Dotted Links, Heegaard Diagrams, and Colored Graphs for PL 4-manifolds

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ABSTRACT

The present paper is devoted to establish a connection between the 4-manifold representation method by dotted framed links (or—in the closed case—by Heegaard diagrams) and the so called *crystallization theory*, which visualizes general PL-manifolds by means of edge-colored graphs.

In particular, it is possible to obtain a crystallization of a closed 4-manifold M^4 starting from a Heegaard diagram $(\#_m(\mathbb{S}^1\times\mathbb{S}^2),\omega)$, and the algorithmicity of the whole process depends on the effective possibility of recognizing $(\#_m(\mathbb{S}^1\times\mathbb{S}^2),\omega)$ to be a Heegaard diagram by crystallization theory.

Key words: PL-manifold, handle-decomposition, dotted framed link, crystallization.

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1. Introduction

The classical way to understand the structure of a closed orientable PL 4-manifold $\bar{M}^4$ is to analyze its handle-decomposition

$$\bar{M}^4 \;=\; H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_{m_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{m_2}^{(2)}) \cup (H_1^{(3)} \cup \cdots \cup H_{m_3}^{(3)}) \cup H^{(4)}$$

where each p-handle ($p \in \{0,1,2,3,4\}$) $H^{(p)} = \mathbb{D}^p \times \mathbb{D}^{4-p}$ is added to the union W of the previous handles by means of an attaching map $h : \partial\mathbb{D}^p \times \mathbb{D}^{4-p} \to \partial W$. Moreover,

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since the attachment of 3- and 4-handles is essentially performed in a unique way, up to PL-homeomorphisms (see [19] and [17]), the attention may be restricted to handles of index $p \leq 2$.

Thus, according to [19], any closed orientable PL 4-manifold may be represented by means of a *Heegaard diagram* $(\#_{m_1}(\mathbb{S}^1 \times \mathbb{S}^2), \omega)$, where ω denotes a framed link in $(\#_{m_1}(\mathbb{S}^1 \times \mathbb{S}^2) = \partial(H^{(0)} \cup (H_1^{(1)} \cup \dots \cup H_{m_1}^{(1)}))$ corresponding to the attaching instructions for the 2-handles. Note that a pair $(\#_{m_1}(\mathbb{S}^1 \times \mathbb{S}^2), \omega)$ is said to be a Heegaard diagram if and only if the result of attaching 2-handles along ω to the handlebody $\mathbb{Y}^4_{m_1} = H^{(0)} \cup (H_1^{(1)} \cup \dots \cup H_{m_1}^{(1)})$ is a (bounded) 4-manifold whose boundary is a connected sum of $m_3 \geq 0$ copies of $\mathbb{S}^1 \times \mathbb{S}^2$, but no general criterion exists to test whether this happens or not.

In an analogous but less restrictive way, César de Sà introduced in [9] the notion of *dotted framed link* in order to identify *any* bounded PL 4-manifold $M^4 = H^{(0)} \cup (H_1^{(1)} \cup \dots \cup H_{m_1}^{(1)}) \cup (H_1^{(2)} \cup \dots \cup H_{m_2}^{(2)})$. Actually, in [9], the term "special framed link" is used, instead of "dotted framed link"; however, the original term has also a different meaning—as it happens in [3] and [4]—and we prefer to avoid confusion. In short, by a dotted framed link $(L^{(d)}, c)$, we mean a framed link consisting of m_1 unknotted and unlinked 0-framed dotted components (which correspond to hypothetic 2-handles giving rise to the same boundary as the 1-handles) and of m_2 framed components (which correspond to the actual 2-handles). Obviously, if $\partial M^4 = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$, the dotted framed link uniquely determines the closed 4-manifold $\bar{M}^4 = M^4 \cup \mathbb{Y}^4_{m_3}$; hence, in this case, having a dotted framed link is perfectly equivalent to having a Heegaard diagram.

The aim of the present paper is to establish a connection between the 4-manifold representation method by dotted framed links (or equivalently—in the closed case—by Heegaard diagrams) and the so called *crystallization theory*, which visualizes general PL-manifolds by means of edge-colored graphs (see [11], [1], [5], [10], [14], [16], [22],...).

In particular, the following subsequent constructions are obtained in sections 3 and 4 respectively.

Construction 1. If $(L^{(d)}, c)$ is any dotted framed link corresponding to a bounded PL 4-manifold $M^4 = M^4(L^{(d)}, c)$, we describe an algorithmic way to construct from $(L^{(d)}, c)$ a 5-colored graph $\tilde{\Lambda}(L^{(d)}, c)$ representing M^4 (see Theorem 3.5).

Note that the boundary $\partial\tilde{\Lambda}(L^{(d)}, c) = \Lambda(L^{(d)}, c)$ of the 5-colored graph $\tilde{\Gamma}(L^{(d)}, c)$ turns out to be a 4-colored graph representing the closed orientable 3-manifold $M^3(L^{(d)}, c) = \partial M^4(L^{(d)}, c)$ obtained from $\mathbb{S}^3$ by Dehn surgery along the framed link underlying $(L^{(d)}, c)$.

Construction 2. If $M^3 = \partial M^4 = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$ (i.e. if $(L^{(d)}, c)$ determines a closed 4-manifold $\bar{M}^4(L^{(d)}, c) = M^4 \cup \mathbb{Y}^4_{m_3}$), then it is always possible to yield from $\tilde{\Lambda}(L^{(d)}, c)$ a 5-colored graph $\bar{\Lambda}(L^{(d)}, c)$ representing $\bar{M}^4(L^{(d)}, c)$ (see Theorem 4.8). In particular, if the 4-colored graph $\Lambda(L^{(d)}, c)$ does satisfy suitable combinatorial conditions (which are known to imply $M^3 = \partial M^4 = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$) the passage from $\tilde{\Lambda}(L^{(d)}, c)$ to $\bar{\Lambda}(L^{(d)}, c)$ is nothing but a boundary identification (see Proposition 4.2).

Unfortunately, $\partial M^4 = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$ is not always sufficient to satisfy the required conditions, as proved in Proposition 4.6. This facts yields a counterexample to a conjecture stated in [16] (see Corollary 4.7).

In other words, the present paper shows how to obtain a crystallization of the closed 4-manifold $\bar{M}^4$ starting from a Heegaard diagram $(\#_{m_1}(\mathbb{S}^1 \times \mathbb{S}^2), \omega)$, and the algorithmicity of the whole process depends on the effective possibility of recognizing $(\#_{m_1}(\mathbb{S}^1 \times \mathbb{S}^2), \omega)$ to be a Heegaard diagram by crystallization theory.

2. Framed links and crystallizations of simply connected 4-manifolds

Throughout the work, a *framed link* is intended to be a pair (L, c), where $L = L_1 \cup \dots \cup L_l$ is a link in $\mathbb{S}^3$ with $l \geq 1$ components and $c = (c_1, \dots, c_l)$ is an l-tuple of integers. According to a wide and well-established literature ([15], [18],...), any framed link (L, c) uniquely represents a simply-connected bounded PL 4-manifold $M^4 = M^4(L, c)$, which is obtained from the 4-disk $\mathbb{D}^4$ by adding 2-handles along the framed link (L, c):

$$M^4 = M^4(L, c) = \mathbb{D}^4 \cup (H_1^{(2)} \cup \dots \cup H_l^{(2)})$$

where the attaching map $f_i : \mathbb{S}^1 \times \mathbb{D}^2 \to \partial \mathbb{D}^4$ of the i-th 2-handle $H_i^{(2)}$ $(i \in \{1, \dots, l\})$ is such that $f_i(\mathbb{S}^1 \times \{0\}) = L_i$ has linking number c_i with $f_i(\mathbb{S}^1 \times \{x\})$, for every $x \in \mathbb{D}^2 - \{0\}$. Moreover, the boundary of $M^4(L, c)$ is the 3-manifold $M^3 = M^3(L, c)$ which is obtained from $\mathbb{S}^3$ by performing a Dehn surgery on (L, c).

Recently, in [7], the above representation of (3- and) 4-manifolds by framed links has been put in closed connection with "crystallization theory": in fact, an edge-colored graph $\tilde{\Lambda}(L, c)$ representing $M^4(L, c)$ is easily obtained from any planar diagram of the link itself.

In order to describe the construction of $\tilde{\Lambda}(L, c)$, it is necessary to assume the link L embedded in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, so that its projection P on the plane $\pi : \mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$ consists of all regular points, and m double points $p_1, \dots p_m$ (the *crossings* of L); thus, $\pi - \mathcal{P}$ results to have exactly $m + 2$ connected components, which are called the *regions* of L. Actually, both the crossings and the regions ought to be referred to a *planar diagram* of L; however, the assumptions about space position allow us to identify the link L and its planar diagram on π.

If an orientation is fixed on each component L_i of L (with $i \in \{1, 2, \dots, l\}$), then L_i is said to have *writhe* $w(L_i)$, where $w(L_i)$ is the algebraic sum of the signs (computed by the rule of Fig. 1) of all the (self-)crossings of L_i. Moreover, if (L, c) is a framed

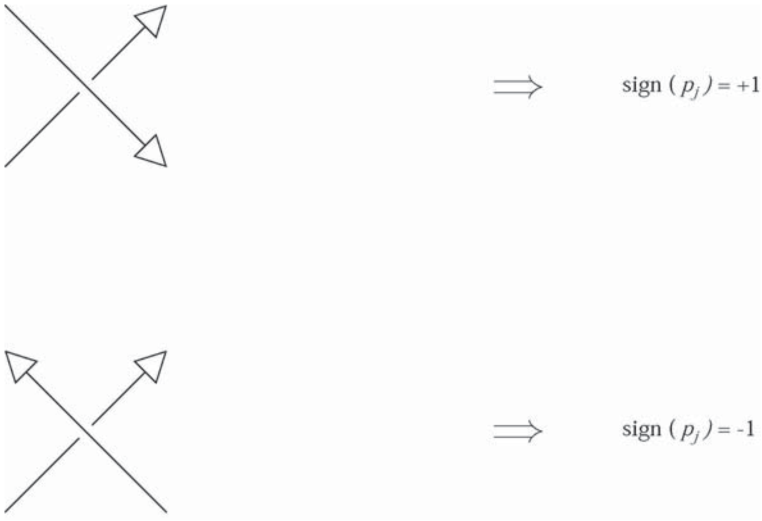


Figure 1

link, then the component L_i of L (with $i \in \{1, 2, \ldots, l\}$) is said to need $t_i = |c_i - w(L_i)|$ *additional curls*, positive or negative according to whether c_i is greater or less than $w(L_i)$ (see Fig. 2).

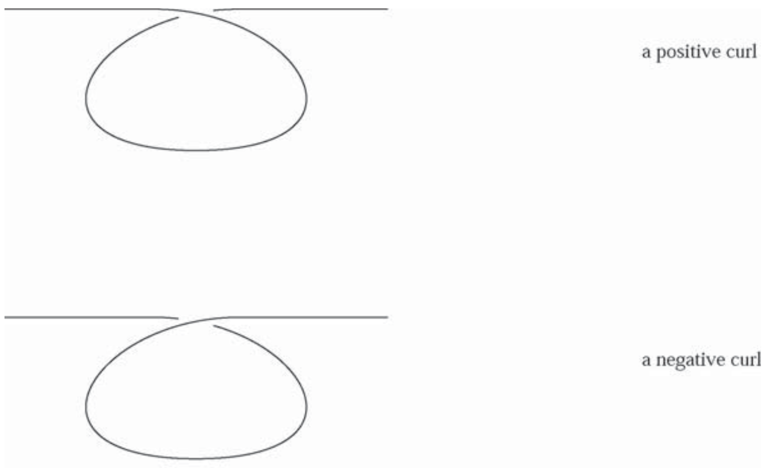
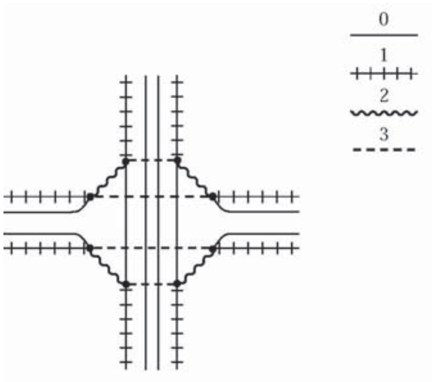


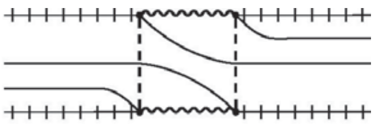
Figure 2

The following rules allow us to construct a 4-colored graph $\Lambda(L, c)$ directly from (L, c).

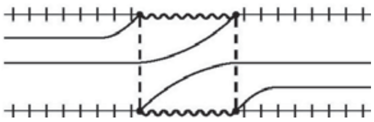
(i) For every crossing p_j of L, construct a partial order eight graph, in the following way:



(ii) For every additional curl, construct one of the following partial order four graphs:



if the curl is a positive one



if the curl is a negative one

(iii) Finally, connect the "hanging" 0- and 1-colored edges, so that every region of L (having r crossings in its boundary) gives rise to a $\{1, 2\}$-colored cycle of length $2r$, and every component of L (having s crossings and t additional curls) gives rise to two $\{0, 3\}$-colored cycles of length $2(s + t)$.

It is not difficult to check that (by possibly adding trivial pairs of opposite additional curls) each component L_i of L gives rise in $\Lambda(L, c)$ to a subgraph $Q^{(i)}$ (a *quadricolor*) with the following structure: $Q^{(i)}$ consists of four vertices $P_0^{(i)}, P_1^{(i)}, P_2^{(i)}, P_3^{(i)}$ and four edges $e_0^{(i)}, e_1^{(i)}, e_2^{(i)}, e_3^{(i)}$, $e_r^{(i)}$ being an r-colored edge between $P_r^{(i)}$ and $P_{r+1}^{(i)}$,

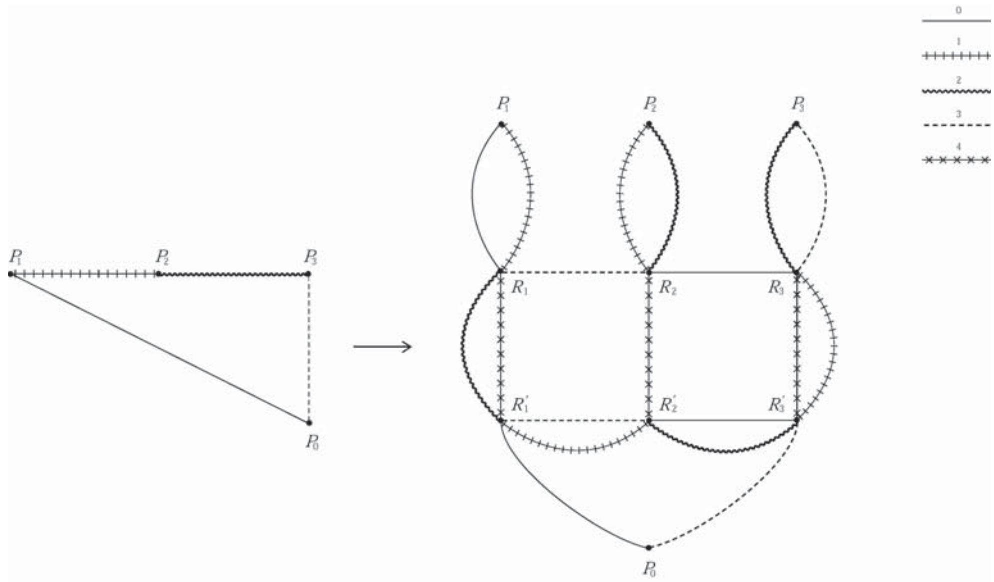


Figure 3

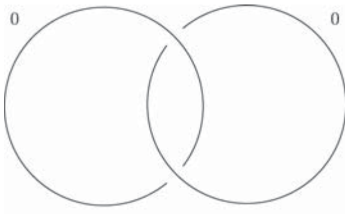
for every $r \in \mathbb{Z}_3$, with the condition that $P_r^{(i)}$ does not belong to the $\{r+1, r+2\}$-colored cycle containing $P_{r+1}^{(i)}, P_{r+2}^{(i)}, P_{r+3}^{(i)}$.

Now, let $\tilde{\Lambda}(L,c)$ be the 5-colored graph directly obtained from the 4-colored graph $\Lambda(L,c)$ by substituting each quadricolor $Q^{(i)}$ ($i \in \{1,\dots,l\}$) with the order ten 5-colored subgraph depicted in Fig. 3. The following result summarizes the meaning of the above described constructions:

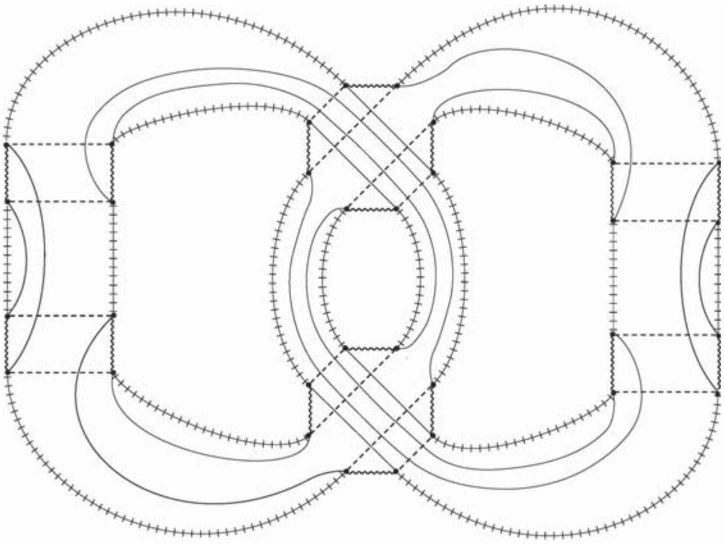
Proposition 2.1 ([7]). *For every framed link* (L,c)*, the* 5*-colored graph* $\tilde{\Lambda}(L,c)$ *represents the simply connected* 4*-manifold* $M^4(L,c)$*. Moreover,* $\tilde{\Lambda}(L,c)$ *admits as its boundary graph (see [11] for details) the* 4*-colored graph* $\Lambda(L,c)$*, which represents the* 3*-manifold* $M^3(L,c)$*.*

Example 2.2. If $(L,(0,0))$ is the 0-framed Hopf link (depicted in Fig. 4(a), then the associated 4-colored graph $\Lambda(L,(0,0))$ (resp. 5-colored graph $\tilde{\Lambda}(L,(0,0))$) is shown in Fig. 4(b) (resp. Fig. 4(c)); by Proposition 2.1, it represents $M^3 = \mathbb{S}^3$ (resp. $M^4 = \mathbb{S}^2 \times \mathbb{S}^2 - \mathbb{D}^4$).

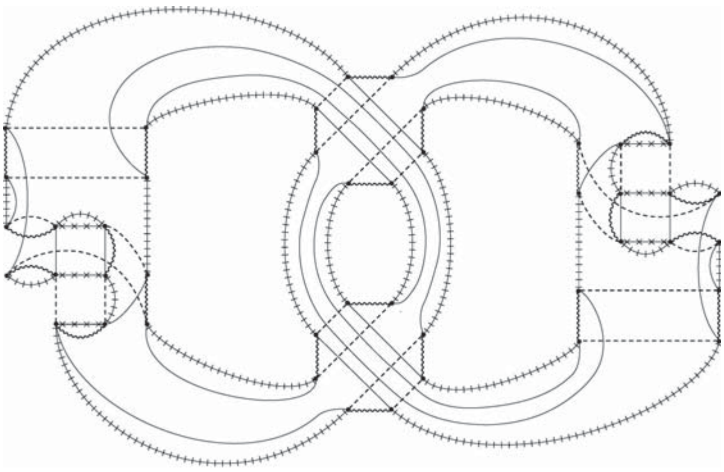
For the purpose of the present work, it is necessary to give a hint of the proof for Proposition 2.1. First, we have to recall some fundamental notions and terminology of crystallization theory; for a much more detailed account, we refer to [11], where a useful bibliography may also be found.



(a)



(b)



(c)

Figure 4

An $(n+1)$-*colored graph* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph (i.e. multiple edges are allowed, while loops are forbidden) and $\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \dots, n\}$ is an edge-coloration, with $\gamma(e) \neq \gamma(f)$ for every pair e, f of adjacent edges; moreover, the vertices of $V(\Gamma)$ may have either degree $n+1$ (*internal* vertices) or n (*boundary* vertices), and in this last case no incident edge can be colored by $n+1$.

Within crystallization theory, each $(n+1)$-colored graph (Γ, γ) is thought of as a visualizing tool for an n-dimensional labeled pseudocomplex (see [13]) $K(\Gamma)$, which is constructed according to the following rules:

(i) For each vertex $v \in V(\Gamma)$, take an n-simplex $\sigma(v)$, with its vertices labeled by $0, 1, \dots, n$.

(ii) For each j-colored edge between v and w ($v, w \in V(\Gamma)$), identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertex labeled by j, so that equally labeled vertices coincide.

If $K(\Gamma)$ triangulates a PL n-manifold M^n, then (Γ, γ) is said to *represent* M^n; in particular, an $(n+1)$-colored graph representing the n-manifold M^n (with empty or connected boundary) is called a *crystallization* of M^n, in case the subgraph $\Gamma_{\hat{j}} = (V(\Gamma), \gamma^{-1}(\Delta_n - \{j\}))$ is connected, for each $j \in \Delta_n$. A basic result of the theory (known as the Pezzana Theorem) states that *every* PL n-manifold admits both $(n+1)$-colored graphs and crystallizations representing it.

Now, we point out that the construction of $K(\Gamma)$ allows us to easily prove that an $(n+1)$-colored graph (Γ, γ) represents a bounded (resp. closed) n-manifold if and only if the n-colored subgraph $\Gamma_{\hat{j}}$ represents a disjoint union of copies of $\mathbb{S}^n$ for $j = n$, and a disjoint union of copies of either $\mathbb{S}^n$ or $\mathbb{D}^n$ for every $j \in \Delta_{n-1}$ (resp. a disjoint union of copies of $\mathbb{S}^n$, for every $j \in \Delta_n$).

In particular, for every framed link (L, c), the subgraph $\tilde{\Lambda}_{\hat{4}}(L, c)$ of $\tilde{\Lambda}(L, c)$ may be proved to represent a colored triangulation $K(L, c) = K(\tilde{\Lambda}_{\hat{4}}(L, c))$ of $\mathbb{S}^3$, whose 1-skeleton contains two copies $L' = L'_1 \cup \dots \cup L'_l$, $L'' = L''_1 \cup \dots \cup L''_l$ of $L = L_1 \cup \dots \cup L_l \subset \mathbb{S}^3$. Further, the linking number between L'_i and L''_i in $K(L, c)$ is equal to c_i, for every $i \in \{1, \dots l\}$.

More precisely, according to notations of Fig. 3, the copy L'_i (resp. L''_i) of the i-th component L_i of L (for every $i \in \{1, \dots l\}$) consists of the two $\{0, 3\}$-labeled edges (resp. $\{1, 2\}$-labeled edges) of tetrahedra $\sigma(R_2^{(i)})$, $\sigma(R_2'^{(i)})$ of $K(L, c)$, having both the same $\{0, 1\}$-labeled edge and the same $\{2, 3\}$-labeled edge. Thus, L'_i and L''_i turn out to be two different longitudes of the same solid torus embedded in $K(L, c)$, i.e. the subcomplex consisting of tetrahedra $\sigma(R_r^{(i)})$ and $\sigma(R_r'^{(i)})$, for $r \in \{1, 2, 3\}$.

At this point, it is not difficult to understand the PL-structure of the 4-dimensional pseudocomplex—$\tilde{K}(L, c)$, say—associated to $\tilde{\Lambda}(L, c)$: since $\tilde{K}(L, c)$ is directly obtained from the cone over $K(L, c)$ (i.e. a 4-disk $\mathbb{D}^4$) by pairwise identification of tetrahedra $\sigma(R_r^{(i)})$ and $\sigma(R_r'^{(i)})$, for $r \in \{1, 2, 3\}$ and $i \in \{1, \dots, l\}$, $\tilde{K}(L, c)$ admits the handle-decomposition $\mathbb{D}^4 \cup H_1^{(2)} \cup \cdots \cup H_l^{(2)}$, with attaching maps $f_i : \mathbb{S}^1 \times \mathbb{D}^2 \to \partial\mathbb{D}^4$ (for every $i \in \{1, \ldots, l\}$) satisfying $f_i(\mathbb{S}^1 \times \{0\}) = L'_i$ and $f_i(\mathbb{S}^1 \times \{x\}) = L''_i$, for some $x \in \mathbb{D}^2 - \{0\}$. This obviously implies that $\tilde{\Lambda}(L,c)$ represents $M^4(L,c)$, as the first part of Proposition 2.1 states. On the other hand, $\Lambda(L,c)$ exactly coincides with the boundary graph $\partial\tilde{\Lambda}(L,c)$ of $\tilde{\Lambda}(L,c)$. In fact, by construction, $\partial\tilde{\Lambda}(L,c)$ has a vertex for every boundary vertex of $\tilde{\Lambda}(L,c)$, and a j-colored edge ($j \in \Delta_3$) for every $\{j,4\}$-colored path in $\tilde{\Lambda}(L,c)$ joining two boundary vertices. Since the boundary graph always represents the boundary manifold (see [11] for details), the second part of Proposition 2.1 follows, too.

Actually, $K(\Lambda(L,c)) = M^3(L,c)$ is also a consequence of the fact that $\Lambda(L,c)$ may be easily obtained from the 4-colored graph (Λ^*, λ^*) described in [16] and [14] (and directly proved to represent $M^3(L,c)$) by a finite sequence of admissible moves, (called *dipole moves*), which are known to link different graphs representing the same manifold.

Recall that, if (Γ,γ) (with $\#V(\Gamma) > 2$) is an $(n+1)$-colored graph representing a PL n-manifold M^n, then an *h-dipole* ($1 \le h \le n$) of (Γ,γ) is a subgraph $\Theta = \{v, w\}$ consisting of two vertices $v, w \in V(\Gamma)$ joined by h edges colored by $j_1, j_2, \ldots, j_h \in \Delta_n$ and satisfying the following conditions:

(i) The vertices v and w belong to different connected components, Ξ_1 and Ξ_2 say, of the graph $\Gamma_{\Delta_n - \{j_1,\ldots,j_h\}} = (V(\Gamma), \gamma^{-1}(\Delta_n - \{j_1, \ldots, j_h\}))$.

(ii) If either v or w is an internal vertex, then either Ξ_1 or Ξ_2 is a regular graph of degree $n + 1 - h$.

The *elimination* of the h-dipole Θ is performed by deleting Θ from (Γ,γ) and welding the "hanging" pairs of edges of the same color $j \in \Delta_n - \{j_1, \ldots, j_h\}$; if (Γ', γ') is the resulting $(n+1)$-colored graph (with $K(\Gamma') = K(\Gamma) = M^n$), then we will also say that (Γ,γ) is obtained from (Γ', γ') by *insertion* of an h-dipole of colors $\{j_1, j_2, \ldots, j_h\}$ and that (Γ,γ) and (Γ', γ') are obtained from each other by a *dipole move*.

3. From dotted framed links to crystallizations of bounded 4-manifolds

The starting point for the notion of dotted framed link is the fact that 1-handles in orientable 4-manifolds may be "traded for" 2-handles (see [9] and [18]).

In short, if the orientable 4-manifold W_1^4 is obtained from W^4 by adding a 1-handle $H^{(1)}$ and if $H^{(2)}$ is the complementary handle of $H^{(1)}$ in W_1^4, then $W_1^4 = W^4 \cup H^{(1)}$ has the same boundary as $W_2^4 = W^4 \cup \overset{*}{H}{}^{(2)}$, where $\overset{*}{H}{}^{(2)}$ is the 2-handle dual to $H^{(2)}$ in W_1^4. Moreover, the surgery instructions for the 2-handle $\overset{*}{H}{}^{(2)}$ always corresponds to an unknotted 0-framed circle in ∂W^4.

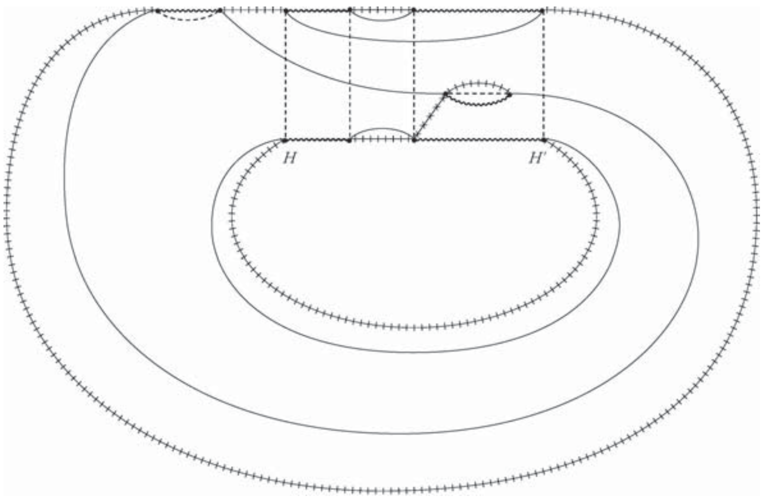


Figure 5

Hence, if a bounded PL 4-manifold admits a handle-decomposition consisting of m_1 1-handles and m_2 2-handles (i.e. $M^4 = H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_{m_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{m_2}^{(2)})$), then it may be represented by an $(m_1 + m_2)$-component link in $\mathbb{S}^3 = \partial H^{(0)}$, with m_1 unknotted and unlinked dotted 0-framed components (which correspond to traded 1-handles) and m_2 (possibly knotted and linked) framed components (which correspond to the surgery instructions for the actual 2-handles). If $(L^{(d)}, c)$ is such a dotted framed link, the present section is devoted to describing an algorithmic way to construct a 5-colored graph representing the associated 4-manifold $M^4 = M^4(L^{(d)}, c)$. A first, minimal step is carried out using the following result.

Proposition 3.1. *Let $(K_0^{(d)}, 0)$ be the dotted framed link consisting of a unique dotted component (i.e. $(K_0^{(d)}, 0)$ is the 0-framed dotted trivial knot). Then, the 5-colored graph $\tilde{\Lambda}(K_0^{(d)}, 0)$ depicted in Fig. 5 represents the 4-manifold $\mathbb{S}^1 \times \mathbb{D}^3 = M^4(K_0^{(d)}, 0)$ and admits the same boundary graph as the 5-colored graph $\tilde{\Lambda}(K_0, 0)$ associated to the underlying framed link (i.e. the 0-framed trivial knot $(K_0, 0)$).*

Proof. It is very easy to check that the subgraph $\{H, H'\}$ of $\tilde{\Lambda}(K_0^{(d)}, 0)$ is a 2-dipole; moreover, the elimination of $\{H, H'\}$ gives rise to the standard 5-colored graph representing $\mathbb{S}^1 \times \mathbb{D}^3$ (see, for example, [2, Theorem 3 (iii)]). On the other hand, the last part of the statement immediately follows by direct construction of the boundary graph. □

Another important step is due to the characteristic structure of graphs $\tilde{\Lambda}(L, c)$.

In order to describe it, we need further definitions and results from crystallization theory.

Definition. Let (Γ', γ') and (Γ'', γ'') be two $(n+1)$-colored graphs and let $v' \in V(\Gamma')$ and $v'' \in V(\Gamma'')$ be two internal (resp. boundary) vertices; moreover, let $\Gamma' \#_{\{v',v''\}} \Gamma''$ be the $(n+1)$-colored graph obtained from Γ' and Γ'' by deleting $\{v', v''\}$ and welding the "hanging" edges of the same color $c \in \Delta_n$ (resp. $c \in \Delta_{n-1}$). The process leading from Γ', Γ'' to $\Gamma' \#_{\{v',v''\}} \Gamma''$ is said to be a *graph connected sum*, while the process leading from $\Gamma' \#_{\{v',v''\}} \Gamma''$ to the disjoint union of Γ' and Γ'' is said to be an *inverse of a graph connected sum*.

Proposition 3.2 ([2]). *If Γ' and Γ'' represent two n-manifolds M_1^n and M_2^n, and if v' and v'' are internal (resp. boundary) vertices, then $\Gamma' \#_{\{v',v''\}} \Gamma''$ represents the n-manifold $M_1^n \# M_2^n$ (resp. $M_1^n \, {}^\partial\# M_2^n$), where $\#$ (resp. ${}^\partial\#$) is the symbol of connected sum (resp. boundary connected sum).*

Let now assume (L, c) is a given framed link, with $l \geq 2$ components, and let $(L^{(\hat{l})}, c^{(\hat{l})})$ be the (possibly disconnected) framed link obtained by deleting the last component (i.e. $L^{(\hat{l})} = L_1 \cup \cdots \cup L_{l-1}$ and $c^{(\hat{l})} = (c_1, c_2, \ldots, c_{l-1})$).

Proposition 3.3. *Let $\tilde{\Lambda}^{(\hat{l})}(L, c)$ be the 5-colored graph obtained from $\tilde{\Lambda}(L, c)$ by deleting the 4-colored edges between $R_r^{(l)}$ and $R_r'^{(l)}$, for $r \in \{1, 2, 3\}$; then, $\tilde{\Lambda}^{(\hat{l})}(L, c)$ represents the simply connected 4-manifold associated to the framed link $(L^{(\hat{l})}, c^{(\hat{l})})$ (or the boundary connected sum of the associated 4-manifolds, in case $(L^{(\hat{l})}, c^{(\hat{l})})$ has a disconnected planar projection). Moreover, a finite sequence of graph moves exists, which consists of dipole eliminations and possibly inverses of graph connected sums, that transforms $\tilde{\Lambda}^{(\hat{l})}(L, c)$ into the possibly disconnected graph $\tilde{\Lambda}(L^{(\hat{l})}, c^{(\hat{l})})$ (resp. $\partial\tilde{\Lambda}^{(\hat{l})}(L, c)$ into the possibly disconnected graph $\Lambda(L^{(\hat{l})}, c^{(\hat{l})})$).*

Proof. Obviously, the first part of the statement is a consequence of the last one, via Proposition 3.2. On the other hand, the 5-colored graph $\tilde{\Lambda}^{(\hat{l})}(L, c)$ immediately appears to contain five 2-dipoles (i.e. the 2-dipoles $\bar{\theta}_1^{(l)} = \{P_1^{(l)}, R_1^{(l)}\}$, $\bar{\theta}_2^{(l)} = \{P_2^{(l)}, R_2^{(l)}\}$, $\bar{\theta}_3^{(l)} = \{P_3^{(l)}, R_3^{(l)}\}$, $\bar{\theta}_4^{(l)} = \{R_1'^{(l)}, R_2'^{(l)}\}$, $\bar{\theta}_5^{(l)} = \{P_0^{(l)}, R_3'^{(l)}\}$), whose eliminations make the quadricolor $Q^{(l)}$ to disappear. Further, the required sequence of graph moves may be easily completed, by simply "following" the subgraph of $\tilde{\Lambda}(L, c)$ (resp. of $\Lambda(L, c)$) corresponding to the l-th component of L. □

Let now $(L^{(d)}, c)$ be a dotted framed link. Without loss of generality, we may order the $l = m_1 + m_2$ (with $m_1, m_2 > 0$) components of L, so that L_i becomes unknotted, unlinked, dotted and 0-framed, for every $i \in \{1, \ldots, m_1\}$. If $\tilde{\Lambda}(L, c)$ is the 5-colored graph associated to the underlying framed link (L, c), set

$$\tilde{\Lambda}^{(d)}(L, c) = \tilde{\Lambda}^{(\widehat{m_1+1}) \cdots (\widehat{m_1+m_2})}(L, c).$$

This means that $\tilde{\Lambda}^{(d)}(L,c)$ is obtained from $\tilde{\Lambda}(L,c)$ by deleting the 4-colored edges corresponding to the undotted components of $(L^{(d)},c)$.

Since $L_i = K_0$ and $c_i = 0$ hold for every $i \in \{1,\dots,m_1\}$, Proposition 3.3 directly yields the following

Corollary 3.4. *With the above notations, we have*

(i) *The* 5*-colored graph* $\tilde{\Lambda}^{(d)}(L,c)$ *represents* ${}^{\partial}\#_{m_1}(\mathbb{S}^2 \times \mathbb{D}^2)$.

(ii) *A well-determined sequence of graph moves exists, which consists of a finite number of dipole eliminations and exactly* $m_1 - 1$ *inverses of graph connected sums, and transforms* $\partial\tilde{\Lambda}^{(d)}(L,c)$ *into* $\bigsqcup_{m_1} \Lambda(K_0,0)$ *(i.e. the disjoint union of* m_1 *copies of the* 4*-colored graph associated to the* 0*-framed trivial knot).*

We are now able to prove the existence of the already stated algorithmic procedure (Construction 1).

Theorem 3.5. *Let* $(L^{(d)},c)$ *be a dotted framed link and* (L,c) *the underlying framed link. Then, there is an algorithm for constructing a* 5*-colored graph* $\tilde{\Lambda}(L^{(d)},c)$ *such that:*

(i) *The graph* $\tilde{\Lambda}(L^{(d)},c)$ *represents the* 4*-manifold* $M^4(L^{(d)},c)$*, obtained from* $\mathbb{D}^4$ *by adding* 1*-handles and* 2*-handles according to* $(L^{(d)},c)$.

(ii) *Its boundary graph* $\partial\tilde{\Lambda}(L^{(d)},c)$ *is exactly* $\Lambda(L,c)$.

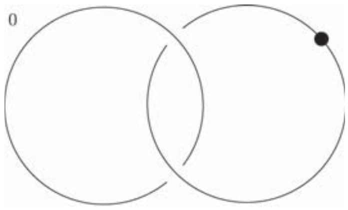
Proof. First, let us state how to construct $\tilde{\Lambda}(L^{(d)},c)$.

Step 1: Consider the disjoint union $\bigsqcup_{m_1} \tilde{\Lambda}(K_0^{(d)},0)$ of m_1 copies of the 5-colored graph of Fig. 5, having $\bigsqcup_{m_1} \Lambda(K_0,0)$ as boundary graph.

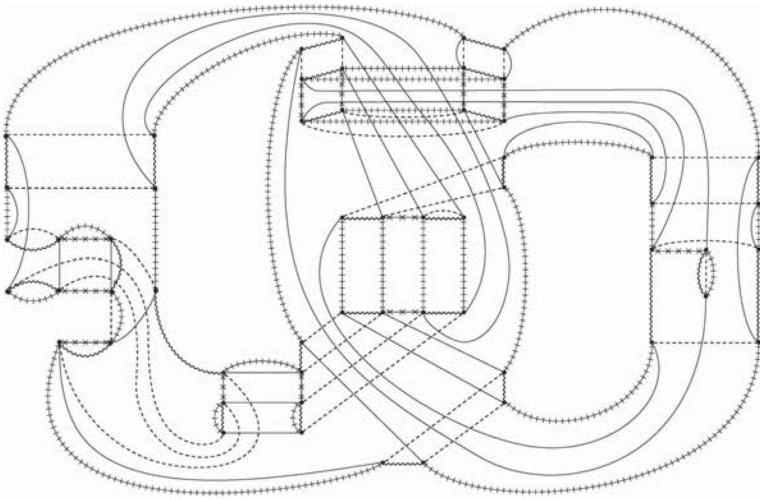
Step 2: By Corollary 3.4 and [8, Lemma B], a well-determined sequence of graph moves exists, which consists of a finite number of dipole insertions and exactly $m_1 - 1$ graph connected sums, and transforms $\bigsqcup_{m_1} \tilde{\Lambda}(K_0^{(d)},0)$ into a 5-colored graph $\Omega(L^{(d)},c)$ of ${}^{\partial}\#_{m_1}(\mathbb{S}^1 \times \mathbb{D}^3) = \mathbb{Y}^4_{m_1}$, having the same boundary as $\tilde{\Lambda}^{(d)}(L,c)$;

Step 3: $\tilde{\Lambda}(L^{(d)},c)$ is simply obtained from $\Omega(L^{(d)},c)$ by adding a 4-colored edge between $R_r^{(i)}$ and $R_r'^{(i)}$, for every $r \in \{1,2,3\}$ and for every $i \in \{m_1+1,\dots,m_1+m_2\}$.

Note that the aim of step 2 is to reproduce on 5-colored graphs (starting from $\bigsqcup_{m_1} \tilde{\Lambda}(K_0^{(d)},0)$, whose boundary graph coincides with $\bigsqcup_{m_1} \Lambda(K_0,0)$) the inverse sequence of moves on 4-colored graphs described in Corollary 3.4. Obviously, no problem arises from graph connected sums (see Proposition 3.2). On the other hand, if Φ is an $(n+1)$-colored graph representing an n-manifold M^n and if Γ is obtained from $\partial\Phi$ by inserting a dipole Θ within the colored subgraph Ξ, then [8, Lemma B] indicates how to obtain another graph $\bar{\Phi}$ of M^n, with $\partial\bar{\Phi} = \Gamma$. If the dipole Θ cannot be directly inserted in Φ, then it may be inserted within the so called "double-layer" over Ξ (which may be added to Φ by a finite sequence of dipole insertions: see [8] for details).



(a)



(b)

Figure 6

Let us now consider the 5-colored graph $\Omega(L^{(d)}, c)$. By construction, it really represents $\mathbb{Y}^4_{m_1} = H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{m_1})$ and has the same boundary as $\tilde{\Lambda}^{(d)}(L, c)$. Thus, for every $i \in \{m_1 + 1, \ldots, m_1 + m_2\}$, the addition of the 4-colored edges between $R^{(i)}_r$ and $R'^{(i)}_r$, for $r \in \{1, 2, 3\}$, has the topological effect of adding a 2-dipole according to the surgery instructions corresponding to the i-th (undotted) component of $(L^{(d)}, c)$. (Recall the hint of proof for Proposition 2.1 given in the second section.) □

Example 3.6. If $(L^{d)}, c)$ is the dotted framed link depicted in Fig. 6(a), then Construction 1 allows us to algorithmically construct the 5-colored graph $\tilde{\Lambda}(L^{(d)}, c)$ of Fig. 6(b). Note that it has the same boundary graph as the 5-colored graph $\tilde{\Lambda}(L, (0, 0))$ shown in Fig. 4(c) (i.e. the 4-colored graph $\Lambda(L, (0, 0))$ shown in Fig. 4(b)). Moreover, the link calculus for 4-manifolds (see [9] or [19]) ensures that $\tilde{\Lambda}(L^{(d)}, c)$ represents the

4-disk $\mathbb{D}^4$.

4. From Heegaard diagrams to crystallizations of closed 4-manifolds

The present section takes into account the case of a dotted framed link $(L^{(d)}, c)$ such that the 3-manifold represented by its underlying framed link is a connected sum of $m_3 \geq 0$ copies of $\mathbb{S}^1 \times \mathbb{S}^2$ (i.e. $(L^{(d)}, c)$ such that $\partial M^4(L^{(d)}, c) = M^3(L, c) = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$, where $\#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$ is intended to indicate the 3-sphere $\mathbb{S}^3$, in case $m_3 = 0$). As already pointed out in the introduction, such a dotted framed link uniquely represents the closed 4-manifold $\bar{M}^4 = M^4(L^{(d)}, c) \cup \mathbb{Y}^4_{m_3}$; in other words—according to [19]—$(L^{(d)}, c)$ turns out to be equivalent to a Heegaard diagram $(\#_{m_1}(\mathbb{S}^1 \times \mathbb{S}^2), \omega)$ of $\bar{M}^4$.

Unfortunately, known results about the characterization of $\mathbb{S}^3$ and/or $\#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$ (see [20] and [21]) are not so useful for concrete applications, both to crystallization theory and to other classical representation methods for 3-manifolds. However, the following combinatorial structures within 4-colored graphs yield interesting information about the associated 3-manifolds.

Definition. Let (Γ, γ) be a 4-colored graph representing a closed orientable 3-manifold M^3.

(i) Two i-colored edges $e, f \in E(\Gamma)$ $(i \in \Delta_3)$ are said to be a *ρ_2-pair* (resp. a *ρ_3-pair*) if e and f belong both to the same $\{i, j\}$-colored cycle and to the same $\{i, k\}$-colored cycle of Γ, with $j, k \in \Delta_3 - \{i\}$ (resp. to the same $\{i, c\}$-colored cycle of Γ, for every $c \in \Delta_3 - \{i\}$).

The *switching* of the ρ_2-pair (resp. ρ_3-pair) is the local process depicted in Fig. 7.

(ii) Four distinctly colored edges $e_0, e_1, e_2, e_3 \in E(\Gamma)$ are said to be a *handle* if they pairwise belong to the same bicolored cycle.

The *breaking* of the handle is the local process depicted in Fig. 8.

Remark. It is very easy to check that every ρ_3-pair implies the existence of a handle, too (see the captions of Fig. 7). On the contrary, if (Γ, γ) contains a handle, another 4-colored graph containing a ρ_3-pair of color i may be obtained by inserting a 1-dipole of color i (see Fig. 9).

Proposition 4.1 ([16]). *Let (Γ, γ) be a 4-colored graph representing a closed orientable 3-manifold M^3.*

(i) *If (Γ', γ') is obtained from (Γ, γ) by switching a ρ_2-pair (resp. ρ_3-pair), then $|K(\Gamma')| = |K(\Gamma)| = M^3$ (resp. $|K(\Gamma)| = M^3 = |K(\Gamma')| \#(\mathbb{S}^1 \times \mathbb{S}^2)$).*

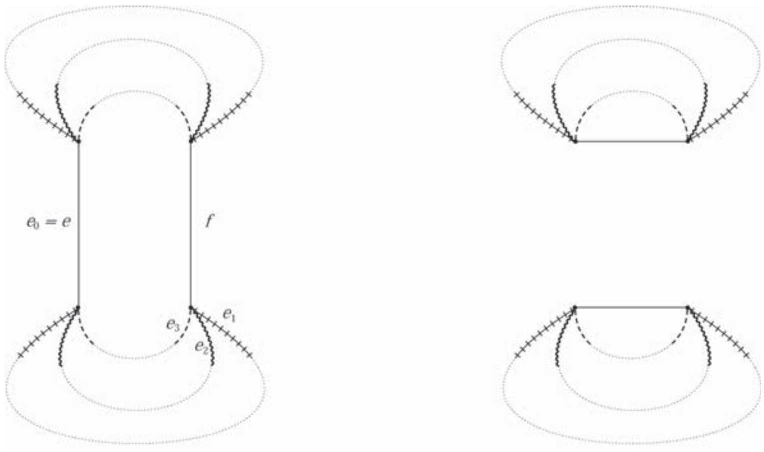


Figure 7

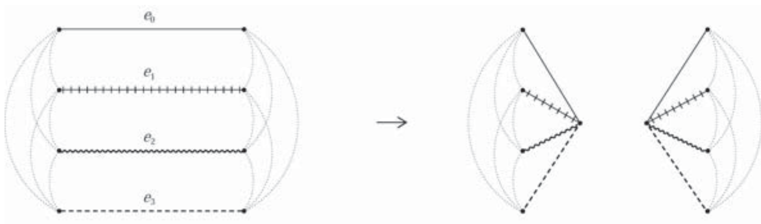


Figure 8

Figure 9

(ii) *If the* 4*-colored graph* (Γ', γ') *obtained from* (Γ, γ) *by breaking a handle is connected, then* $|K(\Gamma)| = M^3 = |K(\Gamma')|\#(\mathbb{S}^1 \times \mathbb{S}^2)$.

Remark. In case the 4-colored graph (Γ', γ') obtained from (Γ, γ) by breaking a handle consists of two connected components Γ'_1 and Γ'_2, then $\Gamma = \Gamma'_1 \# \Gamma'_2$; thus, according to Proposition 3.2, $|K(\Gamma)| = |K(\Gamma'_1)|\#|K(\Gamma'_2)|$.

The following results allow us to algorithmically construct a 5-colored graph $\bar{\Lambda}(L^{(d)}, c)$ representing the closed 4-manifold $\bar{M}^4$ (Construction 2), in case the dotted framed link $(L^{(d)}, c)$ could be recognized as being a Heegaard diagram via ρ_3-pairs and/or handles in the 4-colored graph $\Lambda(L, c)$.

Proposition 4.2. *Let us assume* $\Lambda(L, c)$ *contains* m_3 ρ_3*-pairs of color* i $(i \in \Delta_3)$*, whose switching yields a* 4*-colored graph* H *representing* $\mathbb{S}^3$*. Then,* $\bar{\Lambda}(L^{(d)}, c)$ *is obtained from* $\tilde{\Lambda}(L^{(d)}, c)$ *by simply adding a* 4*-colored edge for every pair of* i*-adjacent vertices in* H*.*

Proof. By Proposition 4.1(i), the hypothesis implies

$$\partial M^4(L^{(d)}, c) = M^3(L, c) = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2).$$

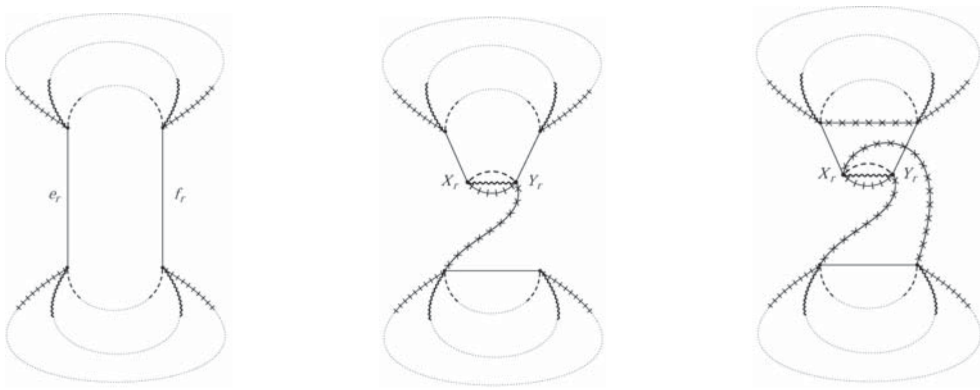
In order to prove the statement, we have to consider the described regular 5-colored graph $\bar{\Lambda} = \bar{\Lambda}(L^{(d)}, c)$ and to check that it represents the unique closed 4-manifold $\bar{M}^4 = M^4(L^{(d)}, c) \cup \mathbb{Y}^4_{m_3}$.

First of all, we construct a 5-colored graph $\tilde{H}$ by applying the following procedure to the graph H (thought of as a 5-colored graph with boundary, representing $\mathbb{D}^4$): for each ρ_3-pair $\{e_r, f_r\}$ $(r \in \{1, \dots, m_3\})$ in $\Lambda = \Lambda(L, c)$, insert a 3-dipole $\Theta_r = \{X_r, Y_r\}$ of colors $\Delta_3 - \{i\}$ and add a 4-colored edge, as indicated in Figs. 10(a), 10(b). By [2, Theorem 3 (iii)], it is easy to check that the resulting 5-colored graph $\tilde{H}$ represents a 4-dimensional handlebody $\mathbb{Y}^4_{m_3}$ of genus m_3; moreover, the boundary graph $\partial\tilde{H}$ exactly coincides with $\partial\tilde{\Lambda}(L^{(d)}, c) = \Lambda$.

Now, let $\bar{H}$ be the regular 5-colored graph obtained from $\tilde{H}$ by adding a 4-colored edge for every pair of i-adjacent vertices in H (see Fig. 10(c)). Note that, for every $r \in \{1, \dots, m_3\}$, $\{X_r, Y_r\}$ are joined in $\bar{H}$ by three edges (colored by $\Delta_3 - \{i\}$), but belonging to the same $\{i, 4\}$-colored cycle of $\bar{H}$; hence, by [2, Theorem 14 (b′)], the 5-colored graph $\bar{H}'$ obtained from $\bar{H}$ by deleting $\{X_r, Y_r\}$ and by welding the "hanging" edges of the same color $c \in \{i, 4\}$, is such that $|K(\bar{H})| = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^3)\#|K(\bar{H}')|$.

Moreover, since $\bar{H}'$ is obtained from the 4-colored graph H (representing $\mathbb{S}^3$) by adding a parallel 4-colored edge for every i-colored edge, then $|K(\bar{H}')| = \mathbb{S}^4$ easily follows (see [12, section 4], where the notion of "suspension graph" is introduced and analyzed).

Hence, the passage from $\tilde{H}$ to $\bar{H}$ has the topological effect of transforming $|K(\tilde{H})| = \mathbb{Y}^4_{m_3}$ into $|K(\bar{H})| = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^3)$. This means that the identification of tetrahedra of $K(\tilde{H})$ associated to i-adjacent vertices in H corresponds to the unique



(a) $\Lambda(L, c)$ (b) $\tilde{H}$ (c) $\bar{H}$

Figure 10

(see [19]) PL-homeomorphism $\phi : \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2) \to \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$ giving rise to the attaching map for 3- and 4-handles.

Finally, since $K(\bar{\Lambda})$ is obtained from $K(\tilde{\Lambda})$ by means of the same identification of boundary tetrahedra, $|K(\bar{\Lambda})| = |K(\tilde{\Lambda})| \cup_\phi \mathbb{Y}^4_{m_3}$ directly follows. □

Example 4.3. If $(K_0^{(d)}, 0)$ is the 0-framed dotted trivial knot, then it is very easy to check that the 5-colored graph $\tilde{\Lambda}(K_0^{(d)}, 0)$ depicted in Fig. 5 (and representing $\mathbb{S}^1 \times \mathbb{D}^3 = \mathbb{Y}^4_1$) satisfies the hypothesis of Proposition 4.2, with $m_3 = 1$ and $i = 1$. Hence, Construction 2 may be easily performed, by a boundary identification. The resulting regular 5-colored graph $\bar{\Lambda}(K_0^{(d)}, 0)$ (representing $\bar{M}^4(K_0^{(d)}, 0) = \mathbb{Y}^4_1 \cup \mathbb{Y}^4_1 = \mathbb{S}^1 \times \mathbb{S}^3$) is shown in Fig. 11.

Example 4.4. If $(L^{d)}, c)$ is the dotted framed link depicted in Fig. 6(a), then the 5-colored graph $\tilde{\Lambda}(L^{(d)}, c)$ shown in Fig. 6(b) (and representing the 4-disk $\mathbb{D}^4$) trivially satisfies the hypothesis of Proposition 4.2, with $m_3 = 0$: hence, Construction 2 may be easily performed, by a boundary identification. The resulting regular 5-colored graph $\bar{\Lambda}(L^{(d)}, c)$ (representing $\bar{M}^4(L^{(d)}, c) = \mathbb{D}^4 \cup \mathbb{D}^4 = \mathbb{S}^4$) is shown in Fig. 12.

Proposition 4.5. *Let us assume $\Lambda(L, c)$ contains m_3 handles, whose breaking yields a connected 4-colored graph representing $\mathbb{S}^3$. Then, a well-determined sequence of dipole moves exists, which transforms $\tilde{\Lambda} = \tilde{\Lambda}(L^{(d)}, c)$ into a 5-colored graph $\tilde{\tilde{\Lambda}}$ with the following properties:*

(i) *The 4-colored graph $\partial\tilde{\tilde{\Lambda}}$ contains m_3 ρ_3-pairs of color i ($i \in \Delta_3$).*

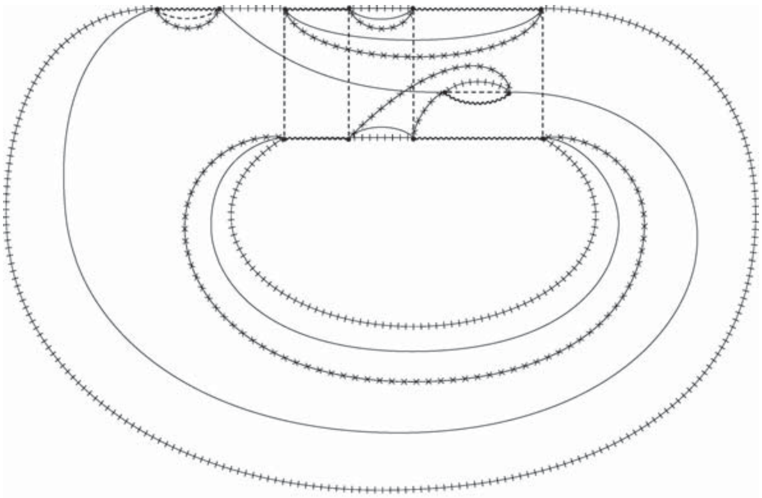


Figure 11

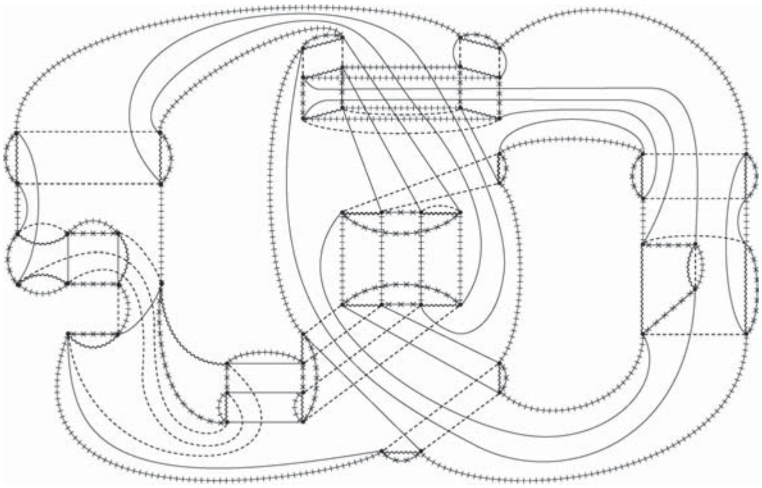


Figure 12

(ii) *The* 5*-colored graph* $\bar{\Lambda}(L^{(d)}, c)$ *may be obtained by suitably adding* 4*-colored edges to* $\tilde{\tilde{\Lambda}}$.

Proof. As a consequence of the Remark before Proposition 4.1, m_3 suitable insertions of 1-dipoles of color i ($i \in \Delta_3$) into $\Lambda(L, c)$ give rise to a 4-colored graph containing m_3 ρ_3-pairs of color i. By [8, Lemma B], the above sequence of dipole insertions may be reproduced on 5-colored graphs, starting from $\tilde{\Lambda}(L^{(d)}, c)$ (whose boundary is exactly $\Lambda(L, c)$). Now, if $\tilde{\tilde{\Lambda}}$ is the resulting 5-colored graph, property (i) is satisfied by construction; on the other hand, property (ii) directly follows by making use of Proposition 4.2. □

Unfortunately, the following statement proves that the assumptions of Proposition 4.2 and/or of Proposition 4.5 are not always satisfied, even if $M^3(L, c) = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$ is assumed to hold.

Proposition 4.6. *Let* (G, g) *be the* 4*-colored graph depicted in Fig. 13(b). Then:*

(i) $|K(G)| = \mathbb{S}^1 \times \mathbb{S}^2$.

(ii) *No handle is contained in* (G, g).

Proof. As far as statement (i) is concerned, it is sufficient to note that $(G, g) = \Lambda(\bar{L}, (0, 0, 0))$, where $\bar{L}$ denotes the "trivial chain with three rings" depicted in Fig. 13(a) (without additional curls). Further, part (ii) follows by direct checking. □

Note that in [16, page 125] a conjecture is stated, which would imply the existence of handles in every 4-colored graph representing $\mathbb{S}^1 \times \mathbb{S}^2$; thus, Proposition 4.6 provides a counterexample to Lins's conjecture:

Corollary 4.7. *Conjecture 5 of [16, page 125] is false.*

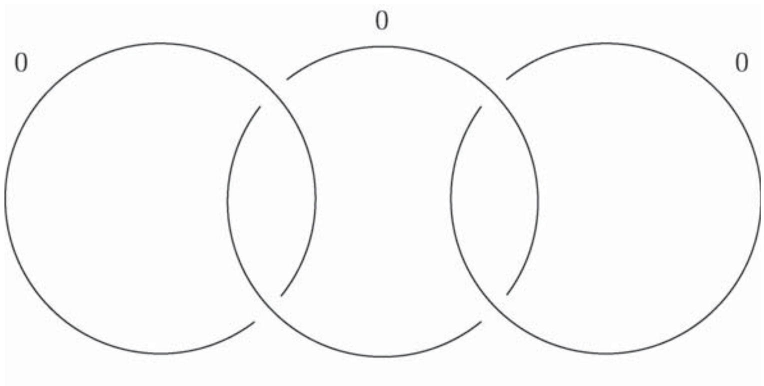
Let us now conclude the paper with the general theorem about Construction 2.

Theorem 4.8. *Let* $(L^{(d)}, c)$ *be any dotted framed link representing a closed 4-manifold* $\bar{M}^4 = \bar{M}^4(L^{(d)}, c)$ *(i.e.* $(L^{(d)}, c)$ *such that* $M^3(L, c) = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$*). Then, a finite sequence of dipole moves exists, which transforms* $\tilde{\Lambda} = \tilde{\Lambda}(L^{(d)}, c)$ *into a* 5*-colored graph* $\tilde{\tilde{\Lambda}}$ *with the following properties:*

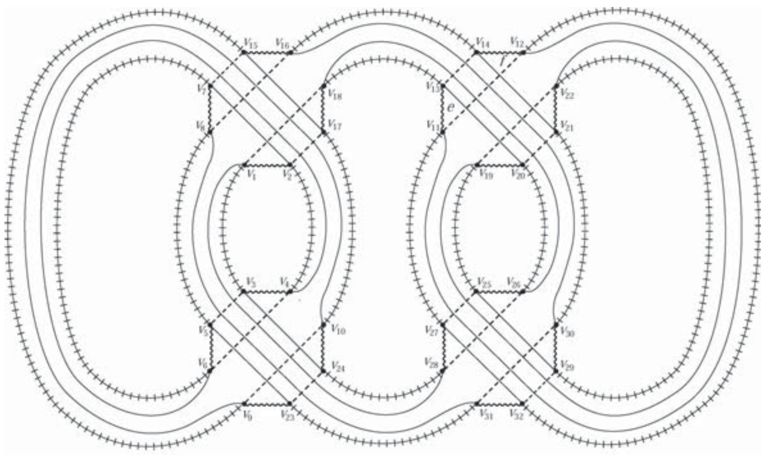
(i) *The* 4*-colored graph* $\partial\tilde{\tilde{\Lambda}}$ *contains* m_3 ρ_3*-pairs of color* i *(*$i \in \Delta_3$*).*

(ii) *The* 5*-colored graph* $\bar{\Lambda}(L^{(d)}, c)$ *may be obtained by suitably adding* 4*-colored edges to* $\tilde{\tilde{\Lambda}}$.

Proof. By hypothesis, the 4-colored graph $\Lambda(L, c)$ represents $M^3 = M^3(L, c) = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$. Obviously, if $\Lambda(L, c)$ contains m_3 ρ_3-pairs of color i ($i \in \Delta_3$), we may set $\tilde{\tilde{\Lambda}} = \tilde{\Lambda}(L^{(d)}, c)$. On the other hand, if $\Lambda(L, c)$ contains m_3 handles, the required



(a)



(b)

Figure 13

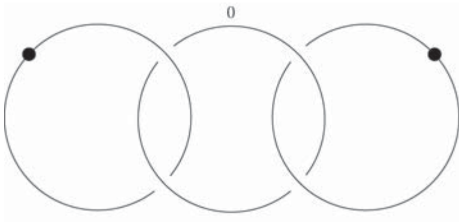


Figure 14

5-colored graph $\tilde{\tilde{\Lambda}}$ is proved to exist (and well-determined) by Proposition 4.5. Otherwise, let $(G^{(m_3)}, g^{(m_3)})$ be a fixed 4-colored graph representing $M^3 = \#_{m_3}(\mathbb{S}^1 \times \mathbb{S}^2)$ and containing m_3 ρ_3-pairs of color i ($i \in \Delta_3$): for example, $(G^{(m_3)}, g^{(m_3)})$ may be obtained by considering m_3 copies of the standard order eight 4-colored graph representing $\mathbb{S}^1 \times \mathbb{S}^2$ and by performing $m_3 - 1$ graph connected sums. The Main Theorem of [6] ensures the existence of a finite sequence of dipole moves which transforms $\Lambda(L, c)$ into $(G^{(m_3)}, g^{(m_3)})$; moreover, by [8, Lemma A and Lemma B], the above sequence of dipole insertions may be reproduced on 5-colored graphs, starting from $\tilde{\Lambda}(L^{(d)}, c)$ (whose boundary is exactly $\Lambda(L, c)$). Now, if $\tilde{\tilde{\Lambda}}$ is the resulting 5-colored graph, property (i) is satisfied by construction, while property (ii) directly follows by making use of Proposition 4.2. □

Example 4.9. If $(L^{(d)}, c)$ is the dotted framed link depicted in Fig. 14, then the associated 5-colored graph $\tilde{\Lambda}(L^{(d)}, c)$ has the 4-colored graph $\Lambda(L, c) = (G, g)$ depicted in Fig. 13(b)) as boundary graph. Since (G, g) does not contain ρ_3-pairs, Proposition 4.2 can not be applied. Notwithstanding this, it is easy to check that a finite sequence of dipole eliminations (more precisely, the subsequent eliminations of 1-dipole $\{v_1, v_2\}$ and 2-dipoles $\{v_3, v_4\}$, $\{v_5, v_6\}$, $\{v_7, v_8\}$, $\{v_9, v_{10}\}$, according to the captions of Fig. 13(b)) transforms (G, g) into a 4-colored graph containing a ρ_3-pair of color 2 (which corresponds to the pair of edges $\{e, f\}$ of (G, g), according to the captions of Fig. 13(b)). Hence, by Theorem 4.8, a regular 5-colored graph $\bar{\Lambda}(L^{(d)}, c)$ of the associated closed 4-manifold $\bar{M}^4$ may be constructed by reproducing on $\tilde{\Lambda}(L^{(d)}, c)$ the above sequence of moves, and finally by applying Proposition 4.2. It is not difficult to check—by making use of [8, Lemma A]—that the resulting 5-colored graph $\bar{\Lambda}(L^{(d)}, c)$ is simply obtained from $\tilde{\Lambda}(L^{(d)}, c)$ by adding a 4-colored edge for every pair of boundary vertices corresponding to vertices of type $\{v_i, v_{i+1}\}$ in (G, g), for any odd index i.

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