Subcanonicity of Codimension Two Subvarieties

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ABSTRACT

We prove that smooth subvarieties of codimension two in Grassmannians of lines of dimension at least six are rationally numerically subcanonical. We prove the same result for smooth quadrics of dimension at least six under some extra condition. The method is quite easy, and only uses Serre's construction, Porteous formula and Hodge index theorem.

Key words: subcanonicity, codimension two, Grassmannians, quadrics.

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Introduction

The main open problem about (complex) subvarieties of small codimension is Hartshorne's conjecture. In the particular case of codimension two, it states that any (smooth) subvariety of codimension two $X \subset \mathbb{P}^n$ must be a complete intersection if $n \geq 6$ (although the original conjecture stated $n > 6$). In [6], Larsen has shown (in a more general framework) that, in these hypotheses, the restriction map $H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is an isomorphism, which implies that X must be subcanonical (previously, Barth had proved in [1] the same result with rational coefficients).

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We are interested in the same problem when changing $\mathbb{P}^n$ with a Grassmannian or a quadric. The interesting case is when the ambient space has dimension six, so that we will focus our attention on $G(1,4)$, the Grassmann variety of lines in $\mathbb{P}^4$, and Q_6, the smooth six-dimensional quadric.

Since on $G(1,4)$ the universal quotient bundle has rank two, we cannot expect that smooth subvarieties of codimension two are complete intersections, although I conjecture that the only other possibility for such a subvariety is to be the zero-locus of a section of a twist of the universal bundle. The only result in this direction is a generalization of Larsen's result for Grassmannians of higher dimension: for a smooth subvariety $X \subset G(1,n)$ of codimension two the restriction map $H^2(G(1,n),\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an isomorphism for $n \geq 6$ (see [2, Theorem 2] for the case $n \geq 7$ and [8] in general). In particular, such X is subcanonical.

On the other hand, Q_6 contains the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$, which is neither a complete intersection nor even subcanonical; I conjecture however that this is the only codimension two subvariety that is not a complete intersection. For higher dimensional quadrics Q_n ($n \geq 7$), [3, Theorem 2.3.11] (or Larsen result for arbitrary codimension) implies that any $X \subset Q_n$ of codimension two is subcanonical.

In this paper we will improve partially the result of Barth and van de Ven by proving that a smooth subvariety $X \subset G(1,4)$ of codimension two is *rationally numerically subcanonical*, i.e. the canonical divisor of X is numerically a rational multiple of the hyperplane section. We will prove the same result for subvarieties of Q_6 but adding an extra numerical condition (obviously some condition was required in order to exclude the counterexample).

The method we will use is very simple and it is implicitly in the literature (see for instance [4]). The idea is to use Serre's construction to regard our variety X as the dependency locus of $r-1$ sections of a vector bundle of rank r on the ambient space. The fact that X is smooth implies that the locus in which these sections have rank $r-3$ is empty, and this will imply numerically that the surface section of X is rationally numerically equivalent to zero. Our results will thus come from Lefschetz hyperplane theorem. These results provide an evidence for the above conjectures, and I hope that this paper will encourage people to work in this direction.

In a first section we will illustrate how the method works in the known case of subvarieties of $\mathbb{P}^n$. In the second section we will prove the result for $G(1,4)$ and $G(1,5)$, while the third section will be devoted for the proof in Q_n. We end with a last section of remarks and conjectures.

1. Subvarieties of projective space

In this section we will see how to prove in a simple way the following corollary of a theorem of Barth and Larsen. This will be the method that we will follow in the rest of the paper when substituting $\mathbb{P}^n$ with a Grassmannian of lines or a smooth quadric.

Proposition 1.1. *Let $X \subset \mathbb{P}^n$ be a smooth subvariety of codimension two. If $n \geq 6$, then X is rationally numerically subcanonical.*

Proof. We consider $S \subset \mathbb{P}^4$ to be the surface obtained as the intersection of X with a general $\mathbb{P}^4 \subset \mathbb{P}^n$. Since S is smooth, we have the relation coming from the double point formula:

$$K_S^2 = \frac{d^2}{2} - \frac{5}{2}d - 5g + 5 + 6\chi(\mathcal{O}_S). \tag{1}$$

Here K_S is the canonical divisor, d is the degree of S (and hence the degree of X) and g is the sectional genus of both S and X.

On the other hand, from Serre's construction, we can obtain X as the dependency locus of $r-1$ sections of a rank r vector bundle F over $\mathbb{P}^n$ (Hartshorne's conjecture is equivalent to say that it is possible to take $r=2$ and that F splits). As usual, we identify the i-th Chern class of F with an integer c_i, in the sense that $c_i(F) = c_i H^i$, where H is the hyperplane class of $\mathbb{P}^n$. We have in particular a Koszul exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus r-1} \to F \to \mathcal{O}_{\mathbb{P}^n}(c_1) \to \mathcal{O}_X(c_1) \to 0. \tag{2}$$

Restricting the above exact sequence to a general $\mathbb{P}^4$ and using Riemann-Roch Theorem we can compute the Hilbert polynomial $\chi(\mathcal{O}_S(l))$ in terms of c_1, c_2, c_3, c_4. Identifying the coefficients of this polynomial with the standard Hilbert polynomial of a surface, we get the relations:

$$c_2 = d,$$
$$c_3 = -c_1 d + 4d + 2g - 2,$$
$$c_4 = \frac{1}{2}d^2 + \frac{25}{2}d + 15g - 15 + dc_1^2 - 8c_1 d - 4c_1 g + 4c_1 + 6\chi(\mathcal{O}_S).$$

And now the tricky —but simple— part comes. Since the locus X in which the map $\mathcal{O}_{\mathbb{P}^n}^{\oplus r-1} \to F$ of (2) has rank at most $r-2$ is smooth, this map cannot have points in which the rank is less than or equal to $r-3$. This means that the corresponding Porteous cycle must be zero in the degree six part of the Chow ring of $\mathbb{P}^n$. By Porteous formula, this means that $c_3^2 - c_2 c_4 = 0$. After making the above substitutions for c_2, c_3, c_4, we first see that this equality does not depend on c_1. But on the other hand, it surprisingly becomes the identity $(K_S H_S)^2 - K_S^2 H_S^2 = 0$ when using (1). Therefore, the Hodge index theorem implies that $K_S - qH_S$ is numerically equivalent to zero for some rational number q. On the other hand, the Lefschetz hyperplane theorem implies that the restriction map $H^2(X, \mathbb{Q}) \to H^2(S, \mathbb{Q})$ is injective. We have just seen that the image of the class of $K_X + (-q + n - 4)H_X$ is zero (recall that $K_S = (K_X + (n-4)H_X)_{|S}$ by the adjunction formula). Therefore, K_X is numerically equivalent to $(q - n + 4)H_X$, as wanted. □

2. Subvarieties of Grassmannians of lines

We prove here the analogue of Proposition 1.1 for Grassmannians of lines. We study first the case in which the Grassmannian has dimension six, i.e. $G(1,4)$, the Grassmannian of lines in $\mathbb{P}^4$.

Theorem 2.1. *Any smooth subvariety $X \subset G(1,4)$ of codimension two is rationally numerically subcanonical.*

Proof. We imitate the proof of Proposition 1.1 for the projective case, but we need to have in mind that there are several natural choices of surfaces inside X, not only linear sections. First of all, recall that the Chow group of degree two of $G(1,4)$ is generated by two elements: the Schubert cycle $\Omega(1,4)$ of all the lines meeting a fixed line of $\mathbb{P}^4$, and the Schubert cycle $\Omega(2,3)$ of all the lines contained in a fixed hyperplane of $\mathbb{P}^4$. Therefore X has a bidegree (a,b), where a is the number of lines of X contained in a hyperplane $H \subset \mathbb{P}^4$ and passing through a point $p \in H$, and b is the number of lines of X contained in a plane of $\mathbb{P}^4$. We can therefore consider the following surfaces of X:

- A general linear section S, which will have, as a surface in the Plücker ambient space, degree $3a+2b$. We will denote by g its sectional genus. By adjunction formula, we have $K_S = (K_X + 2H_X)_{|S}$.

- A surface $S_1 \subset G(1,3)$, consisting of the set of lines of X contained in a general hyperplane $H \subset \mathbb{P}^4$ (we will identify this hyperplane with $\mathbb{P}^3$). As a surface in $G(1,3)$, it has bidegree (a,b) (a being the number of lines of S_1 passing through a given point of $\mathbb{P}^3$, and b being the number of lines of S_1 contained in a given plane of $\mathbb{P}^3$). Its degree as a projective surface is $a+b$, and we will denote with g_1 its sectional genus. This surface is obtained from X as the zero locus of the restriction to X of the section of Q (universal quotient bundle of $G(1,4)$ of rank two) defining H. Therefore, $K_{S_1} = (K_X + H_X)_{|S_1}$.

- A surface S_2 consisting of the set of lines of X meeting a general given line of $\mathbb{P}^4$.

The adjunction properties of the two first surfaces makes reasonable to write all the invariants in terms of the invariants of S and S_1. It will be useful for this purpose to observe that a special linear section S —namely the set of lines meeting two planes Π_1, Π_2 meeting along a line L— splits as $S_1 \cup S_2$, the intersection of S_1 and S_2 being the curve C_1 of lines meeting the line L and contained in the hyperplane spanned by Π_1 and Π_2. The curve C_1 is hence a hyperplane section of S_1.

To obtain the double point formulas for S and S_1, it is enough to multiply the self-intersection formula $c_2(N) = a\Omega(1,4)_{|X} + b\Omega(2,3)_{|X}$ (N being the normal bundle of X in $G(1,4)$) with H_X^2 and $\Omega(1,4)_{|X}$ respectively. We obtain thus

$$K_S^2 = a^2 + ab + \frac{1}{2}b^2 - 2a - \frac{3}{2}b - 3g + 3 + 6\chi(\mathcal{O}_S), \tag{3}$$

$$K_{S_1}^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{3}{2}a - \frac{3}{2}b - 4g_1 + 4 + 6\chi(\mathcal{O}_{S_1}). \tag{4}$$

On the other hand we write X as the dependency locus of $r-1$ sections of a vector bundle F of rank r over $G(1,4)$. We represent the first four Chern classes of F by integers as follows:

- $c_1(F) = c_1\Omega(2,4)$, where $\Omega(2,4)$ is the Schubert cycle of all the lines meeting a given plane, i.e. the hyperplane class H of $G(2,4)$.
- $c_2(F) = c_{21}\Omega(1,4) + c_{22}\Omega(2,3)$.
- $c_3(F) = c_{31}\Omega(0,4) + c_{32}\Omega(1,3)$, where $\Omega(0,4)$ is the Schubert cycle of lines passing through a point, and $\Omega(1,3)$ is the Schubert cycle of lines meeting a line L and contained in a hyperplane $H \supset L$.
- $c_4(F) = c_{41}\Omega(0,3) + c_{42}\Omega(1,2)$, where $\Omega(0,3)$ is the Schubert cycle of lines contained in a hyperplane and passing through a point of it, and $\Omega(1,2)$ is the Schubert cycle of lines contained in a plane.

Restricting the exact sequence $0 \to \mathcal{O}_{G(1,4)}^{\oplus r-1} \to F \to \mathcal{O}_{G(1,4)}(c_1) \to \mathcal{O}_X(c_1) \to 0$ to the intersection of two hyperplanes of $G(1,4)$ and to a Schubert cycle $\Omega(2,3)$ and computing in this way the Hilbert polynomials of S and S_1 we get relations:

$$\begin{aligned}
c_{21} &= a,\\
c_{22} &= b,\\
c_{31} &= -2b - c_1a + 2g - 4g_1 + 2,\\
c_{32} &= 3a + 3b - c_1b - c_1a + 2g_1 - 2,\\
c_{41} &= \frac{a^2}{2} + ab + \frac{a}{2} - 2b - 6c_1a - 2c_1b + 2c_1^2a + c_1^2b + 9g - 12g_1,\\
c_{42} &= \frac{a^2}{2} + \frac{b^2}{2} + \frac{13}{2}a + \frac{13}{2}b - 6c_1a - 6c_1b + c_1^2a + c_1^2b + 12g_1.
\end{aligned}$$

Using these expressions, the equality $c_3(F)^2 - c_2(F)c_4(F) = 0$ becomes

$$\begin{aligned}
-\frac{a^3}{2} - \frac{3}{2}a^2b - \frac{b^3}{2} + \frac{17}{2}a^2 + \frac{27}{2}ab + \frac{13}{2}b^2 - 15a - 8b\\
- 9ag - 8bg + 24ag_1 + 16bg_1 + 4g^2 - 16g_1g + 20g_1^2 + 8g - 24g_1 + 8\\
+ 6a\chi(\mathcal{O}_{S_1}) - 6b\chi(\mathcal{O}_{S_1}) - 6a\chi(\mathcal{O}_S) = 0.
\end{aligned} \tag{5}$$

We are not as lucky as in the proof of Proposition 1.1, since this time equality (5) is not the Hodge inequality for S. Our strategy now is to use Hodge index theorem for S_1 and S_2 but, instead of using their respective canonical divisors, using the restriction to them of the canonical divisor of X (this does not make any difference for S_1, because it is subcanonical, but for S_2 there is a crucial difference). We thus have that $((K_X)_{|S_1}H_{S_1})^2-((K_X)_{|S_1})^2(H_{S_1})^2\geq 0$ becomes

$$\begin{aligned}-\frac{a^3}{2}-\frac{1}{2}a^2b-\frac{1}{2}b^2a-\frac{b^3}{2}+\frac{5}{2}a^2+5ab+\frac{5}{2}b^2+4g_1^2-8g_1+4&\\ -6a\chi(\mathcal{O}_{S_1})-6b\chi(\mathcal{O}_{S_1})\geq 0,&\quad (6)\end{aligned}$$

while $((K_X)_{|S_2}H_{S_2})^2-((K_X)_{|S_2})^2(H_{S_2})^2\geq 0$ becomes

$$\begin{aligned}-a^3-\frac{5}{2}a^2b-ab^2+8a^2+\frac{19}{2}ab+3b^2-6a-3b&\\ -6ag-5bg+12ag_1+8bg_1+4g^2-8g_1g+4g_1^2&\\ -12a\chi(\mathcal{O}_S)-6b\chi(\mathcal{O}_S)+12a\chi(\mathcal{O}_{S_1})+6b\chi(\mathcal{O}_{S_1})\geq 0&\quad (7)\end{aligned}$$

(to obtain these relations we have to use (3) and (4), or even better the self-intersection formula itself).

We now want to compare inequalities (6) and (7) with equality (5). To this purpose we try to eliminate $\chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_{S_1})$. We thus multiply (6) by $b(2a+b)$ and (7) by $a(a+b)$ and subtract (5) multiplied by $(a+b)(2a+b)$. We find the beautiful surprise that the result of this operation, which must be a non negative number, becomes

$$-(-3a^2-5ab-2b^2+4a+2b+2ag+2bg-6ag_1-4bg_1)^2. \qquad (8)$$

Therefore, this last expression is zero (it is not difficult to see that for subcanonical varieties this vanishing holds, since this is equivalent to say that the canonical divisors of the hyperplane sections of S and S_1 are multiples of their respective hyperplane section and these multiples are related with each other). Also the inequalities (6) and (7) become equalities. (It is easy to see that $b>0$, and $a=0$ only if X is a $G(1,3)$, which is obviously subcanonical.) If we compute now $(K_SH_S)^2-K_S^2H_S^2$ it becomes

$$\begin{aligned}-3a^3-5a^2b-\frac{7ab^2}{2}-b^3+15a^2+\frac{41ab}{2}+7b^2+3a+2b-3ag-2bg&\\ +4g^2-8g+4-18a\chi(\mathcal{O}_S)-12b\chi(\mathcal{O}_S)&\end{aligned}$$

which is seen to be (6) multiplied by $\frac{3a+2b}{a+b}$ plus (7) multiplied by $\frac{3a+2b}{2a+b}$ minus (8) divided by $(a+b)(2a+b)$, and hence it is zero. The result follows now from Hodge index theorem and Lefschetz hyperplane theorem, as in the projective case. □

Unfortunately we cannot deduce from the above result the subcanonicity in codimension two for any $G(1,n)$. Indeed, the canonical way would be to use induction on n, regard $G(1,n-1)$ inside $G(1,n)$ as the set of lines lying in a hyperplane of $\mathbb{P}^n$ and restrict any $X \subset G(1,n)$ to $G(1,n-1)$. However, $G(1,n-1)$ is obtained as the zero locus of a section of the rank two quotient bundle $\mathcal{Q}$ on $G(1,n)$. But neither $\mathcal{Q}$ nor its restriction to X are ample, and therefore there is no known analogue of the Lefschetz hyperplane theorem to guarantee that the map $H^2(X,\mathbb{Q}) \to H^2(X \cap G(1,n-1),\mathbb{Q})$ is injective (see [7]).

Nevertheless, as mentioned in the introduction, it is known that, if $n \geq 6$, the Picard group of any smooth subvariety $X \subset G(1,n)$ is generated by its hyperplane section, and hence X is subcanonical (even in the strict sense that the canonical divisor is linearly equivalent to an integral multiple of the hyperplane section). We are therefore left with the case $n = 5$. I include here the numerical details of the proof (whose scheme is the same as in the above proofs), which is due to my student Jorge Caravantes.

Theorem 2.2. *Any smooth subvariety $X \subset G(1,5)$ of codimension two is rationally numerically subcanonical.*

Proof. Let the subvariety X have bidegree (a,b), where a is the number of lines of X contained in a fixed general three-dimensional subspace of $\mathbb{P}^5$ and passing through a fixed general point of it, and b is the number of lines of X contained in a general plane of $\mathbb{P}^5$. We can consider the following surfaces inside X (the last three corresponding to the intersection with the three Schubert cycles of $G(1,5)$ of dimension four):

- The surface S obtained as the intersection of X with four general hyperplanes of $\mathbb{P}^{14}$ (the Plücker space of $G(1,5)$). We will denote with g its sectional genus.
- The surface S_1 consisting of the lines of X passing through a fixed general point of $\mathbb{P}^5$. We will denote with g_1 its sectional genus.
- The surface S_2 consisting of the set of lines of X contained in a hyperplane of $\mathbb{P}^5$ and meeting a line of it.
- The surface S_3 consisting of the lines of X contained in a fixed general three-dimensional subspace of $\mathbb{P}^5$. It is not difficult to see that its sectional genus can be derived from g, g_1, a, and b.

We write again X as the dependency locus of $r-1$ sections of a vector bundle F of rank r over $G(1,5)$. The equation $c_3(F)^2 - c_2(F)c_4(F) = 0$ produces now two identities when intersecting with the two Schubert cycles of codimension two of $G(1,5)$:

$$\begin{aligned}
-\frac{1}{2}a^3 - \frac{3}{2}a^2b - \frac{1}{2}b^3 + \frac{493}{50}a^2 + \frac{141}{10}ab + \frac{13}{2}b^2 - \frac{439}{25}a - \frac{24}{5}b - \frac{38}{25}ag \\
- \frac{8}{5}bg + \frac{477}{25}ag_1 + \frac{32}{5}bg_1 + \frac{4}{25}g^2 - \frac{32}{25}gg_1 + \frac{164}{25}g_1^2 + \frac{24}{25}g - \frac{296}{25}g_1 \\
+ \frac{136}{25} - 2a\chi(\mathcal{O}_S) + 2a\chi(\mathcal{O}_{S_1}) + 4a\chi(\mathcal{O}_{S_3}) - 6b\chi(\mathcal{O}_{S_3}) = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
-\frac{3}{2}a^3 - \frac{3}{2}a^2b - \frac{3}{2}ab^2 + \frac{303}{50}a^2 - \frac{2}{5}ab - 2b^2 \\
+ \frac{136}{25}a + \frac{1}{5}b - \frac{18}{25}ag + \frac{2}{5}bg - \frac{118}{25}ag_1 - \frac{3}{5}bg_1 + \frac{4}{25}g^2 \\
+ \frac{8}{25}gg_1 + \frac{4}{25}g_1^2 + \frac{16}{25}g - \frac{16}{25}g_1 + \frac{16}{25} - 2a\chi(\mathcal{O}_S) - 2b\chi(\mathcal{O}_S) \\
- 4a\chi(\mathcal{O}_{S_1}) + 2b\chi(\mathcal{O}_{S_1}) - 2a\chi(\mathcal{O}_{S_3}) + 4b\chi(\mathcal{O}_{S_3}) = 0.
\end{aligned} \tag{10}$$

On the other hand, the Hodge inequality for the restriction of K_X and H_X to S_1, S_2 and S_3 yield (after using the appropriate double-point formulas) the following three inequalities:

$$-\frac{1}{2}a^3 + \frac{7}{2}a^2 - a + ag_1 + 4g_1^2 - 8g_1 + 4 - 6a\chi(\mathcal{O}_{S_1}) \geq 0, \tag{11}$$

$$\begin{aligned}
-a^3 - \frac{5}{2}a^2b - ab^2 - \frac{236}{25}a^2 - \frac{89}{10}ab - 2b^2 - \frac{14}{25}a + \frac{1}{5}b \\
+ \frac{32}{25}ag + \frac{2}{5}bg - \frac{18}{25}ag_1 - \frac{3}{5}bg_1 + \frac{4}{25}g^2 + \frac{8}{25}gg_1 \\
+ \frac{4}{25}g_1^2 - \frac{16}{25}g - \frac{16}{25}g_1 + \frac{16}{25} - 4a\chi(\mathcal{O}_S) - 2b\chi(\mathcal{O}_S) \\
+ 4a\chi(\mathcal{O}_{S_1}) + 2b\chi(\mathcal{O}_{S_1}) + 8a\chi(\mathcal{O}_{S_3}) + 4b\chi(\mathcal{O}_{S_3}) \geq 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
-\frac{1}{2}a^3 - \frac{1}{2}a^2b - \frac{1}{2}ab^2 - \frac{1}{2}b^3 + \frac{1093}{50}a^2 + \frac{113}{5}ab + \frac{13}{2}b^2 - \frac{264}{25}a - \frac{24}{5}b \\
- \frac{88}{25}ag - \frac{8}{5}bg + \frac{352}{25}ag_1 + \frac{32}{5}bg_1 + \frac{4}{25}g^2 - \frac{32}{25}gg_1 + \frac{64}{25}g_1^2 + \frac{24}{25}g - \frac{96}{25}g_1 \\
+ \frac{36}{25} - 6a\chi(\mathcal{O}_{S_3}) - 6b\chi(\mathcal{O}_{S_3}) \geq 0.
\end{aligned} \tag{13}$$

In this case we can see immediately that the sum of the left-hand sides of (9) and (10) coincides with the sum of the left-hand sides of (11), (12), and (13). Therefore, inequalities (11), (12), and (13) must be in fact equalities.

To see that $(K_SH_S)^2-K_S^2H_S^2$ is zero, we multiply its expression (using the double-point-formulas) by $4a^3+6a^2b+2ab^2$ and subtract the zero expressions $36a^3+74a^2b+48ab^2+10b^3$ times (11), $54a^3+84a^2b+30ab^2$ times (12) and $72a^3+76a^2b+2ab^2$ times (13). In this way we get

$$-\frac{2}{25}(13a+5b)(-27a^2-15ab+16a+10b+2ag-18ag_1-10bg_1)^2.$$

Since it has to be non-negative, this immediately implies that $(K_SH_S)^2-K_S^2H_S^2$ is zero, and the result follows once more by Hodge index theorem and Lefschetz hyperplane theorem. $\square$

Putting together the results of this section we get the following

Corollary 2.3. *Let $X\subset G(1,n)$ be a smooth codimension two subvariety. If $n\geq 4$, then X is rationally numerically subcanonical.*

3. Subvarieties of smooth quadrics

To understand the situation we start with the following example.

Example 3.1. Let X be the image of the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^3$ in $\mathbb{P}^7$. In coordinates, it can be viewed as the set of points $(x_0:\cdots:x_7)$ for which the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3\\ x_4 & x_5 & x_6 & x_7\end{pmatrix}$$

has rank one. Then X is contained in a smooth quadric Q_6, for instance the one of equation $X_0X_5-X_1X_4+X_2X_7-X_3X_6=0$. Therefore one cannot hope that all the smooth subvarieties of Q_6 of codimension two are subcanonical, even numerically and/or rationally. Notice that if in this example we intersect X with the linear space $X_0=X_2=X_4=X_6=0$ (which is contained in Q_6) we get a smooth conic, hence of genus zero, while if we intersect with $X_0=X_1=X_6=X_7=0$ (also contained in Q_6) we get the disjoint union of two lines, hence of genus -1.

Notation. Let $X\subset Q_6$ be a smooth codimension two subvariety of Q_6, the smooth six-dimensional quadric. We will denote by g_1,g_2 the genera of the curves obtained by intersecting X with a three-dimensional linear space of each of the two families of such linear spaces contained in Q_6.

The following result shows how the subcanonicity depends on these two genera.

Theorem 3.2. *Let $X\subset Q_6$ be a smooth codimension two subvariety of Q_6. Then X is rationally numerically subcanonical if and only if $g_1=g_2$.*

Proof. The "only if" part is trivial. Indeed if K_X is numerically equivalent to qH_X, the determinant K_X+6H_X of the normal bundle of X in Q_6 will be numerically equivalent to $(q+6)H_X$. Hence, if C is the intersection of X with a linear space A of dimension three, the determinant of the normal bundle of C in A will have degree $(q+6)\deg(H_C)=(q+6)d$, where d is one half of the degree of X. Since $K_A=-4H_A$, it follows that the degree of the canonical divisor of C is $(q+2)d$, hence independent on which of the two families of linear spaces A belongs to.

For the "if" part we use the same strategy as in the previous sections. Intersecting X with two general hyperplanes of $\mathbb{P}^7$ we obtain a smooth surface S contained in a smooth four-dimensional quadric. The double point formula for S inside this quadric reads

$$K_S^2=d^2-7d-4g_1-4g_2+8+6\chi(\mathcal{O}_S)$$

(recalling that a smooth four-dimensional quadric can be identified with $G(1,3)$, this formula is nothing but (4) with $a=b=d$ and having in mind that the sectional genus of X can be written as g_1+g_2+d-1 by choosing a linear space of codimension three intersecting Q_6 in the union of two linear spaces of dimension three meeting along a plane).

We write now X as the dependency locus of $r-1$ sections of a vector bundle F of rank r over Q_6. As before, if $c_1(F)=c_1H_{Q_6}$, the first Chern classes of F become:

$$\begin{gathered}c_2(F)=dH_{Q_6}^2,\\ c_3(F)=(4d-c_1d+2g_1-2)A_1+(4d-c_1d+2g_2-2)A_2,\end{gathered}$$

where, for $i=1,2$, A_i is the class of a linear space of dimension three meeting X in a curve of genus g_i, and

$$c_4(F)=\big(d^2+25d+2c_1^2d-16c_1d+12g_1+12g_2-24-4c_1g_1-4c_1g_2+8c_1+6\chi(\mathcal{O}_S)\big)A_1H.$$

With these expressions, the relation $c_3(F)^2-c_2(F)c_4(F)=0$ becomes

$$-d^3+7d^2-8d+4dg_1+4dg_2+8g_1g_2-8g_1-8g_2+8-6d\chi(\mathcal{O}_S)=0, \tag{14}$$

while the inequality $(K_SH_S)^2-K_S^2H_S^2\geq 0$ becomes

$$-d^3+7d^2-8d+4dg_1+4dg_2+2g_1^2+4g_1g_2+2g_2^2-8g_1-8g_2+8-6d\chi(\mathcal{O}_S)\geq 0,$$

which, using the equality (14), can be written as $2(g_1-g_2)^2\geq 0$. We therefore have equality in the Hodge inequality if $g_1=g_2$, which proves the result. ☐

4. Final remarks

In the results of this paper, one should expect to be able to replace "rationally numerically equivalent" with "linearly equivalent", as it happens for sufficiently big dimension. In particular I conjecture the following:

Conjecture 4.1. *Let $X \subset G(1,n)$ be a smooth subvariety of codimension two. If $n \geq 4$, then $\operatorname{Pic}(X)$ is generated by the hyperplane section class.*

Conjecture 4.2. *Let $X \subset Q_n$ be a smooth subvariety of codimension two. If $n \geq 6$, then $\operatorname{Pic}(X)$ is generated by the hyperplane section class except if $n = 6$ and $g_1 \neq g_2$.*

Such results would be a first step towards an analogue of Hartshorne's conjecture, which I state in the following way:

Conjecture 4.3. *Let $X \subset G(1,n)$ be a smooth subvariety of codimension two. If $n \geq 4$, then X is either the complete intersection of $G(1,n)$ with two hypersurfaces or the zero locus of a section of a twist of the universal rank-two quotient bundle of $G(1,n)$.*

Conjecture 4.4. *Let $X \subset Q_n$ be a smooth subvariety of codimension two. If $n \geq 6$, then X is the complete intersection of Q_n with two hypersurfaces except if $n = 6$ and $g_1 \neq g_2$.*

Observe that these are not just an analogue of Hartshorne's conjecture, but in fact they would imply it. To see this, it is enough to consider finite maps from $G(1,4)$ and Q_6 to $\mathbb{P}^6$, and then consider the pull-back by these maps of any smooth codimension two subvariety.

In the case of quadrics, Conjectures 4.2 and 4.4 can be strengthen with the following (probably very optimistic):

Conjecture 4.5. *Let $X \subset Q_6$ be a smooth subvariety of codimension two. If $g_1 \neq g_2$, then X is as in Example 3.1.*

A natural question is whether similar results are true for other ambient varieties of dimension at least six. The first natural choice would be arbitrary Grassmannians (as in [2] and [8]). For example, Theorem 2.1 can be interpreted as saying that any smooth codimension two subvariety of $G(2,4)$ is rationally numerically subcanonical (and one can conjecture for instance that moreover its Picard group is generated by its hyperplane section). The problem to extend this to arbitrary Grassmannians of planes is the same we found in section 2 for Grassmannians of lines. The general question is:

Problem 4.6. *Let $X \subset G(k,n)$ be a smooth subvariety. Find conditions under which the restriction maps $H^i(X,\mathbb{Q}) \to H^i(X \cap G(k,n-1),\mathbb{Q})$ and $H^i(X,\mathbb{Z}) \to H^i(X \cap G(k,n-1),\mathbb{Z})$ are injective.*

Observe that $X \cap G(k,n-1)$ is given as the zero locus of the restriction to X of a section of the rank-$(k+1)$ universal quotient bundle of $G(k,n)$. However, in the cases we are interested in, this bundle is not ample, so that we cannot apply [7, Prop. 1.1.6], which would imply a positive answer to the problem.

If Problem 4.6 had a positive answer if $i = 2$ and $\dim G(k, n-1) \geq 6$, this would imply for example the subcanonicity for any subvariety of codimension two in $G(k, n)$ with $\dim G(k, n) \geq 6$. It would be enough to intersect the subvariety with a general $G(k, k+2)$ and apply Corollary 2.3 (or restrict to $G(k, k+1)$ and apply Barth-Larsen result).

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