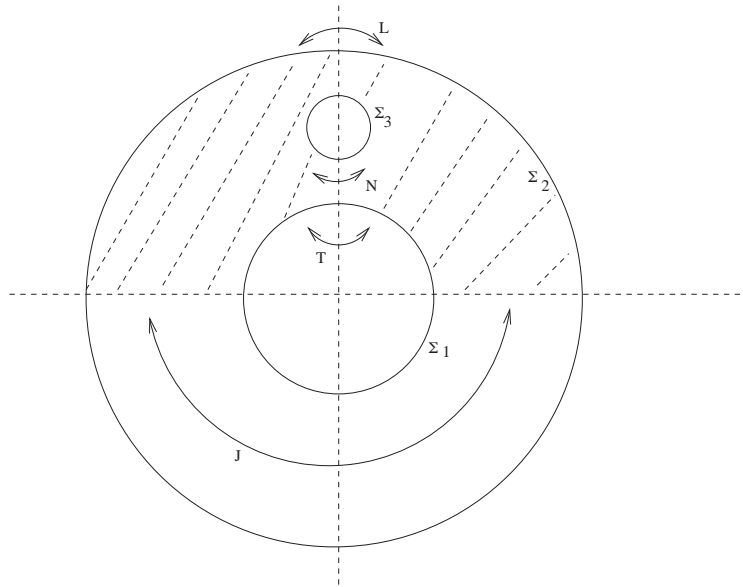


Figure 2

$A = L \circ T$, $B_j = T \circ N_j$, for $j = 3, \dots, \gamma + 1$, then we have $K_1 \triangleleft K$ and of index 2. The quotient $R_0 = \Omega_0/K_1$ is a hyperelliptic closed Riemann surface of genus γ , where the involution T induces the hyperelliptic involution on R_0 .

Inside K_0 we have a Schottky group G_0 , of genus $g = 4(\gamma - 1) + 1$, generated by the following Schottky generators (see figure 3 for $\gamma = 2$):

$$\begin{cases} A_1 = A^2, \\ A_{j-1} = B_j, & j = 3, \dots, \gamma + 1, \\ A_{\gamma+j-2} = J \circ A_{j-1} \circ J, & j = 3, \dots, \gamma + 1, \\ A_{2\gamma+j-3} = T \circ L \circ T \circ A_{j-1} \circ L, & j = 3, \dots, \gamma + 1, \\ A_{3\gamma+j-4} = J \circ A_{2\gamma+j-3} \circ J, & j = 3, \dots, \gamma + 1. \end{cases} \quad (4)$$

It is not hard to see the following facts:

- $G_0 \triangleleft K_0$,
- $G_0 \triangleleft K_1$,
- $K_1/G_0 \cong \mathbb{Z}_2^2$,
- $K_0/G_0 \cong \mathbb{Z}_2^3$,

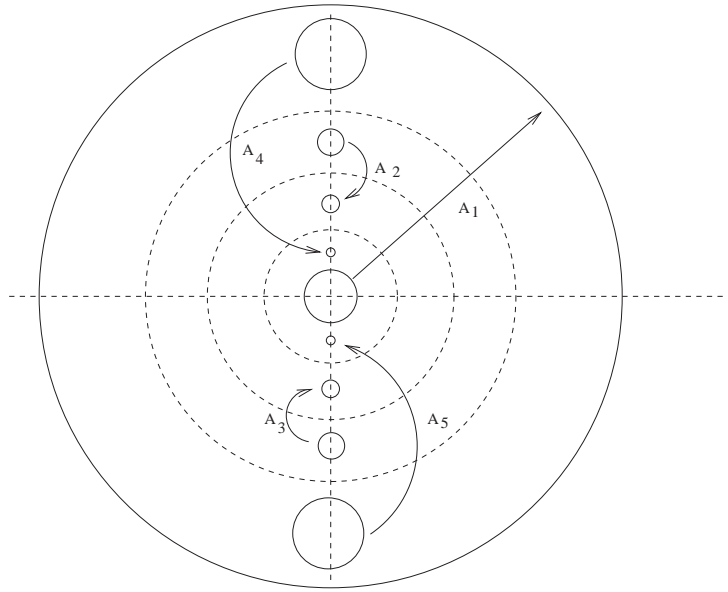


Figure 3

- G_0 is a hyperelliptic Schottky group, of genus g , respect to the involution T ,
- $(\Omega/G_0, K_0/G_0) \in \mathcal{F}_{g,0,0,0}^2$.

Any group obtained by quasiconformal deformation of K_1 will be called a *Klein type Schottky group*. (The reason for the name is that the group K_1 contains a Schottky group G_0 so that K_1/G_0 is the four Klein group.)

It follows from the above that if \hat{K} is a Klein type Schottky group, then

- (i) \hat{K} has a presentation of the form

$$\hat{K} = \langle J, A, B_3, \dots, B_{\gamma+1} : J^2 = J \circ A \circ J \circ A^{-1} = 1 \rangle,$$

where the subgroup

$$\hat{K}^* = \langle A, B_3, \dots, B_{\gamma+1} \rangle,$$

is a Schottky group of genus γ ,

- (ii) \hat{K} contains a Schottky group of genus g , which we denote by $G_{\hat{K}}$, having a set of Schottky generators as in (4),
- (iii) if Ω denotes the region of discontinuity of K , then

$$(S_{\hat{K}} = \Omega/G_{\hat{K}}, H_{\hat{K}} = \hat{K}/G_{\hat{K}}) \in \mathcal{F}_{g,0,0,0}^2,$$

- (iv) if $S_{\widehat{K}}$ is hyperelliptic, then there is a quasiconformal deformation of K_0 , say K , so that \widehat{K} is the quasiconformal image of the respective K_1 . In particular, the Schottky group $G_{\widehat{K}}$ is a hyperelliptic Schottky group respect to the conformal involution corresponding to T . Let us observe that in the case that if we consider the quasiconformal deformation above to be normalized so that it fixes 0, 1, and ∞ , then the transformations J is sent to J and T is sent to a conformal involution T^* commuting with J and having 1 as fixed point. It follows that $T^* = T$.

We have then obtained, as consequence of the above and Theorem 2.3 the following.

Theorem 3.6. *If $g \equiv 1 \pmod 4$ and $g > 1$, then $\mathcal{F}_{g,0,0,0}^2 \neq \emptyset$. If $(S, H) \in \mathcal{F}_{g,0,0,0}^2$, then there is a Klein type Schottky group \widehat{K} , with region of discontinuity $\Omega_{\widehat{K}}$, so that $(\Omega_{\widehat{K}}/G_{\widehat{K}}, \widehat{K}/G_{\widehat{K}})$ is holomorphically equivalent to (S, H) . Moreover, if the surface S/H is hyperelliptic, then the surface S is also hyperelliptic. In particular, for $g = 5$ we have that if $(S, H) \in \mathcal{F}_{5,0,0,0}^2$, then S is hyperelliptic.*

4. Schottky uniformizations for the general situation

4.1. Admissible five-tuples

Let us consider a triple $(r, s, t) \in N_g$. As the values r, s, t all have the same parity, we may write

$$\begin{aligned} r &= n_1 + 2n_2, \\ s &= n_1 + 2n_3, \\ t &= n_1 + 2n_4, \end{aligned} \tag{5}$$

where

$$\gamma, n_2, n_3, n_4 \in \{0, 1, 2, \dots\}, \quad n_1 \in \{0, 1\}.$$

The rest of conditions (2) on the triple (r, s, t) correspond to the following ones:

$$\begin{cases} 0 \leq n_2 \leq n_3 \leq n_4 \leq (g + 1 - n_1)/2; \\ 3n_1 + 2(n_2 + n_3 + n_4) \leq g + 3; \\ 3n_1 + 2(n_2 + n_3 + n_4) \equiv (g + 3) \pmod 4; \\ \text{if } n_1 + 2n_4 = g + 1, \text{ then } n_1 + 2n_3 \leq 2. \end{cases} \tag{6}$$

An admissible five-tuple is a tuple

$$(\gamma, n_1, n_2, n_3, n_4),$$

satisfying properties (5) and (6) for some $g \geq 0$.

Lemma 4.1. *The complete list of admissible five-tuples is the following:*

- (i) $(\gamma, 1, n_2, n_3, n_4)$, with $\gamma \geq 0$ and $n_2 \leq n_3 \leq n_4$,
- (ii) $(\gamma, 0, n_2, n_3, n_4)$, with $\gamma \geq 1$ and $n_2 \leq n_3 \leq n_4$, and
- (iii) $(0, 0, n_2, n_3, n_4)$, with $n_3 \geq 1$.

Proof. (i) Let us consider a tuple $(\gamma, 1, n_2, n_3, n_4)$ so that $n_2 \leq n_3 \leq n_4$. With

$$g = 4\gamma + 2(n_2 + n_3 + n_4),$$

we obtain that such a tuple is in fact admissible.

(ii) Let us consider a tuple $(\gamma, 0, n_2, n_3, n_4)$ so that $\gamma \geq 1$ and $n_2 \leq n_3 \leq n_4$. With

$$g + 3 = 4\gamma + 2(n_2 + n_3 + n_4),$$

we obtain that such a tuple is in fact admissible.

(iii) Let us consider a tuple $(0, 0, n_2, n_3, n_4)$ so that $n_2 \leq n_3 \leq n_4$. If we want such a tuple to be admissible, we need to have the existence of some $g \geq 0$ so that

$$g + 3 = 2(n_2 + n_3 + n_4),$$

in particular, we need to have $n_2 + n_3 + n_4 \geq 2$. All conditions are valid except for the condition $n_4 \leq (g + 1)/2$. To have it valid, we need the condition $n_2 + n_3 \geq 1$, that is, $n_3 \geq 1$. □

4.2. The idea of the construction

We proceed, for any admissible five-tuple $(\gamma, n_1, n_2, n_3, n_4)$, as follows:

- (P1) We construct, by use of the Maskit-Klein combination theorem [17], a Kleinian group \tilde{K} , with region of discontinuity Ω , containing a Schottky group F as a normal subgroup, so that
- (P2) $H = \tilde{K}/F$ is isomorphic to \mathbb{Z}_2^2 , and
- (P3) if $S = \Omega/F$, then S/H has genus γ and the corresponding sets \mathcal{R} , \mathcal{S} and \mathcal{T} have cardinalities $r = n_1 + 2n_2$, $s = n_1 + 2n_3$, $t = n_1 + 2n_4$, respectively.

As a consequence, we have a Schottky uniformization $(\Omega, F, P : \Omega \rightarrow S)$, so that S admits \mathbb{Z}_2^2 as group of conformal automorphisms, which can be lifted by the covering map P . Moreover, the pair $(S, H) \in \mathfrak{F}_{g,r,s,t}$, in particular, this gives an alternative proof of Proposition 2.5.

Once we have an explicit example of such Schottky uniformizations (for each possible admissible five-tuple), we get, as a consequence from theorem 2.2 and the quasiconformal deformation theory, that for every pair $(\hat{S}, \hat{H}) \in \mathfrak{F}_{g,r,s,t}$ there is a quasiconformal deformation of the group K so that (\hat{S}, \hat{H}) can be obtained as before. We shall return to this at the end of the constructions.

4.3. Auxiliary regions and groups

In order to make our constructions, we will need the following auxiliary regions,

$$\begin{aligned} R_1 &= \{z \in \mathbb{C} : |z| < 1, \operatorname{Re}(z) > 0\}, \\ R_2 &= \{z \in R_1 : |z| > \frac{1}{8}\}, \\ R_3 &= \{z \in R_2 : |z - \frac{5}{4}| > \frac{3}{4}\}, \end{aligned}$$

and the following Möbius transformations,

$$J(z) = -z, \quad E(z) = \frac{1}{z}, \quad U(z) = \frac{1}{64z}, \quad \text{and} \quad V(z) = \frac{4z + 3}{-3z - 4}.$$

Let us define the group G_1 to be generated by J and E . We have that G_1 is isomorphic to \mathbb{Z}_2^2 . The region R_1 is a fundamental domain (after adding some boundary points) for G_1 . The group G_2 , generated by G_1 and U , has R_2 (after adding some boundary points) as a fundamental domain. The group G_3 , generated by G_2 and V , has R_3 (after adding some boundary points) as a fundamental domain. These three groups are Kleinian groups of the second kind, geometrically finite, with connected region of discontinuity and every non-loxodromic element is elliptic of order two.

4.4. Case 1

Let us consider the five-tuple $(\gamma, 1, n_2, n_3, n_4)$, with $\gamma \geq 0$, $n_2 \leq n_3 \leq n_4$, and $n_j \geq 1$ for $j = 1, 2, 3, 4$.

Let us consider elliptic transformations of order two, $J_1, \dots, J_{n_2}, E_1, \dots, E_{n_3}, T_1, \dots, T_{n_4}$, and loxodromic transformations C_1, \dots, C_γ , such that all their isometric circles (including of those of the inverses of the loxodromic ones) are pairwise disjoint and contained in the region R_1 . We can apply the first Maskit-Klein combination theorem to obtain a group \tilde{K} , generated by G_1 and all the above transformations. Moreover, it has presentation

$$\tilde{K} = G_1 * K_1 * K_2 * K_3 * K_4,$$

where $K_1 = \langle J_1 \rangle * \dots * \langle J_{n_2} \rangle$, $K_2 = \langle E_1 \rangle * \dots * \langle E_{n_3} \rangle$, $K_3 = \langle T_1 \rangle * \dots * \langle T_{n_4} \rangle$, and $K_4 = \langle C_1 \rangle * \dots * \langle C_\gamma \rangle$.

If F is the smallest normal subgroup of \tilde{K} containing the elements $C_l, J \circ J_i, E \circ E_j$ and $J \circ E \circ T_k$ for all l, i, j, k , then F is a Schottky group. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple.

4.5. Case 2

Let us consider the five-tuple $(\gamma, 1, n_2, n_3, n_4)$, where $\gamma \geq 0$, $n_2 \leq n_3 \leq n_4$, and $n_2 + n_3 + n_4 \leq 2$. We need to consider the following five-tuples:

- (i) $(n_2, n_3, n_4) = (0, 0, 0)$,
- (ii) $(n_2, n_3, n_4) = (0, 0, 1)$, and
- (iii) $(n_2, n_3, n_4) = (0, 1, 1)$.

4.5.1. SUBCASE (i)

We consider the region R_1 and loxodromic elements C_1, \dots, C_γ , so that their isometric circles (including of those of C_i^{-1}) are pairwise disjoint and contained in the above region. As a consequence of the first Maskit-Klein combination theorem, the group \tilde{K} generated by G_1 has presentation

$$\tilde{K} = G_1 * \langle C_1 \rangle * \cdots * \langle C_\gamma \rangle.$$

If F is the smallest normal subgroup of \tilde{K} containing the elements C_1, \dots, C_γ , then F is a Schottky group. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple.

4.5.2. SUBCASE (ii)

We consider the region R_3 and loxodromic elements C_1, \dots, C_γ , such that their isometric circles (including those of the inverses) are pairwise disjoint and contained in the above region. Again Maskit-Klein combination theorem asserts that the group \tilde{K} , generated by the group G_3 and the above loxodromic transformations has presentation

$$\tilde{K} = G_3 * \langle C_1 \rangle * \cdots * \langle C_\gamma \rangle.$$

If F is the smallest normal subgroup of \tilde{K} containing the elements C_1, \dots, C_γ , $J \circ E \circ U$, and $J \circ E \circ V$, then F is a Schottky group. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple.

4.5.3. SUBCASE (iii)

We consider the region R_1 , two elliptic transformations of order two, say T_1 and E_1 , and loxodromic transformations C_1, \dots, C_γ , such the isometric circles of these transformations (including inverses of the loxodromic ones) are pairwise disjoint and contained in the above region. The first Maskit-Klein combination asserts that the

group \tilde{K} , generated by G_1 and all the above transformations, is a geometrically finite Kleinian group and it has presentation

$$\tilde{K} = G_1 * \langle E_1 : E_1^2 = 1 \rangle * \langle T_1 : T_1^2 = 1 \rangle * \langle C_1 \rangle * \cdots * \langle C_\gamma \rangle.$$

If F is the smallest normal subgroup of \tilde{K} containing the elements $C_1, \dots, C_\gamma, E \circ E_1$, and $J \circ E \circ T_1$, then F is a Schottky group. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple.

4.6. Case 3

Let us consider the five-tuple $(\gamma, 0, n_2, n_3, n_4)$, where $\gamma \geq 1$ and $n_2 \leq n_3 \leq n_4$.

4.6.1. SUBCASE $(n_2, n_3, n_4) = (0, 0, 0)$

This case was already considered. Anyway, we give a new description of Schottky uniformizations. We have to consider two possibilities. Let $A(z) = 8z$ and consider loxodromic transformations C_1, \dots, C_γ , with isometric circles (including those of their inverses) pairwise disjoint and contained in the region R_2 . Maskit-Klein combination theorem asserts that the group \tilde{K} , generated by $J, A, C_1, \dots, C_\gamma$, is a geometrically finite Kleinian group that has presentation

$$\tilde{K} = \langle J, A : J^2 = 1, AJ = JA \rangle * \langle C_1 \rangle * \cdots * \langle C_{\gamma-1} \rangle.$$

Let F^1 be the smallest normal subgroup of \tilde{K} containing the elements C_1, \dots, C_γ and A^2 , and F^2 be the smallest normal subgroup of \tilde{K} containing the elements $A^2, A \circ C_1, C_2, \dots, C_\gamma$.

If F is any of the two groups F^1 or F^2 , then F is a Schottky group. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple. The choice of either F_1 or F_2 corresponds to the fact that there are two topological actions of \mathbb{Z}_2^2 acting free fixed points.

4.6.2. SUBCASE $n_i \geq 1$, FOR $i = 2, 3, 4$

We consider the transformation $A(z) = 8z$, elliptic transformations of order two, $J_1, \dots, J_{n_2}, E_1, \dots, E_{n_3}, T_1, \dots, T_{n_4}$, and loxodromic transformations C_1, \dots, C_γ , such that all their isometric circles (included those of inverses of the loxodromic ones) are pairwise disjoint and contained in the region R_2 . The Maskit-Klein combination theorem asserts that the group \tilde{K} , generated by J, A , and all the above transformations, is a geometrically finite Kleinian group that has presentation

$$\tilde{K} = \langle J, A : J^2 = 1, A \circ J = J \circ A \rangle * K_1 * K_2 * K_3 * K_4,$$

where $K_1 = \langle J_1 \rangle * \dots * \langle J_{n_2} \rangle$, $K_2 = \langle E_1 \rangle * \dots * \langle E_{n_3} \rangle$, $K_3 = \langle T_1 \rangle * \dots * \langle T_{n_4} \rangle$, and $K_4 = \langle C_1 \rangle * \dots * \langle C_{\gamma-1} \rangle$.

If F is the smallest normal subgroup of \tilde{K} containing the elements $A^2, C_1, \dots, C_\gamma, J \circ J_1, \dots, J \circ J_{n_2}, A \circ E_1, \dots, A \circ E_{n_3}, J \circ A \circ T_1, \dots, J \circ A \circ T_{n_4}$, then F is a Schottky group. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple.

4.7. Case 4

Let us consider the five-tuple $(0, 0, n_2, n_3, n_4)$, where $n_2 + n_3 + n_4 \geq 2$.

4.7.1. SUBCASE $n_i \geq 1$

We consider elliptic transformations of order two, $J_1, \dots, J_{n_2}, E_1, \dots, E_{n_3}, T_1, \dots, T_{n_4}$, such that their isometric circles are pairwise disjoint. The Maskit-Klein combination theorem asserts that the group \tilde{K} , generated by the above transformations, is a geometrically finite Kleinian group with presentation

$$\tilde{K} = \langle J_1 \rangle * \dots * \langle J_{n_2} \rangle * \langle E_1 \rangle * \dots * \langle E_{n_3} \rangle * \dots * \langle T_1 \rangle * \dots * \langle T_{n_4} \rangle.$$

If F the smallest normal subgroup of \tilde{K} containing the elements $J_1 \circ J_2, \dots, J_1 \circ J_{n_2}, E_1 \circ E_2, \dots, E_1 \circ E_{n_3}, T_1 \circ T_2, \dots, T_1 \circ T_{n_4}$, and $J_1 \circ E_1 \circ T_1$, then F is a Schottky group. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple.

4.7.2. SUBCASE $n_2 + n_3 + n_4 = 2$

We need to consider the case $n_2 = 0, n_3 = n_4 = 1$. Let E_1 and T_1 be two elliptic transformations of order two with disjoint isometric circles. The group \tilde{K} , generated by these two transformations, has presentation

$$\tilde{K} = \langle E_1 : E_1^2 = 1 \rangle * \langle T_1 : T_1^2 = 1 \rangle.$$

If F is the smallest normal subgroup of \tilde{K} containing the element $(T_1 \circ E_1)^2$, then F is a Schottky group of genus 1. The pair of groups \tilde{K} and F are the ones we are looking so that (P1), (P2), and (P3) are satisfied for the chosen admissible five-tuple.

4.8. The quasiconformal deformation technique

Let R be a closed Riemann surface of genus g , and let H^* be a group of conformal automorphisms of R . Assume H^* to be isomorphic to \mathbb{Z}_2^2 . We denote by \tilde{R} the quotient Riemann surface R/H^* of genus γ . Set $\pi : R \rightarrow \tilde{R}$ the natural holomorphic branched covering induced by the action of H^* on R . Choose two different elements of H^* of order two, say a and b . The number of fixed points of a , b , and $a \circ b$ have (by Proposition 2.4) the forms $2(n_1 + 2n_2)$, $2(n_1 + 2n_3)$, and $2(n_1 + n_4)$, respectively,

where n_1 is either zero or one and $n_2, n_3,$ and n_4 are non-negative integers. The Riemann-Hurwitz formula asserts that

$$g = 4(\gamma - 1) + 1 + 3n_1 + 2(n_2 + n_3 + n_4),$$

and we also have that the five-tuple $(\gamma, n_1, n_2, n_3, n_4)$ is admissible.

If γ is greater or equal to one, then we construct on \tilde{R} a set of loops as in Lemma 2.7. If the five-tuple has the form $(\gamma, 0, 0, 0, 0)$, then we have one of two possibilities.

We now consider the Schottky uniformization $(\Omega, F, Q : \Omega \rightarrow S)$ (of genus g) constructed in one of the examples above corresponding to the above five-tuple and, in the case of the tuple $(\gamma, 0, 0, 0, 0)$, the one that corresponds to the correct topological action. Together with this uniformization, we have the Kleinian group \tilde{K} , the Riemann surface S , and the conformal group $H = \tilde{K}/F$.

Theorems 2.2 and 2.3 assert the existence of quasiconformal homeomorphisms $\tilde{f} : S/H \rightarrow \tilde{R}$ and $f : S \rightarrow R$, such that $\pi \circ f = \tilde{f} \circ \pi_S$, where $\pi_S : S \rightarrow S/H$ is the natural holomorphic (branched) covering induced by H .

Let us consider the Beltrami differential μ of the map \tilde{f} . We can lift it as a measurable function on a fundamental domain of \tilde{K} . We extend it to all of Ω by the action of \tilde{K} . On the limit set we extend it as zero. We get in this way a Beltrami coefficient for the group \tilde{K} , which we still denoting by μ .

Let $W : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the unique solution, given by Ahlfors-Bers' theorem, of the Beltrami equation given by

$$\begin{cases} \bar{\partial}W(z) = \mu(z)\partial W(z), & \text{a.e. } \hat{\mathbb{C}}, \\ W(0) = 0, \\ W(1) = 1, \\ W(\infty) = \infty. \end{cases}$$

We conjugate the group \tilde{K} by W to obtain the group of linear fractional transformations $\tilde{H} = W \circ \tilde{K} \circ W^{-1}$. Its region of discontinuity is $W(\Omega)$ and $G = W \circ F \circ W^{-1}$ is a Schottky group. Moreover, \tilde{H} contains G as a normal subgroup, and \tilde{H}/G is isomorphic to \mathbb{Z}_2^2 . By the construction of W and section 1, we have that $(W(\Omega), G, \tilde{f} \circ Q \circ W^{-1} : W(\Omega) \rightarrow R)$ is a Schottky uniformization of R for which H^* lifts. As a consequence of all the above is the following.

Theorem 4.2. *Let S be a closed Riemann surface and $H < \text{Aut}(S)$ a group of conformal automorphisms isomorphic to \mathbb{Z}_2^2 . Then, there is a Schottky uniformization of S which is a quasiconformal deformation of one as described in the previous cases for which H lifts.*

5. Ending remark

Let us assume we have a closed Riemann surface S , of genus $g \geq 2$, and $H < \text{Aut}(S)$ so that there is a Schottky uniformization $(\Omega, G, P : \Omega \rightarrow S)$ for which every element of H lifts. Let us denote by K the Kleinian group obtained by lifting all H under P . In this way, $G \triangleleft K$ and $K/G \cong H$. We have the existence [9] of a canonical homology basis

$$\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$$

for which we have a symplectic representation $\rho : H \rightarrow \text{Sp}_{2g}(\mathbb{Z})$ with

$$\rho(h) = \begin{bmatrix} A_h & B_h \\ 0 & {}^t A_h^{-1} \end{bmatrix}$$

and the integral matrix A_h is given by the automorphism $h^* : G \rightarrow G : x \mapsto h \circ x \circ h^{-1}$. In this way, the Riemann period matrix of S defined by the above canonical homology basis is a solution of the linear equation

$$B_h + Z {}^t A_h^{-1} = A_h Z.$$

In the general situation, to obtain a Riemann period matrix one has to solve quadratic equations. This makes interesting to have groups H as above. In our case, we have that for each $H \cong \mathbb{Z}_2^2$ a symplectic representation as above can be found and then, the algebraic form of a Riemann period matrix can be computed. Of course, this is not enough to compute the Riemann period matrix (still to compute the transcendental part). In the case that we have real structures that commute with H , then this can be done [9].

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