

The Range of a Contractive Projection in $L_p(H)$

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ABSTRACT

We show that the range of a contractive projection on a Lebesgue-Bochner space of Hilbert valued functions $L_p(H)$ is isometric to a ℓ_p -direct sum of Hilbert-valued L_p -spaces. We explicit the structure of contractive projections. As a consequence for every $1 < p < \infty$ the class \mathcal{C}_p of ℓ_p -direct sums of Hilbert-valued L_p -spaces is axiomatizable (in the class of all Banach spaces).

Key words: Contractive projections, Vector-valued L_p -spaces.

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Introduction

It was a remarkable achievement in the isometric theory of Banach spaces of the years 1960's to characterize the contractive linear projections of Lebesgue L_p spaces ($p \neq 2$). In the case of L_p spaces of a probability space it was done by Douglas [4] in the case $p = 1$ and Andô [1] in the case $1 < p < \infty$, $p \neq 2$. They showed that the range of such a contractive projection is itself isometric to a L_p space (for the same p , but a different measure space); if moreover the projection is positive then its range is a sublattice of the initial L_p space and is lattice isomorphically isometric to a L_p space. This was extended to the non-sigma-finite measure space setting by Tzafriri ([17]). In the case of a probability space, the structure of contractive projections is

elucidated by Douglas-Andô works: a general contractive projection P on $L_p(\Omega, \Sigma, \mu)$ has the form

$$P = M_\varepsilon \widehat{P} M_\varepsilon^{-1} + V \quad (1)$$

where M_ε is the multiplication operator by a function ε with $|\varepsilon| = \mathbf{1}$, \widehat{P} is a positive contractive projection, and $V = 0$ if $p > 1$, while if $p = 1$, then V is a contraction from L_1 into the range $R(P)$ of P which vanishes on the band generated by $R(P)$. Moreover \widehat{P} is a weighted conditional expectation, i.e. there exist a sub-sigma algebra \mathcal{B} , an element $B \in \mathcal{B}$ and a nonnegative function $w \in L^p$ such that $\mathbb{E}(w^p \mid \mathcal{B}) = \mathbf{1}$ and

$$\widehat{P}f = w\mathbb{E}(\mathbf{1}_B f \cdot w^{p-1} \mid \mathcal{B})$$

for every $f \in L_p$ (in particular if $P\mathbf{1} = \mathbf{1}$ then P is a conditional expectation). This last formula can also be written

$$\widehat{P}f = w\mathbb{E}_\nu(\mathbf{1}_B f w^{-1} \mid \mathcal{B})$$

where \mathbb{E}_ν is the conditional expectation relative to the measure $\nu = w^p \cdot \mu$. If we denote by S the isometric isomorphism $f \mapsto w \cdot f$ of $L_p(\Omega, \Sigma, \nu)$ onto $L_p(\Omega, \Sigma, \mu)$ and by M_B the multiplication operator by the indicator function $\mathbf{1}_B$, we have:

$$\widehat{P} = S M_B \mathbb{E}_\nu(\ \mid \mathcal{B}) S^{-1}. \quad (2)$$

The structure of contractive projections in the non-sigma finite case was treated by Bernau and Lacey ([3]); their main result can be rephrased in saying that if we assume (as we may) that the measure space (Ω, Σ, μ) is localizable ([7]) then formulas (1) and (2) are still valid; now w is some Σ -measurable positive function, $\nu = w^p \cdot \mu$ and \mathcal{B} is some semi-finite sigma-subalgebra of Σ .

The task of extending these results to various classical spaces was considered by numerous authors; see the recent survey paper [15] and the references inside. Here we are more specifically interested in the case of vector-valued Lebesgue L_p spaces, in particular mixed norm spaces $L_p(L_q)$. Since the survey paper [5] on this specific subject, several partial results appeared. In particular B. Randrianantoanina ([14]) succeeded in solving thoroughly the complex sequential case $\ell_p(\ell_q)$ using hermitian operator techniques introduced in the subject by Kalton and Wood. More recently the case of finite dimensional real Banach spaces with C^2 norm was considered by the authors of [12]; under some additional conditions on the dual norm (in particular it is assumed to be C^2 on the complementary set of the coordinate hyperplanes associated to a distinguished basis) the contractively complemented subspaces are shown to be necessarily generated by a block-basis of the given basis. This can be applied in particular to the real spaces $\ell_p^n(\ell_q^m)$, when $2 < p, q < \infty$ (or by duality when $1 < p, q < 2$), obtaining the same description of their contractively complemented subspaces as in the complex case [16].

In the present paper we examine the case of Lebesgue spaces of Hilbert valued functions $L_p(H)$; this is done in the most general case (without any assumption of

sigma-finiteness of L_p -space or separability of the Hilbert space; in fact we have in mind some applications to the ultrapowers of such spaces, which are neither separable nor sigma-finite). It turns out that the range of a contractive projection is a ℓ_p -direct sum of spaces of the type $L_p(H)$. More precisely:

Theorem 0.1. *Let $1 \leq p < \infty$, $p \neq 2$; H be a Hilbert space and $L_p = L_p(\Omega, \Sigma, \mu)$. The range of every contractive projection $P : L_p(H) \rightarrow L_p(H)$ is isometric to a ℓ_p -direct sum of Hilbert-valued L_p -spaces, i.e.*

$$R(P) \approx_1 \left(\bigoplus_{i \in I} L_p(\Omega_i, \mathcal{B}_i, \mu_i; H_i) \right)_{\ell_p}$$

where $(\Omega_i)_i$ is a family of pairwise almost disjoint members of Σ , each \mathcal{B}_i is a sub-sigma-algebra of the trace Σ_i of Σ on Ω_i ; μ_i is the trace on Ω_i of the measure μ ; and the Hilbert spaces H_i have Hilbertian dimension not greater than the Hilbertian dimension of H .

Conversely a ℓ_p -sum $(\bigoplus_{i \in I} L_p(\Omega_i, \Sigma_i, \mu_i; H_i))_{\ell_p}$ embeds isometrically into $L_p(H)$, where $L_p = (\bigoplus_{i \in I} L_p(\Omega_i, \mathcal{B}_i, \mu_i))_{\ell_p}$ and $H = (\bigoplus_{i \in I} H_i)_{\ell_2}$. Hence a contractively complemented subspace of a ℓ_p -direct sum of Hilbert-valued L_p -spaces is still a ℓ_p -direct sum of Hilbert-valued L_p -spaces. In other words:

Corollary 0.2. *The class \mathcal{C}_p of ℓ_p -direct sums of Hilbert-valued L_p -spaces is stable under contractive projections.*

The structure of the contractive projection P can be easily explained in the case where the space H is separable (the non-separable case is analogous and will be described in Section 5). Recall that given two Banach spaces X, Y , a family of operators $T_\omega : X \rightarrow Y$ is said to be strong-operator Σ -measurable if for every $x \in X$, the map $\omega \mapsto T_\omega x$ is Σ -measurable as a map $\Omega \rightarrow Y$. If moreover $\text{Ess sup}_\omega \|T_\omega\| < \infty$, such a measurable family induces a bounded linear map T from $L_p(\Omega, \Sigma, \mu; X)$ into $L_p(\Omega, \Sigma, \mu; Y)$ by the equation:

$$(Tf)(\omega) = T_\omega(f_\omega)$$

Theorem 0.3. *Under the conditions of Thm. 0.1, if moreover H is separable, then*

$$P = \sum_{i \in I} S_i(\tilde{P}_i \otimes \text{Id}_{H_i})S_i^\sharp M_{\Omega_i} + V$$

where \tilde{P}_i is a positive contractive projection in $L_p(\Omega_i, \Sigma_i, \mu_i)$; S_i is an isometric embedding of $L_p(\Omega_i, \Sigma_i, \mu_i; H_i)$ into $L_p(\Omega_i, \Sigma_i, \mu_i; H)$ associated with a (strong-operator)-measurable family $(S_{i,\omega})_{\omega \in \Omega_i}$ of isometric embeddings $H_i \rightarrow H$, while S_i^\sharp is associated with the adjoint family $(S_{i,\omega}^*)_{\omega \in \Omega_i}$ of projections $H \rightarrow H_i$; $M_{\Omega_i} : L_p(\Omega; H) \rightarrow L_p(\Omega_i; H_i)$ is the multiplication operator by the indicator function $\mathbf{1}_{\Omega_i}$; and $V = 0$ if $p > 1$, while if $p = 1$ then V is a contraction of $L_1(\Omega, \Sigma, \mu; H)$ vanishing on every $L_1(\Omega_i, \Sigma_i, \mu_i; H)$ and taking values in the range of P .

Let us present shortly an application of the Thm. 0.1 which was in fact our main motivation for starting this study. If X, Y are Banach spaces, we say that X is an *ultraroot* of Y if Y is isometric to some ultrapower of X . Recall that a Banach space X embeds canonically isometrically in every of its ultrapowers $X_{\mathcal{U}}$, and that if X is reflexive, then this canonical image is contractively complemented in $X_{\mathcal{U}}$. As a consequence of Thm. 0.1 we see that every ultraroot of a $L_p(H)$ space, $p > 1$ is a member of \mathcal{C}_p . By Cor. 0.2 the same is true for ultraroots of members of \mathcal{C}_p . On the other hand it was proved in [13] that every ultraproduct of $L_p(H)$ spaces is isometric to a ℓ_p -direct sum of Hilbert-valued L_p -spaces. More generally every ultraproduct of members of \mathcal{C}_p is itself isometric to a member of \mathcal{C}_p . Hence we obtain:

Corollary 0.4. *For every $1 < p < \infty$ the class \mathcal{C}_p of ℓ_p -direct sums of Hilbert-valued L_p -spaces is stable under ultraproducts and ultraroots.*

In other words the class \mathcal{C}_p is *axiomatizable* in the sense of Henson-Iovino [9] in their language of normed spaces structures (see [9], Thm. 13.8).

The paper is organized as follows: after a section devoted to definitions, notations and a general result on orthogonally complemented subspaces of $L_p(H)$, we have two sections of preliminary results distinguishing the case $p = 1$ (Section 2) from the case $p > 1$ (Section 3). In these sections it is proved that if f belongs to the range of a contractive projection P , then the whole subspace $Z_f := \overline{L_\infty(\Omega) \cdot f}$ is preserved by P (i.e. $PZ_f \subset Z_f$) which suggests clearly a possible reduction to the scalar case. It is also proved that the “orthogonal projection” onto Z_f preserves the range of P . This allows to find an “orthogonal system” in $R(P)$ which generates $Z_P := \overline{L_\infty(\Sigma) \cdot R(P)}$ over $L_\infty(\Sigma)$ which will furnish the orthogonal bases of the Hilbert spaces H_i of Thm. 0.1. Section 4 is devoted to the proof of Thm. 0.1; a key point consists in proving that the different subalgebras of Σ given by the scalar theorem (applied to each Z_f) are induced by the same sigma-subalgebra \mathcal{F} of Σ . Finally Thm 0.3 is proved in Section 5 (in a more general version not requiring separability).

1. General preliminaries

1.1. Definitions and notations

Let $1 \leq p < \infty$, H be an Hilbert space and (Ω, Σ, μ) be a measure space. In the following we denote (when there is no ambiguity) by $L_p(H)$ the Lebesgue-Bochner space $L_p(\Omega, \Sigma, \mu; H)$ of classes of H -valued p -integrable functions (for μ -a.e. equality). Similarly $L_\infty(H)$ will be the space of classes of Bochner measurable, essentially bounded H -valued functions. These spaces can be defined directly from the Banach lattices L_p (resp. L_∞) and the Hilbert space H , but we adopt the functional point of view for the simplicity of the exposition. In the case where (Ω, Σ, μ) is not sigma-finite, it is preferable to suppose that this measure space is localizable: the measure μ is semifinite (every set in Σ of positive measure contains a further one of positive and

finite measure) and $L_\infty(\Omega, \Sigma, \mu)$ is order complete. In particular every family $(A_i)_{i \in I}$ in Σ has a supremum A , denoted by $\bigvee_{i \in I} A_i$. The set A is defined (up to a μ -null set) by the conditions:

$$A \dot{\supset} A_i \text{ for every } i \in I,$$

$$\text{If } B \in \Sigma \text{ and } B \dot{\supset} A_i \text{ for every } i \in I \text{ then } B \dot{\supset} A,$$

where $B \dot{\supset} A$ means $\mu(A \setminus B) = 0$ (define similarly $A \dot{\subset} B$ and $A \dot{=} B$). We say that B, C are almost disjoint if $A \cap B \dot{=} \emptyset$.

To every $f \in L_p(H)$ we associate its “random norm” $N(f) \in L_p^+$ defined by $N(f)(\omega) = \|f(\omega)\|_H$, its *vectorial function support* $\mathbf{VS}(f) = \text{Supp}(N(f))$ and its “random direction”, i.e. the element u_f of $L_\infty(H)$ defined by $u_f(\omega) = \frac{f(\omega)}{N(f)(\omega)}$ if $\omega \in \mathbf{VS}(f)$, $= 0$ if $\omega \notin \mathbf{VS}(f)$. If $M \subset L_p(H)$ we set $\mathbf{VS}(M) = \bigvee \{ \mathbf{VS}(f) \mid f \in M \}$. If $f \in L_p(H)$, $g \in L_q(H)$ we define their random scalar product $\langle\langle f, g \rangle\rangle \in L_r$ (where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$) by $\langle\langle f|g \rangle\rangle(\omega) = \langle f(\omega)|g(\omega) \rangle_H$, where $\langle \cdot | \cdot \rangle_H$ denotes the scalar product in H (which we suppose left linear, right antilinear in the complex case). When p, q are conjugate ($\frac{1}{p} + \frac{1}{q} = 1$), we obtain a sesquilinear pairing

$$\langle f, g \rangle = \int_\Omega \langle\langle f | g \rangle\rangle d\mu \tag{3}$$

which gives rise to a canonical antilinear identification of $L_q(H)$ with $L_p(H)^*$ (if $1 < p, q < \infty$; the case $p = 1, q = \infty$ is more delicate); it is the usual duality pairing in the real spaces case. We have also

$$\forall f \in L_p(H), \langle\langle f|u_f \rangle\rangle = N(f).$$

We say that two elements $f, g \in L_p(H)$ are *orthogonal*, and we write $f \perp g$ if $\langle\langle f | g \rangle\rangle = 0$. A related notation is the following. We set

$$\{ f \perp g \} = \{ \omega \in \Omega \mid \langle\langle f | g \rangle\rangle(\omega) = 0 \}$$

We have then $f \perp g \iff \{ f \perp g \} \dot{=} \Omega$.

Let H, K two Hilbert spaces. We say that a linear operator $T : L_p(H) \rightarrow L_p(K)$ is Σ -*modular* iff $T(\varphi \cdot f) = \varphi \cdot Tf$ for every $f \in L_p(H)$ and $\varphi \in L_\infty(\Omega, \Sigma, \mu)$. It is *modularly contractive*, resp. *modularly isometric* iff $N(Tf) \leq N(f)$, resp. $N(Tf) = N(f)$ for every $f \in L_p(H)$: it is then automatically Σ -modular (and, of course, contractive, resp. isometric). If H is separable, then a modularly contractive, resp. modularly isometric operator T is associated with a measurable family of contractions, resp. isometries $T_\omega : H \rightarrow K$.

Let \mathcal{F} be a sub-sigma-algebra of Σ ; a linear subspace Z of $L_p(H)$ is a $L_\infty(\mathcal{F})$ -*submodule* iff $\varphi \cdot f \in Z$ for every $f \in Z$ and $\varphi \in L_\infty(\Omega, \mathcal{F}, \mu)$. To every $f \in L_p(H)$ we associate the bounded Σ -modular operator:

$$E_f : L_p(H) \rightarrow L_p(H), \quad g \mapsto \langle\langle g|u_f \rangle\rangle u_f.$$

We have $N(E_f g) = |\langle\langle g|u_f \rangle\rangle| \mathbf{1}_{\mathbf{VS}(f)} \leq N(g)$, hence E_f is modularly contractive.

We have clearly $E_f(f) = N(f)u_f = f$. Consequently for every $\varphi \in L_\infty$, we have

$$E_f((\varphi N(f)) \cdot u_f) = E_f(\varphi f) = \varphi f = (\varphi N(f)) \cdot u_f$$

and by density we deduce that $E_f(\psi \cdot u_f) = \psi \cdot u_f$ for every $\psi \in L_p$. In particular $E_f(E_f g) = E_f g$, so E_f is a projection (with range $R(E_f) = L_p(\Omega) \cdot u_f$). It is not hard to see that $R(E_f)$ is exactly the closed $L_\infty(\Sigma)$ -submodule generated by f . Note also that if $f, g \in L_p(H)$,

$$f \perp g \iff E_f g = 0 \iff E_g f = 0.$$

1.2. Orthogonal projections

We end this section by considering a special class of contractive projections, namely the orthogonal ones. A projection Q in $L_p(H)$ is said to be *orthogonal* if $(f - Qf) \perp Qf$ for every $f \in L_p(H)$. Such a projection is trivially modularly contractive since

$$N(f)^2 = N(Qf)^2 + N((I - Q)f)^2 \geq N(Qf)^2.$$

Note that by polarization we have for every $f, g \in L_p(H)$:

$$\langle\langle f | g \rangle\rangle = \langle\langle Qf | Qg \rangle\rangle + \langle\langle (I - Q)f | (I - Q)g \rangle\rangle$$

Replacing g by Qg , we have

$$\langle\langle f | Qg \rangle\rangle = \langle\langle Qf | Qg \rangle\rangle$$

that is $(I - Q)f \perp Qg$; hence $\ker Q = R(I - Q) \perp R(Q)$.

Conversely if $f \perp R(I - Q)$ then $f - Qf \perp R(I - Q)$ and in particular $f - Qf \perp f - Qf$, i.e. $f = Qf \in R(Q)$. Hence $R(Q) = \ker Q^\perp := \{f \in L_p(H) \mid f \perp \ker Q\}$ and similarly (exchanging the roles of Q and $I - Q$) we have: $\ker Q = R(Q)^\perp$.

If A is a subset of $L_p(H)$ then A^\perp is a closed $L_\infty(\Sigma)$ -submodule of $L_p(H)$. In particular the range of any orthogonal projection in $L_p(H)$ is a closed $L_\infty(\Sigma)$ -submodule. The converse is true:

Lemma 1.1. *If Z is a closed $L_\infty(\Sigma)$ -submodule of $L_p(\Omega, \Sigma, \mu; H)$ there exists a unique orthogonal projection Q_Z in $L_p(H)$ with range Z .*

Proof. Let $(f_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal non zero elements of Z . For every family $(\varphi_\alpha)_\alpha$ in $L_p(\Omega)$ and every finite subset B of A we have

$$\left\| \sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha} \right\|_{L_p(H)} = \left\| N \left(\sum_{\alpha \in B} \varphi_\alpha u_{f_\alpha} \right) \right\|_p = \left\| \left(\sum_{\alpha \in B} \mathbf{1}_{\mathbf{VS}(f_\alpha)} |\varphi_\alpha|^2 \right)^{1/2} \right\|_p.$$

Hence, by Cauchy's criterion, $\sum_{\alpha \in A} \varphi_\alpha u_{f_\alpha}$ converges in $L_p(H)$ iff $(\sum_{\alpha \in A} \mathbf{1}_{\mathbf{vs}(f_\alpha)} |\varphi_\alpha|^2)^{1/2}$ exists in L_p and

$$\left\| \sum_{\alpha \in A} \varphi_\alpha u_{f_\alpha} \right\|_{L_p(H)} = \left\| \left(\sum_{\alpha \in A} \mathbf{1}_{\mathbf{vs}(f_\alpha)} |\varphi_\alpha|^2 \right)^{1/2} \right\|_p.$$

If now $f \in L_p(H)$ and B is a finite subset of A we have

$$\begin{aligned} N\left(\sum_{\alpha \in B} \langle\langle f|u_{f_\alpha} \rangle\rangle u_{f_\alpha}\right)^2 &= \sum_{\alpha \in B} |\langle\langle f|u_{f_\alpha} \rangle\rangle|^2 \\ &= \left\langle\left\langle f, \sum_{\alpha \in B} \langle\langle f|u_{f_\alpha} \rangle\rangle u_{f_\alpha} \right\rangle\right\rangle \leq N(f) N\left(\sum_{\alpha \in B} \langle\langle f|u_{f_\alpha} \rangle\rangle u_{f_\alpha}\right), \end{aligned}$$

whence

$$N\left(\sum_{\alpha \in B} \langle\langle f|u_{f_\alpha} \rangle\rangle u_{f_\alpha}\right) = \left(\sum_{\alpha \in B} |\langle\langle f|u_{f_\alpha} \rangle\rangle|^2\right)^{1/2} \leq N(f),$$

so

$$\left(\sum_{\alpha \in A} |\langle\langle f|u_{f_\alpha} \rangle\rangle|^2\right)^{1/2} \leq N(f).$$

Consequently $Qf := \sum_{\alpha \in A} \langle\langle f|u_{f_\alpha} \rangle\rangle u_{f_\alpha} = \sum_{\alpha \in A} E_{f_\alpha} f$ converges in $L_p(H)$ (with $\|Qf\| \leq \|f\|$). Since $R(E_{f_\alpha})$ is the closed $L_\infty(\Sigma)$ -submodule generated by f_α , we have $R(E_{f_\alpha}) \subset Z$ for each α and consequently $Qf \in Z$ for every $f \in L_p(H)$. The map Q is modular for the action of $L_\infty(\Omega)$, and clearly $Qf_\beta = f_\beta$ for every $\beta \in A$. It results easily that $Qf = f$ for every $f = \sum_{\alpha \in A} \varphi_\alpha u_{f_\alpha}$ (when this series converges), i.e. Q is a contractive projection in $L_p(H)$ with range

$$\begin{aligned} R(Q) &= \left\{ \sum_{\alpha \in A} \varphi_\alpha u_{f_\alpha} \mid \left(\sum_{\alpha} |\varphi_\alpha|^2\right)^{1/2} \in L_p(\Omega) \right\} \\ &= \left\{ \sum_{\alpha \in A} \psi_\alpha f_\alpha \mid \left(\sum_{\alpha} |\psi_\alpha|^2 N(f_\alpha)^2\right)^{1/2} \in L_p(\Omega) \right\}. \end{aligned}$$

Since clearly $\langle\langle Qf|f_\alpha \rangle\rangle = \langle\langle f|f_\alpha \rangle\rangle$ for every $\alpha \in A$ we have $(f - Qf) \perp f_\alpha$ for every $\alpha \in A$. By maximality of the system (f_α) we deduce that

$$f = Qf \text{ for every } f \in Z$$

so $R(Q)$ contains Z , hence coincides with Z . Note also that $f - Qf \perp Z$ for all $f \in L_p(H)$, and so Q is orthogonal.

The unicity of the orthogonal projection onto Z is a consequence of the fact that its image and kernel are uniquely determined ($R(Q) = Z$ and $\ker Q = Z^\perp$). \square

2. Preliminary results: the case $p = 1$

Lemma 2.1. *Let P be a contractive projection in $L_1(H)$. Then for every $f \in R(P)$ we have*

$$PE_f = E_fPE_f$$

Proof. For every $\varphi \in L_1(\Omega)$ with $0 \leq \varphi \leq N(f)$ we have

$$\begin{aligned} \|f\| - \|\varphi \cdot u_f\| &= \int N(f) d\mu - \int N(\varphi u_f) d\mu = \int (N(f) - \varphi) d\mu \\ &= \|(N(f) - \varphi) \cdot u_f\| = \|f - \varphi \cdot u_f\| \\ &\geq \|P(f - \varphi \cdot u_f)\| = \|f - P(\varphi \cdot u_f)\| \\ &\geq \|f\| - \|P(\varphi \cdot u_f)\| \\ &\geq \|f\| - \|\varphi \cdot u_f\|. \end{aligned}$$

Hence all the inequalities are equalities, and in particular

$$\|f - P(\varphi \cdot u_f)\| = \|f\| - \|P(\varphi \cdot u_f)\|,$$

that is,

$$\int N(f - P(\varphi \cdot u_f)) d\mu = \int [N(f) - N(P(\varphi \cdot u_f))] d\mu.$$

Note that the function in the left-hand integral is greater than the one in the right-hand integral. Thus,

$$N(f - P(\varphi \cdot u_f)) = N(f) - N(P(\varphi \cdot u_f))$$

(equality as elements of $L_1(\Omega)$). Since H is strictly convex this implies that

$$P(\varphi \cdot u_f) = \alpha \cdot f$$

for some $\alpha \in L_\infty^+(\Omega)$. Hence

$$E_fP(\varphi \cdot u_f) = E_f(\alpha \cdot f) = \alpha \cdot f = P(\varphi \cdot u_f).$$

This property has been proved for $\varphi \in L_1(\Omega)$ with $0 \leq \varphi \leq N(f)$; it is extended by linearity and density to every $\varphi \in L_1(\Omega)$. In particular if we take $\varphi = \langle\langle h|u_f \rangle\rangle$, we obtain

$$\forall h \in L_1(H), \quad E_fPE_fh = PE_fh,$$

that is, $E_fPE_f = PE_f$. □

Lemma 2.2. *Let P be a contractive projection in $L_1(H)$. Then for every $f, g \in R(P)$ we have: $E_gf \in R(P)$. In other words $E_gP = PE_gP$.*

Proof. We have $(f - E_g f) \perp g$, while (by Lemma 2.1) $E_g f - PE_g f = E_g(f - PE_g f) \in L_1(\Omega) \cdot u_g$. Hence $(f - E_g f) \perp (E_g f - PE_g f)$. It results that

$$N(f - PE_g f) = [N(f - E_g f)^2 + N((E_g f - PE_g f)^2)]^{1/2} \geq N(f - E_g f). \quad (4)$$

Hence:

$$\begin{aligned} \|f - PE_g f\| &\geq \|f - E_g f\| \\ &\geq \|P(f - E_g f)\| \\ &= \|f - PE_g f\| \end{aligned}$$

Hence the inequalities are equalities. In view of (4), the equality $\|f - PE_g f\| = \|f - E_g f\|$ implies

$$N(f - PE_g f) = [N(f - E_g f)^2 + N((E_g f - PE_g f)^2)]^{1/2} = N(f - E_g f),$$

which implies in turn that $N(E_g f - PE_g f) = 0$, that is $E_g f = PE_g f$. So $E_g f \in R(P)$. \square

3. Preliminary results: the case $p > 1$

Notations. Let p_* be the conjugate exponent of p . If $T : L_p(H) \rightarrow L_p(H)$ is a bounded operator, we define its adjoint $T^* : L_{p_*}(H) \rightarrow L_{p_*}(H)$ by

$$\forall f \in L_{p_*}(H), \forall g \in L_p(H) \quad \langle T^* f, g \rangle = \langle f, Tg \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the sesquilinear pairing given by eq. (3).

If $f \in L_p(H)$, $f \neq 0$, let $Jf \in L_{p_*}(H)$ be the unique norm-one element such that $\langle f, Jf \rangle = \|f\|$. In fact it will be easier to consider the $(p-1)$ -homogeneous functional $J_p(h) = \|h\|^{p-1} J(h)$. We have $J_p(h) = N(h)^{p-1} \cdot u_h = N(h)^{p-2} h$, hence J_p is random direction preserving. Note that pJ_p is the derivative of the p^{th} power of the norm.

Lemma 3.1. *Let $1 < p < \infty$, $p \neq 2$, and P be a contractive projection in $L_p(H)$. Then for every $f, g \in R(P)$ the function $F(f, g) := \text{sgn}\langle g | f \rangle f + \gamma_p \mathbf{1}_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$, where γ_p is a positive constant depending only on p .*

Proof. a) Case $2 < p < \infty$.

Recall that since $L_p(H)$ is smooth the duality map J maps $R(P)$ into $R(P^*)$ (see e.g. [6, Lemma 4.8]); hence $J_p(f + tg) \in R(P^*)$ for every $t \geq 0$. The derivative $\frac{\partial}{\partial t} J_p(f + tg)$ exists at $t = 0$ (since the norm to the power p is twice differentiable) and it belongs to $R(P^*)$ too. We have

$$\frac{\partial}{\partial t} J_p(f + tg) = N(f + tg)^{p-2} g + \left(\frac{p-2}{2} \frac{\partial}{\partial t} N(f + tg)^2 \right) N(f + tg)^{p-4} (f + tg).$$

Hence

$$\begin{aligned} A(f, g) &:= \frac{\partial}{\partial t} J_p(f + tg) |_{t=0} = N(f)^{p-2}g + (p - 2) \operatorname{Re}(\langle f | g \rangle) N(f)^{p-4}f \\ &= N(f)^{p-2}[g + (p - 2) \operatorname{Re}(\langle u_f, g \rangle) u_f] \in R(P^*) \end{aligned} \tag{5}$$

In the complex case, replacing f by if , we obtain

$$B(f, g) := N(f)^{p-2}[g - i(p - 2) \operatorname{Im}(\langle u_f, g \rangle) u_f] \in R(P^*) \tag{5bis}$$

adding

$$N(f)^{p-2}[2g + (p - 2)\langle g, u_f \rangle u_f] \in R(P^*)$$

With $E_f g = \langle g, u_f \rangle u_f$ we obtain

$$N(f)^{p-2}[2(g - E_f g) + pE_f g] \in R(P^*).$$

In the case of a real space (5) is valid without the symbol Re and we obtain

$$N(f)^{p-2}[(g - E_f g) + (p - 1)E_f g] \in R(P^*).$$

If $h \in R(P^*)$ then $J_{p^*} h = N(h)^{p^*-1} u_h \in R(P)$, hence if we set $Tg = \alpha_p(g - E_f g) + E_f g$, with $\alpha_p = \frac{2}{p}$ in the complex case, $\alpha_p = \frac{1}{p-1}$ in the real case, we obtain:

$$\Phi(g) := N(f)^{(p-2)(p^*-1)} N(Tg)^{(p^*-1)} u_{Tg} \in R(P).$$

Since T is Σ -modular we have $u_{T(\varphi \cdot u_h)} = \mathbf{1}_{\operatorname{Supp} \varphi} \cdot u_{Th}$ for every $h \in L_p(H)$ and $\varphi \in L_p$, and more generally $u_{T^k(\varphi \cdot u_h)} = \mathbf{1}_{\operatorname{Supp} \varphi} \cdot u_{T^k h}$ for every $k \geq 1$. It is easily deduced that: $u_{T^k \Phi(g)} = \mathbf{1}_{\mathbf{vs}(f)} \cdot u_{T^{k+1}g}$ for every $k \geq 0$. Then

$$\begin{aligned} u_{\Phi^n(g)} &= u_{\Phi(\Phi^{n-1}(g))} = \mathbf{1}_{\mathbf{vs}(f)} \cdot u_{T\Phi^{n-1}(g)} \\ &= \mathbf{1}_{\mathbf{vs}(f)} \cdot u_{T\Phi(\Phi^{n-2}(g))} = \mathbf{1}_{\mathbf{vs}(f)} \cdot u_{T^2\Phi^{n-2}(g)} \cdots \\ &= \mathbf{1}_{\mathbf{vs}(f)} \cdot u_{T^n g} \end{aligned} \tag{6}$$

for every $n \geq 1$. If $E_f g(\omega) \neq 0$ we have

$$u_{T^n g}(\omega) = \frac{\alpha_p^n (g - E_f g)(\omega) + E_f g(\omega)}{N(\alpha_p^n (g - E_f g) + E_f g)(\omega)} \longrightarrow \frac{E_f g(\omega)}{N(E_f g)(\omega)} = u_{E_f g}(\omega) \tag{7}$$

(norm convergence in H) while if $E_f g(\omega) = 0$

$$u_{T^n g}(\omega) = \frac{(g - E_f g)(\omega)}{N((g - E_f g)(\omega))} = u_{(g - E_f g)}(\omega) = u_g(\omega). \tag{7'}$$

Since $g - E_f g \perp E_f g$ we have $N(Tg) \leq N(g)$. Hence

$$N(\Phi(g)) = N(f)^{2-p^*} N(Tg)^{p^*-1} \leq N(f)^{2-p^*} N(g)^{p^*-1}. \tag{8}$$

In particular

$$N(\Phi(g)) \leq \max(N(f), N(g)). \tag{9}$$

Reiterating (8) we obtain for every $n \geq 1$

$$N(\Phi^n(g)) \leq N(f)^{(2-p_*)\sum_{k=0}^{n-1} (p_*-1)^k} N(g)^{(p_*-1)^n} = N(f)^{1-(p_*-1)^n} N(g)^{(p_*-1)^n}.$$

Since $0 < p_* - 1 < 1$ we obtain

$$\overline{\lim}_{n \rightarrow \infty} N(\Phi^n(g)) \leq \mathbf{1}_{\mathbf{VS}(g)} N(f). \tag{10}$$

We try now to be more precise. If $E_f g(\omega) = 0$ we have $N(Tg)(\omega) = \alpha_p N(g)(\omega)$. Hence

$$N(\Phi(g))(\omega) = N(f)(\omega)^{2-p_*} (\alpha_p N(g)(\omega))^{p_*-1}.$$

Moreover, since in this case $u_{\Phi^n(g)}(\omega) = u_g(\omega)$, we have $E_f \Phi^n(g)(\omega) = 0$ for every n , and we can reiterate. We obtain

$$N(\Phi^n(g))(\omega) = (\alpha_p^{p_*-1} N(f)(\omega)^{(2-p_*)\sum_{k=0}^{n-1} (p_*-1)^k} N(g)(\omega)^{(p_*-1)^n}).$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} N(\Phi^n(g))(\omega) &= \alpha_p^{(p_*-1)/(2-p_*)} \mathbf{1}_{\mathbf{VS}(g)}(\omega) N(f)(\omega) \\ &= \alpha_p^{1/(p-2)} \mathbf{1}_{\mathbf{VS}(g)}(\omega) N(f)(\omega). \end{aligned} \tag{11}$$

If now $E_f(g)(\omega) \neq 0$, we have also $E_f(\Phi^n(g))(\omega) \neq 0$ for every $n \geq 0$. Set

$$\beta_n(\omega) = \frac{N(E_f \Phi^n(g))(\omega)}{N(\Phi^n(g))(\omega)}$$

We have then

$$N(T\Phi^n(g))(\omega) \geq \beta_n(\omega) N(\Phi^n(g))(\omega)$$

and consequently:

$$N(\Phi^{n+1}(g))(\omega) \geq N(f)^{2-p_*} (\beta_n(\omega) N(\Phi^n(g))(\omega))^{p_*-1}. \tag{12}$$

On the other hand

$$\beta_n(\omega) = |\langle u_{\Phi^n(g)}, u_f \rangle(\omega)| = |\langle u_{T^n(g)}, u_f \rangle(\omega)| = \frac{N(E_f T^n(g))(\omega)}{N(T^n(g))(\omega)} = \frac{N(E_f g)(\omega)}{N(T^n(g))(\omega)}$$

and since $N(T^n g) = (\alpha_p^{2n} N(g - E_f g)^2 + N(E_f g)^2)^{1/2} \searrow N(E_f g)$ pointwise (as $\alpha_p < 1$) we have $\beta_n(\omega) \nearrow 1$ on the set $\{\omega \mid E_f g(\omega) \neq 0\}$. Reiterating (12) from the step $n = n_0$ we obtain then

$$\underline{\lim}_{n \rightarrow \infty} N(\Phi^n(g))(\omega) \geq (\beta_{n_0}(\omega))^{1/(p-2)} \mathbf{1}_{\mathbf{VS}(\Phi_{n_0}(g))}(\omega) N(f)(\omega)$$

and letting $n_0 \rightarrow \infty$, we have, since $\mathbf{VS}(\Phi_n(g)) = \mathbf{VS}(g) \cap \mathbf{VS}(f)$ for every n ,

$$\liminf_{n \rightarrow \infty} N(\Phi^n(g))(\omega) \geq \mathbf{1}_{\mathbf{VS}(g)}(\omega)N(f)(\omega). \tag{13}$$

From (6), (7), (7'), and (11), (10), (13) we deduce that

$$\Phi^n(g) \rightarrow N(f)[u_{E_f(g)} + \alpha_p^{1/(p-2)} \mathbf{1}_{\{f \perp g\}} u_g] \tag{14}$$

almost everywhere in H -norm, hence in $L_p(H)$ -norm by (9) and Lebesgue's Theorem. Hence the right-hand member of (14) belongs to $R(P)$. Since $u_{E_f g} = \text{sgn}\langle\langle g | f \rangle\rangle u_f$ the right member of (14) can be written

$$\text{sgn}\langle\langle g | f \rangle\rangle f + \gamma_p \mathbf{1}_{\{f \perp g\}} N(f) u_g = F_p(f, g) \tag{15}$$

where we have set $\gamma_p = \alpha_p^{1/(p-2)}$.

b) Case $1 < p < 2$.

This case is treated by duality. Set $\gamma_p = \gamma_{p_*}^{p_*-1}$ and define $F_p(f, g)$ by the formula (15). If $g = J_{p_*} g'$, $f = J_{p_*} f'$ with $f', g' \in L_{p_*}(H)$ we have

$$\text{sgn}\langle\langle g | f \rangle\rangle = \text{sgn}\langle\langle g' | f' \rangle\rangle.$$

Hence

$$\text{sgn}\langle\langle g | f \rangle\rangle f = J_{p_*}(\text{sgn}\langle\langle g' | f' \rangle\rangle f')$$

and similarly

$$N(f) = N(J_{p_*} f') = N(f')^{p_*-1}.$$

Hence

$$N(f) u_g = N(f')^{p_*-1} u_g = J_{p_*}(N(f') u_{g'}).$$

Finally, since $\{f \perp g\} = \{f' \perp g'\}$ and J_{p_*} is additive on elements with disjoint functional supports, and positively homogeneous of degree $p_* - 1$,

$$F_p(f, g) = J_{p_*}(F_{p_*}(f', g')).$$

Then since $f' = J_p f, g' = J_p g$ belong to $R(P^*)$, the function $F_{p_*}(f', g')$ belongs to $R(P^*)$ too by the case (a), and $F_p(f, g)$ belongs to $R(P)$. \square

Corollary 3.2. *Let p and P be as in Lemma 3.1. Then for every $f, g \in R(P)$ the three elements $\text{sgn}\langle\langle g | f \rangle\rangle f$, $\mathbf{1}_{\{f \perp g\}} f$ and $\mathbf{1}_{\{f \perp g\}} N(f) u_g$ belong to $R(P)$.*

Proof. The set Λ of scalars λ such that the set $\{\omega \in \mathbf{VS}(f) \mid \frac{\langle\langle g | f \rangle\rangle(\omega)}{\langle\langle f | f \rangle\rangle(\omega)} = -\lambda\}$ has positive measure is at most countable. This set is also the set of λ 's such that $\{(g + \lambda f) \perp f\} \cap \mathbf{VS}(f)$ has positive measure. Choose a sequence (ε_n) of positive numbers not in $\Lambda \cup (-\Lambda)$ which converges to 0. Then by Lemma 3.1

$$\text{sgn}\langle\langle g \pm \varepsilon_n f | f \rangle\rangle f \in R(P)$$

for every $n \geq 1$. Since

$$\operatorname{sgn}\langle\langle g \pm \varepsilon_n f \mid f \rangle\rangle(\omega) \rightarrow \begin{cases} \operatorname{sgn}\langle\langle g \mid f \rangle\rangle(\omega) & \text{if } \langle\langle g \mid f \rangle\rangle(\omega) \neq 0, \\ \pm 1 & \text{if } \langle\langle g \mid f \rangle\rangle(\omega) = 0 \text{ and } f(\omega) \neq 0, \end{cases}$$

we have

$$\operatorname{sgn}\langle\langle g \mid f \rangle\rangle f \pm \mathbf{1}_{\{f \perp g\}} f = \lim_n \operatorname{sgn}\langle\langle g \pm \varepsilon_n f \mid f \rangle\rangle f \in R(P)$$

and consequently $\operatorname{sgn}\langle\langle g \mid f \rangle\rangle f$ and $\mathbf{1}_{\{f \perp g\}} f$ belong to $R(P)$. Then $F_p(f, g) - \operatorname{sgn}\langle\langle g \mid f \rangle\rangle f = \gamma_p \mathbf{1}_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$ too. \square

Corollary 3.3. *Let p and P be as in Lemma 3.1. Then for every $f, g \in R(P)$ we have $\mathbf{1}_{\mathbf{vs}(g)} f \in R(P)$.*

Proof. By Cor. 3.2, $h := G(f, g) := \mathbf{1}_{\{f \perp g\}} N(f) u_g$ belongs to $R(P)$. Then $G(h, f) = \mathbf{1}_{\{f \perp g\}} \mathbf{1}_{\{u_g \neq 0\}} N(f) u_f = \mathbf{1}_{\mathbf{vs}(g) \cap \{f \perp g\}} f$ belongs to $R(P)$ too. By Cor. 3.2, $f - \mathbf{1}_{\{f \perp g\}} f = \mathbf{1}_{\{f \not\perp g\}} f \in R(P)$, thus $\mathbf{1}_{\mathbf{vs}(g)} f = \mathbf{1}_{\{f \not\perp g\}} f + \mathbf{1}_{\mathbf{vs}(g) \cap \{f \perp g\}} f \in R(P)$. \square

Remark 3.4. In the complex case, for every $f, g \in R(P)$ the elements $\operatorname{sgn}(\operatorname{Re}\langle\langle g \mid f \rangle\rangle) f$ and $\mathbf{1}_{\{\operatorname{sgn}(\operatorname{Re}\langle\langle g \mid f \rangle\rangle)=0\}} f$ belong to $R(P)$ too. Indeed, $L_p(H)$ is a real Hilbert-valued $L_p(K)$ space, where K is the real vector space H equipped with the scalar product $(x, y)_K = \operatorname{Re}(x \mid y)_H$. As a consequence, the element $\mathbf{1}_{\{\operatorname{Re}\langle\langle g \mid f \rangle\rangle > 0\}} f = \frac{1}{2}(\operatorname{sgn} \operatorname{Re}\langle\langle g \mid f \rangle\rangle + \mathbf{1}_{\{\operatorname{sgn}(\operatorname{Re}\langle\langle g \mid f \rangle\rangle) \neq 0\}}) f$ belongs to $R(P)$.

Lemma 3.5. *Let p and P be as in Lemma 3.1. For every $f, g \in R(P)$ denote by $\Sigma_{f,g}$ the σ -field generated by the element $\frac{\langle\langle g \mid f \rangle\rangle}{\langle\langle f \mid f \rangle\rangle}$. Then for every $\Sigma_{f,g}$ -measurable function φ such that $\varphi \cdot N(f) \in L_p(\Omega, \Sigma, \mu)$, the element $\varphi \cdot f$ belongs to $R(P)$.*

Proof. Since $R(P)$ is a closed linear subspace, it is sufficient to prove this for indicator functions of $\Sigma_{f,g}$ -measurable sets. The sigma-algebra $\Sigma_{f,g}$ is generated by the sets $\{\frac{\operatorname{Re}\langle\langle g \mid f \rangle\rangle}{\langle\langle f \mid f \rangle\rangle} > \lambda\}$, $\{-\frac{\operatorname{Re}\langle\langle g \mid f \rangle\rangle}{\langle\langle f \mid f \rangle\rangle} > \lambda\}$, $\{\frac{\operatorname{Im}\langle\langle g \mid f \rangle\rangle}{\langle\langle f \mid f \rangle\rangle} > \lambda\}$, and $\{-\frac{\operatorname{Im}\langle\langle g \mid f \rangle\rangle}{\langle\langle f \mid f \rangle\rangle} > \lambda\}$, $\lambda \in \mathbb{R}_+$. If $A_{f,g,\lambda} = \{\frac{\operatorname{Re}\langle\langle g \mid f \rangle\rangle}{\langle\langle f \mid f \rangle\rangle} > \lambda\}$ we have $A_{f,g,\lambda} = \{\operatorname{Re}\langle\langle g - \lambda f \mid f \rangle\rangle > 0\}$, hence $\mathbf{1}_{A_{f,g,\lambda}} f \in R(P)$ by Rem. 3.4. The conclusion is the same for the three others kinds of sets (replacing g by $-g$ or $\pm ig$). Now if $\mathbf{1}_B f \in R(P)$ then $A_{f,g,\lambda} \cap B = A_{f',g,\lambda}$ with $f' = \mathbf{1}_B f$, hence $\mathbf{1}_{A_{f,g,\lambda} \cap B} f = \mathbf{1}_{A_{f',g,\lambda}} f' \in R(P)$. It results that the class \mathcal{C} of the sets $A \in \Sigma$ such that $\mathbf{1}_A f \in R(P)$ contains finite intersections of sets of the four preceding types. Since \mathcal{C} is closed by complementation and monotone limits, it contains the sigma-algebra $\Sigma_{f,g}$. \square

Corollary 3.6. *Let p and P be as in Lemma 3.1. For every $f \in R(P)$ we have $E_f P = P E_f$.*

Proof. Let $g \in R(P)$. Applying Lemma 3.5 to the function $\varphi = \frac{\langle\langle g|f \rangle\rangle}{\langle\langle f|f \rangle\rangle}$ we obtain that $E_f g \in R(P)$. Hence for every $h \in L_p(H)$, we have $E_f P h \in R(P)$, i.e. $E_f P h = P E_f P h$; thus $E_f P = P E_f P$. Similarly, reasoning with the contractive projection P^* in $L_{p^*}(H)$, and the element $J_p f$ of $R(P^*)$, we have $E_{J_p f} P^* = P^* E_{J_p f} P^*$. Dualizing we obtain $P E_{J_p f}^* = P E_{J_p f}^* P$. We claim that $E_f^* = E_{J_p f}$. This will show that $P E_f = P E_f P = E_f P$. Let us show this claim. Since $u_{J_p f} = u_f$, we have for every $g \in L_p(H)$ and $h' \in L_{p^*}(H)$

$$\begin{aligned} \langle E_f g, h' \rangle &= \int \langle\langle E_f g, h' \rangle\rangle d\mu = \int \langle\langle \langle\langle g, u_f \rangle\rangle u_f, h' \rangle\rangle d\mu \\ &= \int \langle\langle g, u_f \rangle\rangle \langle\langle u_f, h' \rangle\rangle d\mu \\ &= \int \langle\langle g, \langle\langle h', u_f \rangle\rangle u_f \rangle\rangle d\mu \\ &= \int \langle\langle g, E_{J_p f} h' \rangle\rangle d\mu = \langle g, E_{J_p f} h' \rangle \quad \square \end{aligned}$$

Remark. The preceding proof of Cor. 3.6 is essentially a real one. In the complex case it can be replaced by a shorter one, of more algebraic nature, due to Arazy and Friedman in the context of spaces C_p (see [2]). It seemed interesting to us to reproduce this proof in the Annex (see §6), after simplifying it considerably by eliminating the unnecessary non-commutative apparatus.

4. The range of a contractive projection

This section is devoted to the proof of Thm. 0.1, which consists in four lemmas.

Lemma 4.1. *The closed $L_\infty(\Sigma)$ -module Z generated by $R(P)$ in $L_p(H)$ is generated (as L_∞ -module) by a family $(f_\alpha)_{\alpha \in A}$ of pairwise orthogonal elements of $R(P)$. We have in fact a Schauder (orthogonal) decomposition*

$$Z = \bigoplus_{\alpha \in A} L_p(\Omega) \cdot u_{f_\alpha}$$

Proof. Let $(f_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal non zero elements of $R(P)$ and Z_0 be the closed $L_\infty(\Sigma)$ -submodule generated by the family $(f_\alpha)_{\alpha \in A}$. Let Q_{Z_0} be the orthogonal projection onto Z_0 . By the proof of Lemma 1.1 we know that $Q_{Z_0} = \sum_{\alpha \in A} E_{f_\alpha}$ (convergence in strong operator topology). Hence, by Lemma 2.2 if $p = 1$, resp. Cor. 3.6 if $p > 1$, $Q_{Z_0} f \in R(P)$ for every $f \in R(P)$. Since Q_{Z_0} is orthogonal and $f_\alpha \in R(Q_{Z_0})$ we have $(f - Q_{Z_0} f) \perp f_\alpha$ for every $\alpha \in A$. By maximality of the system (f_α) we deduce that

$$f = Q_{Z_0} f \text{ for every } f \in R(P)$$

i.e. $Q_{Z_0}P = P$. Then $Z_0 = R(Q_{Z_0})$ is a closed L_∞ -module containing $R(P)$ and generated by a subset of $R(P)$; hence it coincides with the closed L_∞ -module generated by $R(P)$. \square

Lemma 4.2. *There exists a sub- σ -algebra \mathcal{F} of Σ containing the vectorial function supports of all elements of $R(P)$ such that for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$, the product $\varphi \cdot u_f$ belongs to $R(P)$ iff $\mathbf{1}_{\mathbf{VS}(f)}N(f)^{-1}\varphi$ is \mathcal{F} -measurable. In particular $R(P)$ is a $L_\infty(\Omega, \mathcal{F}, \mu)$ -submodule.*

Proof. Since $PE_f = E_fPE_f$ by Lemma 2.1 (if $p = 1$) or by Cor. 3.6 (if $p > 1$), we have $P(\varphi \cdot u_f) \in L_p(\Omega) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. We may write $P(\varphi \cdot u_f) = (\tilde{P}_f\varphi) \cdot u_f$, with $\text{Supp}(\tilde{P}_f\varphi) \subset \mathbf{VS}(f)$. Clearly \tilde{P}_f is linear, $\tilde{P}_f^2 = \tilde{P}_f$ and

$$\|\tilde{P}_f\varphi\|_p = \|P(\varphi \cdot u_f)\| \leq \|\varphi \cdot u_f\| \leq \|\varphi\|_p,$$

hence \tilde{P}_f is a contractive projection in $L_p(\Omega, \Sigma, \mu)$. Moreover $\tilde{P}_f(N(f)) = N(f)$ and $\tilde{P}_f\psi = 0$ for every $\psi \in L_p(\Omega, \Sigma, \mu)$ disjoint from $N(f)$.

It results from Douglas' theorem (in case $p = 1$) or Andô's theorem (in case $p > 1$) that \tilde{P}_f is positive and

$$\tilde{P}_f(\varphi) = N(f)\mathbb{E}_{\nu_f}^{\mathcal{F}_f} \left(\frac{\mathbf{1}_{\text{Supp}(N(f))}\varphi}{N(f)} \right)$$

where $\mathbb{E}_{\nu_f}^{\mathcal{F}_f}$ is the conditional expectation with respect to some subalgebra \mathcal{F}_f of Σ containing $\mathbf{VS}(f)$ and to the measure $\nu_f = N(f)^p d\mu$. (We may assume that $\Omega \setminus \mathbf{VS}(f)$ is an atom of \mathcal{F}_f). In particular $L_p(\Omega, \Sigma, \mu) \cdot u_f \cap R(P) = L_p(\Omega, \mathcal{F}_f, \nu_f) \cdot f$ is a $L_\infty(\Omega, \mathcal{F}_f, \mu)$ -module.

Let us denote $\mathbb{E}^f\psi = \mathbb{E}_{\nu_f}^{\mathcal{F}_f}(\mathbf{1}_{\mathbf{VS}(f)}\psi)$, we have then $P(\psi \cdot f) = \mathbb{E}^f(\psi) \cdot f$ for every $\psi \in L_\infty(\Omega, \Sigma, \mu)$. Let now $f, g \in R(P)$. If $g = h \cdot u_f$ with $h \in L^p(\Omega)$ then $\frac{h}{N(f)}$ is \mathcal{F}_f -measurable and for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$ we have

$$\mathbb{E}^g(\varphi) \cdot g = P(\varphi h \cdot u_f) = N(f)\mathbb{E}^f \left(\frac{\varphi \cdot h}{N(f)} \right) \cdot u_f = h\mathbb{E}^f(\varphi) \cdot u_f = \mathbb{E}^f(\varphi) \cdot g,$$

Hence

$$\mathbb{E}^g(\varphi) = \mathbf{1}_{\mathbf{VS}(g)} \cdot \mathbb{E}^f(\varphi) = \mathbf{1}_{\text{Supp } h} \mathbb{E}^f(\varphi). \tag{16}$$

Let now g be a general element of $R(P)$. For every $\varphi \in L_\infty(\Omega)$ the equation

$$P(\varphi \cdot (f + g)) = P(\varphi \cdot f) + P(\varphi \cdot g)$$

is equivalent to

$$\mathbb{E}^{f+g}(\varphi) \cdot (f + g) = \mathbb{E}^f(\varphi) \cdot f + \mathbb{E}^g(\varphi) \cdot g. \tag{17}$$

Let $g = h \cdot u_f + g'$ be the orthogonal decomposition, i.e. $h = \langle\langle g \mid u_f \rangle\rangle$ and $g' \perp f$. Note that $h \cdot u_f = E_f g \in R(P)$. Set $A = \mathbf{VS}(f)$, $B = \mathbf{VS}(g)$ and $B' = \mathbf{VS}(g')$. Taking the images of both sides of (17) by the orthogonal projection $I - E_f$ we obtain

$$\mathbb{E}^{f+g}(\varphi) \cdot g' = \mathbb{E}^g(\varphi) \cdot g',$$

hence $\mathbf{1}_{B'} \mathbb{E}^{f+g}(\varphi) = \mathbf{1}_{B'} \mathbb{E}^g(\varphi)$. Then by (17) again, $\mathbf{1}_{B'} \mathbb{E}^{f+g}(\varphi) f = \mathbf{1}_{B'} \mathbb{E}^f(\varphi) f$ and finally

$$\mathbf{1}_{A \cap B'} \mathbb{E}^{f+g}(\varphi) = \mathbf{1}_{A \cap B'} \mathbb{E}^f(\varphi) = \mathbf{1}_{A \cap B'} \mathbb{E}^g(\varphi). \tag{18}$$

On the other hand similarly to (17) we have

$$\mathbb{E}^{h \cdot u_f - g}(\varphi) \cdot (h \cdot u_f - g) = \mathbb{E}^{hu_f}(\varphi) \cdot hu_f - \mathbb{E}^g(\varphi) \cdot g.$$

Since $h \cdot u_f - g = -g'$ we deduce that

$$\mathbf{1}_{\Omega \setminus B'} \mathbb{E}^{hu_f}(\varphi) \cdot hu_f = \mathbf{1}_{\Omega \setminus B'} \mathbb{E}^g(\varphi) \cdot g,$$

hence

$$\mathbf{1}_{B \setminus B'} \mathbb{E}^{hu_f}(\varphi) = \mathbf{1}_{B \setminus B'} \mathbb{E}^g(\varphi). \tag{19}$$

We have $\mathbb{E}^{hu_f}(\varphi) = \mathbf{1}_{\text{Supp } h} \mathbb{E}^f(\varphi)$ by eq. (16). Hence since $B \setminus B' \subset \text{Supp } h$, eq. (19) gives

$$\mathbf{1}_{B \setminus B'} \mathbb{E}^f(\varphi) = \mathbf{1}_{B \setminus B'} \mathbb{E}^g(\varphi)$$

which together with eq. (18) gives

$$\mathbf{1}_{A \cap B} \mathbb{E}^f(\varphi) = \mathbf{1}_{A \cap B} \mathbb{E}^g(\varphi)$$

for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$. In particular

$$\begin{aligned} \mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)} &= \mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)} \mathbb{E}^g(\mathbf{1}_{\mathbf{VS}(g)}) \\ &= \mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)} \mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(g)}) \\ &= \mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)} \mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)}), \end{aligned}$$

hence

$$\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)}) \geq \mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)}$$

and since \mathbb{E}^f is a contraction in $L_p(\Omega, \Sigma, N(f)^p \cdot \mu)$ we have in fact

$$\mathbb{E}^f(\mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)}) = \mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)},$$

that is, $\mathbf{VS}(f) \cap \mathbf{VS}(g) \in \mathcal{F}_f$. In particular $\mathbf{1}_{\mathbf{VS}(g)} \cdot f = \mathbf{1}_{\mathbf{VS}(f) \cap \mathbf{VS}(g)} \cdot f \in R(P)$. More generally for every $A \in \mathcal{F}_g$ its trace $\mathbf{VS}(f) \cap A$ belongs to \mathcal{F}_f (as is easily seen by treating separately the cases $A \subset \mathbf{VS}(g)$ and $A = \Omega \setminus \mathbf{VS}(g)$). Let \mathcal{F} be the σ -algebra consisting of sets $A \in \Sigma$ such that $A \cap \mathbf{VS}(f)$ belongs to \mathcal{F}_f for every $f \in R(P)$. Then for every $f \in R(P)$ and $\varphi \in L_0(\Omega, \Sigma, \mu)$ the function $\mathbf{1}_{\mathbf{VS}(f)} \varphi$ is \mathcal{F} measurable iff it is \mathcal{F}_f -measurable, and the Lemma follows. \square

Lemma 4.3. *There is a weight $w \in L_0(\Omega, \Sigma, \mu)$ with support $\mathbf{VS}(R(P))$ such that for every $f \in R(P)$, $w^{-1}N(f)$ is \mathcal{F} -measurable.*

Proof. a) First we claim that for every $f, g \in R(P)$ then $\mathbf{1}_{\mathbf{VS}(f)} \frac{N(g)}{N(f)}$ is \mathcal{F} -measurable. Since $E_f g = \langle\langle g \mid u_f \rangle\rangle u_f \in R(P)$ by Lemma 2.2, it results from Lemma 4.2 that $N(f)^{-1} \langle\langle g \mid u_f \rangle\rangle = N(f)^{-2} \langle\langle g \mid f \rangle\rangle$ is \mathcal{F} -measurable; hence its absolute value $N(f)^{-2} |\langle\langle g \mid f \rangle\rangle|$ is \mathcal{F} -measurable, and similarly $N(g)^{-2} |\langle\langle f \mid g \rangle\rangle|$ is \mathcal{F} -measurable too. Then the ratio of these functions, that is $\mathbf{1}_{\text{Supp}\langle\langle g \mid f \rangle\rangle} N(g)^2 N(f)^{-2}$ is \mathcal{F} -measurable, and so is its square root $\mathbf{1}_{\text{Supp}\langle\langle g \mid f \rangle\rangle} N(g) N(f)^{-1}$. Replacing g by $g_\varepsilon = g + \varepsilon f$, $\varepsilon > 0$ we obtain that $\mathbf{1}_{\text{Supp}\langle\langle g_\varepsilon \mid f \rangle\rangle} N(g_\varepsilon) N(f)^{-1}$ is \mathcal{F} measurable. When $\varepsilon \rightarrow 0$ we have $g_\varepsilon \rightarrow g$, $N(g_\varepsilon) \rightarrow N(g)$ (in L_p -norm) and $\text{Supp}\langle\langle g_\varepsilon \mid f \rangle\rangle \rightarrow \text{Supp} N(f) = \mathbf{VS}(f)$ (in probability). At the limit $\mathbf{1}_{\mathbf{VS}(f)} \frac{N(g)}{N(f)}$ is \mathcal{F} -measurable.

b) Let $(f_i)_{i \in I}$ be a maximal family of non zero elements in $R(P)$ with pairwise almost disjoint functional supports $\mathbf{VS}(f_i)$. Then $\mathbf{VS}(R(P)) = \bigvee_{i \in I} \mathbf{VS}(f_i)$: if $f \in R(P)$ then, since $S = \bigvee_{i \in I} \mathbf{VS}(f_i)$ belongs to \mathcal{F} , so does its complementary set S^c , and thus $\mathbf{1}_{S^c} f \in R(P)$; then, by maximality of the family (f_i) , we have $\mathbf{1}_{S^c} f = 0$, that is, $f = \mathbf{1}_S \cdot f$. We set $w = \sum_{i \in I} N(f_i)$ (which converges in $L_0(\Omega, \Sigma, \mu)$): this is a Σ -measurable weight with support $\mathbf{VS}(R(P))$. For every $f \in R(P)$ and every $i \in I$, $\mathbf{1}_{\mathbf{VS}(f_i)} w^{-1} N(f) = \mathbf{1}_{\mathbf{VS}(f_i)} N(f_i)^{-1} N(f)$ is \mathcal{F} -measurable; hence $w^{-1} N(f) = \sum_{i \in I} \mathbf{1}_{\mathbf{VS}(f_i)} w^{-1} N(f)$ is \mathcal{F} -measurable. \square

We can now give the

Proof of the Thm. 0.1. Consider the new measure $\nu = w^p \cdot \mu$, which has support $\Omega_P = \mathbf{VS}(R(P))$ and set $T : L_p(\Omega_P, \Sigma_P, \mu) \rightarrow L_p(\Omega_P, \Sigma_P, \nu)$, defined by $Tf = w^{-1} f$ (we denote by Σ_P the trace of Σ on Ω_P). Then T is an isometry; $Y := (T \otimes \text{Id}_H)(R(P))$ is a $L_\infty(\mathcal{F}_P)$ -module isometric to $R(P)$ and for every $f \in Y$ its new random norm $\tilde{N}(f) = w^{-1} N(f)$ belongs to $L_p(\Omega_P, \mathcal{F}_P, \nu)$. It results from an argument in [13] that Y is isometric to $(\bigoplus_{i \in I} L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}; H_i))_{\ell_p}$, for some families (Ω_i) of pairwise almost disjoint sets in \mathcal{F} and (H_i) of Hilbert spaces. Set then $\hat{w}_i = (\mathbb{E}(\mathbf{1}_{\Omega_i} \cdot w^p \mid \mathcal{F}))^{1/p}$, and define an isometry $S_i : L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}) \rightarrow L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \mu_{|\Omega_i})$ by $S_i f = \hat{w}_i \cdot f$. Then each $S_i \otimes \text{Id}_H$ is an onto isometry $L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}; H_i) \rightarrow L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \mu_{|\Omega_i}; H_i)$; the collection of these isometries induces an isometry of the corresponding ℓ_p -direct sums. The proof of Thm. 0.1 is complete. \square

Let us finally adapt to the present situation the argument of [13] for the commodity of the reader (and for further reference in Section 5).

Lemma 4.4. *Let (Ω, Σ, ν) be a localizable measure space, \mathcal{F} be a sub-sigma algebra such that $(\Omega, \mathcal{F}, \nu)$ is still localizable and H be a Hilbert space. Let Y be a closed $L_\infty(\mathcal{F})$ -submodule of $L_p(\Omega, \Sigma, \nu; H)$ such that for every $f \in Y$ its random norm $N(f)$ is \mathcal{F} -measurable. Then there exist a family $(\Omega_i)_{i \in I}$ of pairwise almost disjoint members of \mathcal{F} , a family (\mathcal{H}_i) of Hilbert spaces (of lower Hilbertian dimension than H) and a random norm preserving isometry from Y onto $(\bigoplus_{i \in I} L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu_{|\Omega_i}; \mathcal{H}_i))_{\ell_p}$.*

Proof. Note that by an elementary polarization argument all the scalar products $\langle\langle f | g \rangle\rangle$, $f, g \in Y$ are \mathcal{F} -measurable. Hence for every $f \in Y$, the projection E_f restricts to a projection from Y onto $L_p(\Omega, \mathcal{F}) \cdot u_f$. It results that for every closed $L_\infty(\mathcal{F})$ -submodule Z of Y there is an orthogonal projection from Y onto Z (which is the restriction of the orthogonal projection from $L_p(\Omega, \Sigma; H)$ onto the closed $L_\infty(\Sigma)$ -submodule generated by Z). In particular $Y = Z \oplus (Z^\perp \cap Y)$.

Remark that if $A \in \mathcal{F}$ is ν -sigma-finite and $M \subset Y$ is a closed $L_\infty(\mathcal{F})$ -submodule such that $\mathbf{VS}(M) \supset A$ then there exists $g \in M$ such that $\mathbf{VS}(g) = A$: take a maximal family (g_n) in M of norm-one elements with almost disjoint functional supports included in A ; this family is necessarily at most countable and $\bigvee_n \mathbf{VS}(g_n) = A$; then set $g = \sum_n 2^{-n} g_n$.

Now we claim that for every $A \in \mathcal{F}$, $A \subset \mathbf{VS}(Y)$ with positive measure, there exists a \mathcal{F} -measurable subset B of A of positive measure and a family of pairwise orthogonal element $(f_\gamma)_{\gamma \in \Gamma_B}$, such that $\mathbf{VS}(f_\gamma) = B$ for every $\gamma \in \Gamma_B$, which generates $\mathbf{1}_B \cdot Y$ as closed $L_\infty(\mathcal{F})$ -submodule. For, let $A' \subset A$ be a sigma-finite \mathcal{F} -measurable subset with positive measure, and $(g_\gamma)_{\gamma \in \Gamma}$ be a maximal family of pairwise orthogonal elements of Y with $\mathbf{VS}(g_\gamma) = A'$. If this family generates $\mathbf{1}_{A'} \cdot Y$ as closed $L_\infty(\mathcal{F})$ -submodule we can take $B = A'$. If not, consider the set $M = \{f \in Y \mid f \perp g_\gamma, \forall \gamma \in \Gamma\}$. Then M is a closed $L_\infty(\mathcal{F})$ -submodule of Y , and $\mathbf{VS}(M) \not\supset A'$ by the maximality of $(g_\gamma)_{\gamma \in \Gamma}$ (and the preceding remark). Let $B = A' \setminus \mathbf{VS}(M)$, then $(\mathbf{1}_B g_\gamma)_{\gamma \in \Gamma}$ is a maximal family in $\mathbf{1}_B \cdot Y$ of nonzero, pairwise orthogonal elements of $\mathbf{1}_B \cdot Y$. Consequently it generates $\mathbf{1}_B \cdot Y$ as $L_\infty(\mathcal{F})$ -submodule, and moreover $\mathbf{VS}(\mathbf{1}_B g_\gamma) = B$ for every $\gamma \in \Gamma$.

Let now $(\Omega_i)_{i \in I}$ be a maximal family of \mathcal{F} -measurable almost disjoint subsets of $\mathbf{VS}(Y)$ of positive measure, such that there exists for each $i \in I$ a family $(f_\gamma^i)_{\gamma \in \Gamma_i}$ of pairwise orthogonal elements with $\mathbf{VS}(f_\gamma^i) = \Omega_i$ for every $\gamma \in \Gamma_i$, which generates $\mathbf{1}_{\Omega_i} \cdot Y$ as closed $L_\infty(\mathcal{F})$ -submodule. By the claim, we have $\bigvee_{i \in I} \Omega_i = \mathbf{VS}(Y)$. Every $f \in \mathbf{1}_{\Omega_i} \cdot Y$ can be written $f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f_\gamma^i$ with $\varphi_\gamma \in L_0((\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)$; then $N(f) = (\sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2 N(f_\gamma^i)^2)^{1/2} \in L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu)$.

Note that, by refining if necessary the "partition" (Ω_i) we may suppose that each Ω_i has finite ν -measure. Then, replacing each f_γ^i by $u_{f_\gamma^i} = N(f_\gamma^i)^{-1} f_\gamma^i$, we may assume that $N(f_\gamma^i) = \mathbf{1}_{\Omega_i}$. We have then $N(f) = (\sum_{\gamma \in \Gamma_i} |\varphi_\gamma|^2)^{1/2}$ for each $f = \sum_{\gamma \in \Gamma_i} \varphi_\gamma f_\gamma^i$ in $\mathbf{1}_{\Omega_i} \cdot Y$. Let $\mathcal{H}_i = \ell^2(\Gamma_i)$. Then $T_i : L_p(\Omega_i, \mathcal{F}_{|\Omega_i}, \nu; \mathcal{H}_i) \rightarrow \mathbf{1}_{\Omega_i} \cdot Y$, $\sum_{\gamma \in \Gamma_i} \varphi_\gamma e_\gamma \mapsto \sum_{\gamma \in \Gamma_i} \varphi_\gamma u_{f_\gamma^i}$ is an (onto) isometry (preserving the random norm), and finally Y is isometric to $(\bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i))_{\ell_p}$.

For proving the assertion about the Hilbertian dimension of \mathcal{H}_i , suppose that for some $i \in I$, the Hilbertian dimension d_H of H is strictly smaller than that of \mathcal{H}_i , $d_{\mathcal{H}_i}$. We distinguish two cases:

- (i) if H is finite dimensional: select a finite subset Γ'_i of Γ_i with cardinality $d_H + 1$; since $\langle\langle f_\gamma^i | f_\delta^i \rangle\rangle = 0$ for every $\gamma \neq \delta \in \Gamma'_i$, there exists $\omega \in \Omega$ such that $\langle\langle f_\gamma^i | f_\delta^i \rangle\rangle(\omega) = 0$, i.e. the vectors $f_\gamma^i(\omega)$, $\gamma \in \Gamma'_i$ of H are pairwise orthogonal: a

contradiction.

- (ii) if H is infinite dimensional: for every $x \in H$ the set $\{\gamma \in \Gamma_i \mid \langle\langle f_\gamma^i \mid \mathbf{1}_{\Omega_i} x \rangle\rangle \neq 0\}$ is at most countable (since $\sum_\gamma |\langle\langle f_\gamma^i \mid \mathbf{1}_{\Omega_i} x \rangle\rangle|^2 \leq N(\mathbf{1}_{\Omega_i} \cdot x)^2 = \mathbf{1}_{\Omega_i} \|x\|^2$), hence if D is a dense set in H of cardinality d_H , the set $\{\gamma \in \Gamma_i \mid \exists x \in D, \langle\langle f_\gamma^i \mid \mathbf{1}_{\Omega_i} x \rangle\rangle \neq 0\}$ has cardinality $d_H < d_{\mathcal{H}_i} = \#\Gamma_i$. Hence there exists some $\gamma \in \Gamma_i$ such that $f_\gamma^i \perp \mathbf{1}_{\Omega_i} x$ for every $x \in D$, and thus for every $x \in H$, which means $f_\gamma^i = 0$, a contradiction. \square

Remark 4.5. The final argument in the proof of Lemma 4.4 shows indeed that if $L_p(\Omega, \Sigma, \nu; \mathcal{H})$ embeds in $L_p(\Omega, \Sigma, \nu; H)$ by a modularly isometric map then $\dim \mathcal{H} \leq \dim H$.

Remark 4.6. In a forthcoming paper ([10]) it will be proved that contractively complemented sublattices of $L_p(L_q)$ are isometric to “abstract $L_p(L_q)$ spaces”, i.e. bands in $L_p(L_q)$ spaces. Let us show how this permits to deduce shortly the essence of Thm. 0.1 from Lemma 4.1.

As in the proof of Lemma 4.1 let $(f_\alpha)_{\alpha \in A}$ be a maximal family of non zero, pairwise orthogonal elements of $R(P)$ and $Z = \bigoplus_\alpha L_p(\Omega, \Sigma, \mu) \cdot u_{f_\alpha}$ be the closed $L_\infty(\Sigma)$ -submodule generated by $R(P)$. There is clearly a Σ -modular isometry U from the closed submodule Z onto a band Y of the Banach lattice $L_p(\Omega, \Sigma, \mu; \mathcal{H})$ where \mathcal{H} is the discrete Banach lattice $\ell_2(A)$, such that $Ue_\alpha = N(f_\alpha)e_\alpha$, where $(e_\alpha)_{\alpha \in A}$ is a Hilbertian basis of \mathcal{H} . Then $P|_Z$ is similar by U to a contractive projection \widehat{P} of Y which preserves the spaces $Y_\alpha = L_p(A_\alpha) \cdot e_\alpha$ (where $A_\alpha = \text{Supp } N(f_\alpha)$) by Lemma 2.1 if $p = 1$ and Cor. 3.6 if $p > 1$, as well as the elements $N(f_\alpha) \cdot e_\alpha$. By the classical (scalar) theorem of Douglas if $p = 1$, and if $p > 1$, $\widehat{P}|_{Y_\alpha}$ is positive and its image is a sublattice of Y_α . Since $Y = \bigoplus_\alpha Y_\alpha$ is a decomposition in disjoint subbands, \widehat{P} is itself positive and its range is a sublattice of Y , hence of $L_p(\mathcal{H})$. By the analysis of contractive projections on sublattices in $L_p(L_q)$ -spaces developed in [10], the range $R(\widehat{P})$ is an abstract $L_p(L_2)$ -space, hence by [13] it is Banach-isometric to a ℓ_p -direct sum $\bigoplus_{i \in I} L_p(\Omega_i, H_i)$, where the H_i are Hilbert spaces. \square

5. Structure of the contractive projections

Theorem 5.1. *Let $1 \leq p < \infty$, $p \neq 2$. For every contractive projection P of $L_p(H)$ there exist a family $(u_\gamma)_{\gamma \in \Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, a positive contractive projection \widetilde{P} of $L_p(\Omega)$ and, if $p = 1$, a contractive linear operator $V : L_1(H) \rightarrow L_1(H)$ verifying $\ker V \supset \mathbf{1}_A L_1(H)$ where $A = \bigvee_{\gamma \in \Gamma} \mathbf{V}\mathbf{S}(u_\gamma)$, and $R(V) \subset \sum_\gamma R(\widetilde{P}) \cdot u_\gamma$, such that:*

$$Pf = \begin{cases} \sum_\gamma \widetilde{P}(\langle\langle f \mid u_\gamma \rangle\rangle) u_\gamma & \text{if } p \neq 1, \\ \sum_\gamma \widetilde{P}(\langle\langle f \mid u_\gamma \rangle\rangle) u_\gamma + V(f) & \text{if } p = 1, \end{cases} \tag{20}$$

for every $f \in L_p(H)$.

Conversely for every family $(u_\gamma)_{\gamma \in \Gamma}$ of pairwise orthogonal elements of $L_\infty(H)$, every positive contractive projection \tilde{P} of $L_p(\Omega)$ [and every linear contraction V of $L_1(H)$ satisfying the previous conditions of kernel and range in the case $p = 1$], the formula (20) defines a contractive projection P of $L_p(H)$.

Moreover if $p \neq 1$ the inequality $N(Pf) \leq \tilde{P}(N(f))$ holds for every $f \in L_p(H)$ [this happens also for a contractive projection of $L_1(H)$ for which the operator V of formula (20) is zero]. (\tilde{P} is a “majorizing L_p -contraction” for P in the terminology of [8]).

The proof of Thm. 5.1 will require the two following Lemmas, the first of which is specific to the $p = 1$ case:

Lemma 5.2. *Let P be a contractive projection in $L_1(H)$. Then $Pf = 0$ for every $f \in L_1(H)$ with $\mathbf{VS}(f) \subset \mathbf{VS}(R(P))$ and $f \perp R(P)$.*

Proof. Assume that $f \perp R(P)$ and $\mathbf{VS}(f) \subset \mathbf{VS}(h)$ for some $h \in R(P)$. Then $g := Pf + \mathbf{1}_{(\mathbf{VS}(Pf))^c}h$ belongs to $R(P)$ and $\mathbf{VS}(g) \supset \mathbf{VS}(f) \cup \mathbf{VS}(Pf)$. We have for every $t > 0$:

$$\begin{aligned} \int (N(g)^2 + t^2 N(f)^2)^{1/2} d\mu &= \|g + tf\| \\ &\geq \|P(g + tf)\| = \|g + tPf\| \\ &= (1 + t)\|Pf\| + \|\mathbf{1}_{(\mathbf{VS}(Pf))^c} \cdot h\| \\ &= \|g\| + t\|Pf\|. \end{aligned}$$

Hence:

$$\|Pf\| \leq \lim_{t \rightarrow 0} \left(\frac{\|g + tf\| - \|g\|}{t} \right) = \lim_{t \rightarrow 0} \int \frac{(N(g)^2 + t^2 N(f)^2)^{1/2} - N(g)}{t} d\mu = 0. \quad \square$$

Lemma 5.3. *Let P be a contractive projection in $L_p(H)$. There exists a positive contractive projection \tilde{P} on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi \cdot u_f) = (\tilde{P}\varphi) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$.*

Proof. Let \mathcal{F} be the σ -algebra of Lemma 4.2 and w be the weight of Lemma 4.3. Define $\tilde{P}_f(\varphi)$ as in the proof of Lemma 4.2. Recall that for every $f \in R(P)$ the function $w^{-1}N(f)$ is \mathcal{F} -measurable. We have then for every $h \in L_\infty(\Omega, \mathcal{F}, \mu)$:

$$\begin{aligned} \int \tilde{P}_f(\varphi)hN(f)^{p-1} d\mu &= \int \mathbb{E}_{N(f)^p \cdot \mu}^{\mathcal{F}}(N(f)^{-1}\mathbf{1}_{\mathbf{VS}(f)}\varphi)hN(f)^p \cdot d\mu \\ &= \int \mathbf{1}_{\mathbf{VS}(f)}\varphi \cdot hN(f)^{p-1} \cdot d\mu \end{aligned}$$

$$\begin{aligned} &= \int (\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1} \varphi) \cdot (\mathbf{1}_{\mathbf{VS}(f)} h(w^{-1} N(f))^{p-1}) w^p \cdot d\mu \\ &= \int \mathbb{E}_{w^p \mu}^{\mathcal{F}} (\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1} \varphi) \mathbf{1}_{\mathbf{VS}(f)} h(w^{-1} N(f))^{p-1} w^p \cdot d\mu \\ &= \int \mathbf{1}_{\mathbf{VS}(f)} w \mathbb{E}_{w^p \mu}^{\mathcal{F}} (\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1} \varphi) h N(f)^{p-1} d\mu. \end{aligned}$$

Hence $\tilde{P}_f \varphi = \mathbf{1}_{\mathbf{VS}(f)} \tilde{P} \varphi$ if we set $\tilde{P} \varphi = w \mathbb{E}_{w^p \mu}^{\mathcal{F}} (\mathbf{1}_{\mathbf{VS}(R(P))} w^{-1} \varphi)$ for every $\varphi \in L_p(\Omega, \Sigma, \mu)$. Then \tilde{P} is a positive contractive projection in $L_p(\Omega, \Sigma, \mu)$ and $P(\varphi \cdot u_f) = \tilde{P}(\varphi) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. \square

Proof of Thm. 5.1. Let Q be the orthogonal projection from $L_p(H)$ onto the closed submodule generated by $R(P)$. It results from Lemma 4.1 that if $(f_\gamma)_{\gamma \in \Gamma}$ is a maximal family of pairwise orthogonal elements of $R(P)$ then $Q = \sum_{\gamma \in \Gamma} E_{f_\gamma}$ (convergence for s.o.t.), hence $PQ = \sum_{\gamma \in \Gamma} PE_{f_\gamma}$. If $p > 1$ we know by Cor. 3.6 that $E_{f_\gamma} P = PE_{f_\gamma}$ for every γ , hence $P = QP = PQ$. If $p = 1$ let $\Pi : L_1(H) \rightarrow L_1(H)$ the projection defined by $\Pi f = \mathbf{1}_{\mathbf{VS}(R(P))} \cdot f$, then Π and $I - \Pi$ are contractive. We have $Q\Pi = \Pi Q = Q$ and it results from the preceding Lemma 5.2 that $P(I - Q)\Pi = 0$. Hence $P = PQ + V$, where $V = P(I - \Pi)$.

Let us express now PE_f when $f \in R(P)$. If \tilde{P} is the positive projection in $L_p(\Omega, \Sigma, \mu)$ defined in Lemma 5.3 we have for every $g \in L_p(\Omega, \Sigma, \mu; H)$

$$PE_f g = P(\langle\langle g | u_f \rangle\rangle \cdot u_f) = \tilde{P}(\langle\langle g | u_f \rangle\rangle) \cdot u_f$$

The formula (20) in Thm. 5.1 is now clear if we set $u_\gamma = u_{f_\gamma}$.

Conversely given (u_γ) , \tilde{P} and V , let us prove first that P is a contraction. We have for every finite subset G of Γ (using the positivity of \tilde{P}):

$$\begin{aligned} N\left(\sum_{\gamma \in G} \tilde{P}(\langle\langle f | u_\gamma \rangle\rangle) u_\gamma\right) &= \left(\sum_{\gamma \in G} |\tilde{P}(\langle\langle f | u_\gamma \rangle\rangle)|^2\right)^{1/2} \\ &= \bigvee \left\{ \left| \sum_{\gamma \in G} a_\gamma \tilde{P}(\langle\langle f | u_\gamma \rangle\rangle) \right| \mid a_\gamma \in \mathbb{C}, \sum_{\gamma \in G} |a_\gamma|^2 \leq 1 \right\} \\ &\leq \tilde{P} \left(\bigvee \left\{ \left| \sum_{\gamma \in G} a_\gamma \langle\langle f | u_\gamma \rangle\rangle \right| \mid a_\gamma \in \mathbb{C}, \sum_{\gamma \in G} |a_\gamma|^2 \leq 1 \right\} \right) \\ &= \tilde{P} \left(\left(\sum_{\gamma \in G} |\langle\langle f | u_\gamma \rangle\rangle|^2 \right)^{1/2} \right). \end{aligned}$$

Hence $\|\sum_{\gamma \in G} \tilde{P}(\langle\langle f | u_\gamma \rangle\rangle) u_\gamma\|^p \leq \int (\sum_{\gamma \in G} |\langle\langle f | u_\gamma \rangle\rangle|^2)^{p/2} d\mu$ and the sum $P_0 f := \sum_{\gamma \in \Gamma} \tilde{P}(\langle\langle f | u_\gamma \rangle\rangle) u_\gamma$ converges in $L_p(H)$. Moreover

$$N(P_0 f) \leq \tilde{P} \left(\left(\sum_{\gamma \in \Gamma} |\langle\langle f | u_\gamma \rangle\rangle|^2 \right)^{1/2} \right) \leq \tilde{P}(N(\mathbf{1}_A \cdot f))$$

(see section 1.2 and the proof of Lemma 1.1) and

$$\|P_0 f\| \leq \|\mathbf{1}_A \cdot f\|$$

where $A = \bigvee_{\gamma} \mathbf{V}\mathbf{S}(u_{\gamma})$. That P_0 is a projection follows immediately from the fact that \tilde{P} is. If $p = 1$ we have to care with the contraction V . Since $\|Vf\| \leq \|\mathbf{1}_{A^c} \cdot f\|$ we obtain $\|Pf\| \leq \|\mathbf{1}_A \cdot f\| + \|\mathbf{1}_{A^c} \cdot f\| = \|f\|$. Then since $VP_0 = 0$, $P_0V = V$, it follows clearly that $P = P_0 + V$ is a projection. \square

We can now give the structure theorem for contractive projections:

Theorem 5.4. *For every contractive projection P of $L_p(\Omega, \Sigma, \mu; H)$ ($1 \leq p < \infty$, $p \neq 2$) there exist:*

- a modularly isometric automorphism W of $L_p(H)$;
- a family $(\Omega_i)_{i \in I}$ of pairwise almost disjoint Σ -measurable subsets of Ω of positive measure;
- a family $(\mathcal{H}_i)_{i \in I}$ of Hilbert spaces;
- for every $i \in I$ a (strong operator) measurable family $(U_{i,\omega})_{\omega \in \Omega}$ of isometric embeddings of \mathcal{H}_i into H ;
- a positive contractive projection \tilde{P} of $L_p(\Omega, \Sigma, \mu)$ commuting with the band projections associated with the sets Ω_i ;
- and if $p = 1$ a contraction V from $L_1(S, \Sigma|_S, \mu|_S; H)$ into $R(P)$, where $S = \Omega \setminus \bigvee_i \Omega_i$

such that (setting $V = 0$ if $p > 1$):

$$P = WU \left(\sum_i \tilde{P}M_{\Omega_i} \otimes \text{Id}_{\mathcal{H}_i} \right) U^{\sharp} W^{-1} + V$$

where M_{Ω_i} denotes the multiplication operator by the characteristic function $\mathbf{1}_{\Omega_i}$; U is the modularly isometric embedding of $\bigoplus L_p(\Omega_i, \Sigma|_{\Omega_i}, \mu|_{\Omega_i}; \mathcal{H}_i)$ into $L_p(H)$ naturally associated with the family $(U_{i,\omega})_{i \in I, \omega \in \Omega}$ by mean of the formula:

$$(Uf)(\omega) = U_{i,\omega}(f(\omega)) \quad \text{when } \omega \in \Omega_i$$

and similarly $U^{\sharp} : L_p(H) \rightarrow \bigoplus L_p(\Omega_i, \Sigma|_{\Omega_i}, \mu|_{\Omega_i}; \mathcal{H}_i)$ is the modularly contractive map associated with the family $(U_{i,\omega}^*)_{i \in I, \omega \in \Omega}$.

Remark 5.5. In fact the families $(U_{i,\omega})_{\omega \in \Omega}$ may be chosen locally constant, i.e. there is a partition of Ω_i in Σ -measurable subsets of positive measure on which $U_{i,\omega}$ is constant.

Remark 5.6. In the case where H is separable, it is a standard (and easy) fact that every modularly isometric automorphism W of $L_p(H)$ is associated with a measurable family $(W_\omega)_{\omega \in \Omega}$ of unitary operators on H ; so we recover the Theorem 0.3 of the Introduction.

Proof. By the proof of Thm. 0.1 in Section 4, there are a sub- σ -algebra \mathcal{F} of Σ , a family $(\Omega_i)_{i \in I}$ of pairwise almost disjoint elements of \mathcal{F} , a positive weight w on Ω with support $\bigvee_{i \in I} \Omega_i$, a family $(\mathcal{H}_i)_{i \in I}$ of Hilbert spaces and for every $i \in I$ an isometry T_i from $L_p(\Omega_i, \mathcal{F}|_{\Omega_i}, \nu|_{\Omega_i}; \mathcal{H}_i)$ into $L_p(\Omega_i, \mathcal{F}|_{\Omega_i}, \nu|_{\Omega_i}; H)$ such that $R(P) = \bigoplus_{i \in I} w \cdot R(T_i)$ and moreover $N(T_i f) = N(f)$ for all $f \in L_p(\Omega_i, \mathcal{F}|_{\Omega_i}, \nu|_{\Omega_i}; \mathcal{H}_i)$ (recall that $\nu = w^p \cdot \mu$). Moreover P commutes with the action of $L_\infty(\mathcal{F})$, in particular with the multiplication operators M_{Ω_i} .

Each T_i extends uniquely to a modularly isometric map \tilde{T}_i from $L_p(\Omega_i, \Sigma|_{\Omega_i}, \nu|_{\Omega_i}; \mathcal{H}_i)$ onto the closed $L_\infty(\Sigma)$ -submodule generated by $R(T_i)$ in $L_p(\Omega, \Sigma, \nu; H)$: set simply $\tilde{T}_i(\sum_k \varphi_k f_k) = \sum_k \varphi_k T_i(f_k)$ when $\varphi_1, \dots, \varphi_n \in L_\infty(\Omega_i, \Sigma|_{\Omega_i})$ and $f_1, \dots, f_k \in L_p(\Omega_i, \mathcal{F}|_{\Omega_i}, \nu|_{\Omega_i}; \mathcal{H}_i)$ and verify that $N(\sum_k \varphi_k \tilde{T}_i(f_k)) = N(\sum_k \varphi_k f_k)$ (since T_i preserves the random scalar products).

Now define $S_i : L_p(\Omega_i, \Sigma|_{\Omega_i}, \mu|_{\Omega_i}; \mathcal{H}_i) \rightarrow L_p(\Omega, \Sigma, \mu; H)$ by $S_i f = w \tilde{T}_i(w^{-1} f)$: the range $R(S_i) = w R(\tilde{T}_i)$ is exactly $\mathbf{1}_{\Omega_i} \cdot Z$, where Z is the closed $L_\infty(\Sigma)$ -submodule generated by $R(P)$. We can glue up the maps S_i and obtain a modularly isometric embedding S from $\bigoplus_{i \in I} L_p(\Omega_i, \Sigma|_{\Omega_i}, \mu|_{\Omega_i}; \mathcal{H}_i)$ into $L_p(\Omega, \Sigma, \mu; H)$, with range $R(S) = Z$.

By Lemma 5.3 there exists a positive projection \tilde{P} on $L_p(\Omega, \Sigma, \mu)$ such that $P(\varphi \cdot u_f) = (\tilde{P}\varphi) \cdot u_f$ for every $f \in R(P)$ and $\varphi \in L_p(\Omega, \Sigma, \mu)$. Note that \tilde{P} is \mathcal{F} -modular, in particular it commutes with every multiplication operator M_{Ω_i} , $i \in I$.

If $A \in \mathcal{F}$ is a ν -integrable subset of Ω_i and $e \in \mathcal{H}_i$ we have $S_i(\mathbf{1}_A w \otimes e) = w T_i(\mathbf{1}_A \otimes e) \in R(P)$, and $N(S_i(\mathbf{1}_A \cdot w \otimes e)) = N(\mathbf{1}_A \cdot w \otimes e) = \mathbf{1}_A \cdot w$, and consequently for $f = S_i(\mathbf{1}_A w \otimes e)$ we have $f = w \cdot u_f$. Thus for every $\psi \in L_\infty(\Omega, \Sigma, \mu) \cap L_p(\Omega, \Sigma, \mu)$ we have

$$\begin{aligned} P S_i(\psi \mathbf{1}_A w \otimes e) &= P(\psi \cdot w u_f) = \tilde{P}(\psi \cdot w \mathbf{1}_{\Omega_i}) \cdot u_f = \tilde{P}(\psi \cdot w \mathbf{1}_{\Omega_i}) \cdot w^{-1} S_i(\mathbf{1}_A w \otimes e) \\ &= S_i(\tilde{P}(\psi \cdot w \mathbf{1}_{\Omega_i}) \cdot w^{-1} \cdot \mathbf{1}_A w \otimes e) = S_i(\tilde{P}(\psi \cdot w \mathbf{1}_{\Omega_i}) \mathbf{1}_A \otimes e), \end{aligned}$$

hence by linearity and density we have for every $\varphi \in L_p(\Omega_i, \Sigma|_{\Omega_i}, \mu|_{\Omega_i})$ and $e \in \mathcal{H}_i$:

$$P S_i(\varphi \otimes e) = S_i(\tilde{P}(\varphi) \otimes e),$$

that is, the restriction of P to $\mathbf{1}_{\Omega_i} \cdot Z$ is similar by S_i to the projection $\tilde{P} \otimes \text{id}_{\mathcal{H}_i}$; consequently the restriction of P to Z is similar by S to the projection $\sum_{i \in I} \tilde{P} M_{\Omega_i} \otimes \text{id}_{\mathcal{H}_i}$.

In the case where $Z = L_p(\Omega, \Sigma, \mu; H)$ we have necessarily $\dim H = \dim \mathcal{H}_i$ for every $i \in I$ since S_i is a modularly isometric map from $L_p(\Omega_i; \mathcal{H}_i)$ onto $\mathbf{1}_{\Omega_i} \cdot Z =$

$L_p(\Omega_i; H)$ (see Remark 4.5). Thus we may assume that $\mathcal{H}_i = H$ and the conclusion of Thm. 5.4 is obtained with $W = S$ and $U = \text{Id}$.

In the general case we apply Lemma 4.4 to the $L_\infty(\Sigma)$ submodule Z^\perp . We find a family $(\Omega'_j)_{j \in J}$ of pairwise almost disjoint members of Σ , a family $(\mathcal{K}_j)_{j \in J}$ of Hilbert spaces and a modularly isometric map S' from $(\bigoplus_{j \in J} L_p(\Omega'_j, \Sigma|_{\Omega'_j}, \mu|_{\Omega'_j}; \mathcal{K}_j))_{\ell_p}$ onto Z^\perp . Note that now the sets Ω'_j have no reason to belong to the smaller σ -algebra \mathcal{F} . We have $\bigvee_j \Omega'_j = \mathbf{VS}(Z^\perp)$. For the commodity of the notation we may assume $\bigvee_j \Omega'_j = \Omega$, adding if necessary one extra set $\Omega'_0 = \Omega \setminus \bigvee_j \Omega'_j$ for which we set $\mathcal{K}_0 = \{0\}$, the 0-dimensional Hilbert space. Similarly, up to the cost of adding one extra set $\Omega_0 = \Omega \setminus \Omega_P$ and setting $\mathcal{H}_0 = \{0\}$, we may assume that $\bigvee_i \Omega_i = \Omega$. We may also refine the partition (Ω'_j) by setting $\Omega'_{ij} = \Omega_i \cap \Omega'_j$ and removing the Ω'_{ij} which are almost void. This operation gives a doubly indexed family $(\Omega'_{ij})_{i \in I, j \in J_i}$.

For every $i \in I, j \in J_i$, set $L_{ij} = \mathcal{H}_i \oplus \mathcal{K}_j$ (direct Hilbertian sum). Then $L_p(\Omega'_{ij}; \mathcal{H}_i)$ and $L_p(\Omega'_{ij}; \mathcal{K}_j)$ identify naturally to a pair of mutually orthogonal $L_\infty(\Sigma)$ -submodules of $L_p(\Omega'_{ij}; L_{ij})$: if u_{ij}^0 and u_{ij}^0 are the inclusion maps of \mathcal{H}_i , resp. \mathcal{K}_j into L_{ij} then $U_{ij}^0 = \text{id} \otimes u_{ij}^0$ and $U_{ij}^0 = \text{id} \otimes u_{ij}^0$ are the corresponding embeddings of $L_p(\Omega'_{ij}; \mathcal{H}_i)$ and $L_p(\Omega'_{ij}; \mathcal{K}_j)$ into $L_p(\Omega'_{ij}; L_{ij})$. Since u_{ij}^{0*} and u_{ij}^{0*} are the orthogonal projections $L_{ij} \rightarrow \mathcal{H}_i$, resp. $L_{ij} \rightarrow \mathcal{K}_j$, we see that $U_{ij}^{0\#}$ and $U_{ij}^{0\#}$ are the orthogonal projections (in the sense given in Section 1.2) onto $L_p(\Omega'_{ij}; \mathcal{H}_i)$, resp. $L_p(\Omega'_{ij}; \mathcal{K}_j)$.

Now define $W_{ij}^0 : L_p(\Omega'_{ij}; L_{ij}) \rightarrow L_p(\Omega'_{ij}; H)$ by $W_{ij}^0 f = S_i(U_{ij}^{0\#} f) + S'_j(U_{ij}^{0\#} f)$: we have

$$N(W_{ij}^0 f)^2 = N(S_i(U_{ij}^{0\#} f))^2 + N(S'_j(U_{ij}^{0\#} f))^2 = N(U_{ij}^{0\#} f)^2 + N(U_{ij}^{0\#} f)^2 = N(f)^2$$

since S_i and S'_j are modularly isometric and have values in orthogonal subspaces Z , resp. Z^\perp . Hence W_{ij}^0 is modularly isometric and $R(W_{ij}^0) = \mathbf{1}_{\Omega'_{ij}} Z + \mathbf{1}_{\Omega'_{ij}} Z^\perp = L_p(\Omega'_{ij}; H)$.

We know by the proof of Thm. 5.1 that $P = PQ + V$, where Q is the orthogonal projection onto Z . Since V satisfies the requirements of the theorem, we look only for a representation of $P_0 = PQ$. From the first part of the proof we know that $P_0 S_i = S_i(\tilde{P} \otimes \text{id}_{\mathcal{H}_i})$; on the other hand $P_0 S'_j = 0$ since $R(S'_j) \subset Z^\perp = \ker Q$. Hence, for every $f \in L_p(\Omega_{ij}; L_{ij})$,

$$P_0 W_{ij}^0 f = P_0 S_i U_{ij}^{0\#} f + P_0 S'_j U_{ij}^{0\#} f = S_i(\tilde{P} \otimes \text{id}_{\mathcal{H}_i}) U_{ij}^{0\#} f = W_{ij}^0 U_{ij}^0 (\tilde{P} \otimes \text{id}_{\mathcal{H}_i}) U_{ij}^{0\#} f$$

i.e. P_0 is similar by W_{ij}^0 to $U_{ij}^0 (\tilde{P} M_{\Omega'_{ij}} \otimes \text{id}_{\mathcal{H}_i}) U_{ij}^{0\#}$.

Since $L_p(\Omega_{ij}; L_{ij})$ is modularly isometric to $L_p(\Omega_{ij}; H)$ (by W_{ij}^0), we have $\dim L_{ij} = \dim H$ by Rem. 4.5, so we may identify L_{ij} with H by an isomorphism θ_{ij} . This isomorphism induces in turn a modular isometry $\Theta_{ij} = \text{Id} \otimes \theta_{ij}$ from $L_p(\Omega'_{ij}; L_{ij})$ onto $L_p(\Omega'_{ij}; H)$. Set $W_{ij} = W_{ij}^0 \Theta_{ij}^{-1}$: then W_{ij} is a modular automorphism of $L_p(\Omega'_{ij}; H)$. Let also $u_{ij} = \theta_{ij} \circ u_{ij}^0$ be the embedding of \mathcal{H}_i into H resulting from this identification

and $U_{ij} = \text{id}_{L_p(\Omega_{ij})} \otimes u_{ij} = \Theta_{ij} U_{ij}^0$ be the associated embedding of $L_p(\Omega_{ij}; \mathcal{H}_i)$ into $L_p(\Omega_{ij}; \mathcal{H})$. Since $\Theta_{ij}^{-1} = \Theta_{ij}^\sharp$ we see that P_0 is similar by W_{ij} to $U_{ij}(\tilde{P}M_{\Omega'_{ij}} \otimes \text{id}_{\mathcal{H}_i})U_{ij}^\sharp$.

Finally we glue up the automorphisms W_{ij} to an automorphism W of $L_p(\Omega; \mu; H)$ by setting

$$Wf = \sum_{i \in I} \sum_{j \in J_i} W_{ij} M_{\Omega'_{ij}} f$$

and similarly we glue up the embeddings U_{ij} to an embedding U of $\bigoplus_{i \in I} L_p(\Omega_i; \mathcal{H}_i)$ into $L_p(\Omega; H)$. The maps W and U are still modularly isometric and P_0 is similar by W to $U(\sum_{i \in I} \tilde{P}M_{\Omega_i} \otimes \text{id}_{\mathcal{H}_i})U^\sharp$. \square

6. Annex: a proof of Corollary 3.6 specific to the complex case

The following proof is an adaptation of that of Thm. 4.1 in [2]. We assume that $2 < p < \infty$ (the case $1 < p < 2$ follows by duality).

If $f \in R(P)$ we introduce besides the projection E_f (defined in §1) the operators F_f and G_f defined by

$$F_f g = \mathbf{1}_{\text{vs}(f)^c} g ; G_f g = \mathbf{1}_{\text{vs}(f)} g - E_f g.$$

Then E_f, G_f and F_f are commuting modularly contractive projections in $L_p(H)$ with $E_f + F_f + G_f = I$.

Let $f, g \in R(P)$, then the elements $A(f, g)$ and $B(f, g)$ defined in §3 (eqs. (5) and (5bis)) belong to the range of P^* ; so do the sum and difference: $M_f(g) := A(f, g) + B(f, g)$ and $\Gamma_f(g) := \frac{p}{p-2}[A(f, g) - B(f, g)]$ belong to $R(P^*)$. Set

$$Q_f g = \langle\langle u_f, g \rangle\rangle u_f.$$

We have then

$$\begin{aligned} M_f(g) &= N(f)^{p-2}[2g + (p-2)E_f g] \\ \Gamma_f(g) &= pN(f)^{p-2}Q_f g \end{aligned}$$

Then M_f , resp. Γ_f are bounded linear, resp. antilinear operators from $L_p(H)$ into $L_{p^*}(H)$, and Q_f is a contractive antilinear endomorphism of $L_p(H)$ such that $Q_f^2 = E_f$; moreover:

$$M_f P = P^* M_f P, \quad \Gamma_f P = P^* \Gamma_f P. \tag{21}$$

Consider the positive symmetric bounded bilinear form defined on $L_p(H)$ by

$$(g, h)_f := \langle M_f(g), h \rangle = \int N(f)^{p-2} \langle\langle (2I + (p-2)E_f)g \mid h \rangle\rangle d\mu$$

Note that $\Gamma_f = M_f Q_f = Q_{J_p f} M_f$ and $Q_f^* = Q_{J_p f}$; then Q_f is hermitian for $(\cdot, \cdot)_f$ since

$$\begin{aligned} (Q_f g, h)_f &= \langle M_f Q_f g, h \rangle = \langle Q_f g, M_f h \rangle \\ &= \langle Q_{J_p f} M_f h, g \rangle = \langle M_f Q_f h, g \rangle = (Q_f h, g)_f \end{aligned}$$

On the other hand P is hermitian for $(\cdot, \cdot)_f$ since (using (21))

$$\begin{aligned} (P g, h)_f &= \langle M_f P g, h \rangle = \langle P^* M_f P g, h \rangle = \langle M_f P g, P h \rangle = (P g, P h)_f \\ &= \overline{(P h, P g)_f} = \overline{(P h, g)_f} = (g, P h)_f \end{aligned}$$

Let N_f be the kernel of the form $(\cdot, \cdot)_f$: we have $g \in N_f$ iff $(g, g)_f = 0$ iff $(g, h)_f = 0$ for all $h \in L_p(H)$ (by Cauchy-Schwartz inequality). Then $P N_f \subset N_f$ since

$$(P g, P g)_f = (g, P g)_f = 0 \quad \text{if } g \in N_f$$

On the other hand the operator $2 \cdot \mathbf{1}_{\mathbf{VS}(f)} + (p-2)E_f$ maps $L_p(H)$ onto $\mathbf{1}_{\mathbf{VS}(f)} L_p(H)$; hence $g \in N_f$ iff $\langle N(f)^{p-2} g, h \rangle = 0$ for every $h \in \mathbf{1}_{\mathbf{VS}(f)} L_p(H)$ iff $N(f)^{p-2} g = 0$ iff $\mathbf{1}_{\mathbf{VS}(f)} g = 0$.

We have thus $R(F_f) = N_f$ and consequently

$$P F_f = F_f P F_f$$

Since $L_p(H)$ is a strictly convex Banach space as well as its dual, we have by the auxiliary Lemma 6.1 below:

$$P F_f = F_f P$$

Let us show that $Q_f P$ is hermitian for $(\cdot, \cdot)_f$, using eq. (21) again:

$$\begin{aligned} (Q_f P g, h)_f &= \langle M_f Q_f P g, h \rangle = \langle \Gamma_f P g, h \rangle \\ &= \langle P^* \Gamma_f P g, h \rangle = \langle \Gamma_f P g, P h \rangle \\ &= (Q_f P g, P h)_f = (Q_f P h, P g)_f \\ &= (Q_f P h, g)_f \end{aligned}$$

Since Q_f and P are separately hermitian for $(\cdot, \cdot)_f$ we have

$$(Q_f P g, h)_f = (P Q_f h, g)_f,$$

hence $(P Q_f - Q_f P)h \in N_f$, i.e. $(I - F_f)P Q_f = (I - F_f)Q_f P$. Composing on the left by G_f and on the right by Q_f , or conversely, we obtain

$$G_f P E_f = 0 = E_f P G_f.$$

Since, on the other hand,

$$F_f P E_f = P F_f E_f = 0 = E_f F_f P = E_f P F_f,$$

we obtain

$$PE_f = E_fPE_f = E_fP. \quad \square$$

We state now and give a proof of the announced auxiliary Lemma.

Lemma 6.1. *Let X be a strictly convex Banach space with strictly convex dual, and P, Q two contractive projections on X . The following conditions are equivalent:*

- (i) PQ is a projection.
- (ii) $PQ = QPQ$.
- (iii) $PQ = PQP$.

If moreover the complementary projection Q^\perp is contractive too then $PQ = QP$.

Proof. If (ii) is verified then $(PQ)^2 = PQPQ = P \cdot PQ = PQ$; while if (iii) is verified then $(PQ)^2 = PQPQ = PQ \cdot Q = PQ$. Hence both (ii) and (iii) imply (i) (without any contractiveness assumption). Conversely if (i) is verified then for every $x \in R(PQ)$ we have $x = Qx = PQx$ (by [2, Prop. 1.1 (iii)]); only the strict convexity of X is needed) so $x \in R(P) \cap R(Q)$. Since the converse is trivial, we see that $R(PQ) = R(P) \cap R(Q)$; in particular $QPQ = PQ$ and (ii) is verified. Dualizing we have that P^*, Q^* and Q^*P^* are contractive projections in X^* ; hence $Q^*P^* = P^*Q^*P^*$, so $PQ = PQP$ and (iii) is verified. Now (iii) implies $PQ^\perp = PQ^\perp P$, and if Q^\perp is contractive this implies $PQ^\perp = Q^\perp PQ^\perp$ by the preceding. Then

$$Q = PQ + PQ^\perp = QPQ + Q^\perp PQ^\perp$$

which in turn implies $QP = PQ = QPQ$. □

Remark. The final assertion $PQ = QP$ of Lemma 6.1 is stated in [2] (for $X = C_p$) as Cor. 1.7 without the assumption that the complementary projection Q^\perp is contractive. This statement is not correct: if $p \neq 2$ it is easy to construct rank 1 contractive projections P, Q in $X = \ell_p$ or C_p such that $PQ = 0 \neq QP$: choose non zero elements $a, b \in X$ such that their norming functionals Ja, Jb verify $\langle Ja, b \rangle = 0$ and $\langle Jb, a \rangle \neq 0$ and set $P = a \otimes Ja, Q = b \otimes Jb$.

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