Tail and free poset algebras

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This work is dedicated to the philosopher Ibn Tophail (1100–1185).

ABSTRACT

We characterize free poset algebras $F(P)$ over partially ordered sets and show that they can be represented by upper semi-lattice algebras. Hence, the uniqueness, in decomposition into normal form, using symmetric difference, of non-zero elements of $F(P)$ is established. Moreover, a characterization of upper semi-lattice algebras that are isomorphic to free poset algebras is given in terms of a selected set of generators of $B(T)$.

Key words: Free poset algebra, tail algebra, semigroup algebra

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1. Characterization of $F(P)$

The main result of this section is Theorem 1.2 that gives an algebraic characterization of free poset algebras. Our approach in studying these algebras is different from [1] and [8]. Indeed, we show that the class of free poset algebras is actually a subclass of upper semi-lattice algebras. Therefore, non-zero elements has a unique decomposition into normal form. Two elements a and b of a poset $\langle P, \leq\rangle$ are *comparable* whenever $a \leq b$ or $b \leq a$. We say $a \parallel b$, whenever a and b are not comparable in $\langle P, \leq\rangle$. A subset A of $\langle P, \leq\rangle$ of non comparable elements is called an *anti-chain* of $\langle P, \leq\rangle$. $\mathcal{I}d(P)$ shall denote the set of ideals of P. ($J \subseteq P$ is an ideal if J is an non empty initial segment of P such that every $p, q \in J$ there is $r \in J$ such that $p, q \leq r$). For a non empty set X, we denote by $[X]^{<\omega}$ the set all finite subsets of X. Thus, we set $\mathrm{Ant}(P) = \{\sigma \in [P]^{<\omega} : \sigma \text{ is an antichain of } P\}$.

Now, a subset F of P is a *final segment* of P, whenever it is closed upwards i.e., if $(a, b) \in F \times P$ and $a \leq b$, then $b \in F$. E.g., for each $p \in P$, $P^{\geq p} := \{q \in P : p \leq q\}$ is a final segment of P. For a finite subset σ of P, $P^{\geq \sigma} := \bigcup_{p \in \sigma} P^{\geq p}$ is a final segment of P generated by σ. Define $\preceq$ be the binary relation on $[P]^{<\omega}$ defined by $\sigma \preceq \tau$ if $P^{\geq \sigma} \supseteq P^{\geq \tau}$. i. e., for every $q \in \tau$ there is $p \in \sigma$ such that $p \leq q$. So $\preceq$ is a reflexive and transitive relation. For $\sigma \in [P]^{<\omega}$ let $\min(\sigma)$ be the set of minimal elements of σ. So $P^{\geq \sigma} = P^{\geq \min(\sigma)}$. It is trivial that $\preceq$ restricted to $\mathrm{Ant}(P)$ is a partial order. Also for $\sigma, \tau \in \mathrm{Ant}(P)$, $P^{\geq \sigma} \cup P^{\geq \tau} = P^{\geq \sigma \cup \tau} = P^{\geq \min(\sigma \cup \tau)}$.

Throughout this paper $\langle P, \leq \rangle$ shall denote a partially ordered set (*poset*) with greatest element ∞. Notice that this assumption on $\langle P, \leq \rangle$ is not restrictive but it's rather convenient for expository purposes; indeed, let $\langle Q, \leq \rangle$ be a poset and denote by $\mathrm{Fs}^{+}(Q)$ the set of all final segments of Q, and $\mathrm{Fs}(Q) := \mathrm{Fs}^{+}(Q) \setminus \{\emptyset\}$ then $\mathrm{Fs}(Q)$ and $\mathrm{Fs}(Q \cup \{\infty\})$ are homeomorphic spaces; for more see e. g. [3]. $\mathrm{Fs}(P)$ the set of all non-empty final segments of P is a closed subspace of $\{0,1\}^P$ by identifying $\wp(P)$ with $\{0,1\}^P$ via characteristic functions of sets and taking the the Cantor topology on $\{0,1\}^P$. For a topological space X, $\mathrm{clop}(X)$ shall denote the Boolean algebra of clopen (*closed and open*) subsets of X. Thus, $\mathrm{clop}(\mathrm{Fs}(P))$ is the *free poset algebra* over P, denoted by $F(P)$. For $p \in P$, set $V_p := \{F \in \mathrm{Fs}(P) : p \in F\}$. So V_p is a clopen subset of $\mathrm{Fs}(P)$. Also, for any finite subset σ of P. (†) $\bigcap_{p \in \sigma} V_p = \{F \in \mathrm{Fs}(P) : F \supseteq P^{\geq \sigma}\}$. Hence a basis of $\mathrm{Fs}(P)$ is $\langle \bigcap_{p \in \sigma} V_p \cap \bigcap_{q \in \tau} -V_q, \sigma, \tau \in [P]^{<\omega} \rangle$, where $-V_q$ denotes the complement of V_q in $\mathrm{Fs}(P)$; (i.e., $-V_q = \mathrm{Fs}(P) \setminus V_q$). Notice that $V_p \in F(P)$ and that every member of $F(P)$ is a finite union of $\bigcap_{p \in \sigma} V_p \cap \bigcap_{q \in \tau} -V_q$. Finally, recall that $\mathrm{cl}_B(S)$ denotes the subalgebra of B generated by S for any Boolean algebra B and any $S \subseteq B$. Next, $\mathcal{U}lt(B)$, the set of ultrafilters of B, shall denote the Stone space of B.

Lemma 1.1. *Let $(P, \leq)$ be a poset, with greatest element ∞.*

(i) *If $p < q$ then $V_p \subseteq V_q$.*

(ii) $\bigcap_{p \in P} V_p = \{P\} \neq \emptyset$.

(iii) $V_\infty = \mathrm{Fs}(P)$.

(iv) *For finite subsets σ, τ of P, the following properties are equivalent:*

  (a) $\bigcap_{p \in \sigma} V_p \cap \bigcap_{q \in \tau} -V_q = \emptyset$.

  (b) There are $p \in \sigma$, $q \in \tau$ such that $p \leq q$.

(v) *If* $\bigcap_{i=1}^{n} V_{p(i)} \subseteq \bigcup_{k=1}^{m} \left(\bigcap_{j=1}^{m(k)} V_{q(k,j)} \right)$ *then*

  (a) there is $k \in \{1, \ldots, m\}$, such that $\bigcap_{i=1}^{n} V_{p(i)} \subseteq \bigcap_{j=1}^{m(k)} V_{q(k,j)}$, and

  (b) for any $j \in \{1, \ldots, m(k)\}$ there is $i \in \{1, \ldots, n\}$ so that $p(i) \leq q(k,j)$.

Proof. (i)–(iii) are easy to check.

To prove (iv), set $W := \bigcap_{p\in\sigma} V_p \cap \bigcap_{q\in\tau} -V_q$. Then,

$$W = \{F \in \mathrm{Fs}(P) : \sigma \subseteq F \text{ and } \tau \cap F = \emptyset\}.$$

Suppose that there are $p \in \sigma$, $q \in \tau$ and $p \le q$. So every final segment containing σ contains q. Thus, $W = \emptyset$. Conversely, suppose that for every $p \in \sigma$, $q \in \tau$, and $p \not\le q$ we have $P^{\ge\sigma} \cap \tau = \emptyset$. It follows then, $P^{\ge\sigma} \in W$; thus $W \ne \emptyset$.

To prove (v), set $\sigma := \{p(1), \dots, p(n)\}$, $F := P^{\ge\sigma}$. So, $F \in \bigcap_{i=1}^{n} V_{p(i)}$. Choose k so that $F \in \cap_{j=1}^{m(k)} V_{q(k,j)}$. Note that $q(k,j) \in F$ for every $j \in \{1, \dots, m(k)\}$. Now, if $G \in \bigcap_{i=1}^{n} V_{p(i)}$ then $F \subseteq G$; thus $q(k,j) \in G$ for every $j \in \{1, \dots, m(k)\}$. Hence, $\bigcap_{i=1}^{n} V_{p(i)} \subseteq \bigcap_{j=1}^{m(k)} V_{q(k,j)}$. Next, since $F \in \cap_{j=1}^{m(k)} V_{q(k,j)}$ it follows that for every $q(k,j)$ there is $p(i)$ such that $p(i) \le q(k,j)$. □

The following theorem characterizes free poset algebras and will be of use later on in the paper.

Theorem 1.2. *The following statements are equivalent for any Boolean algebra B.*

(i) *B is isomorphic to a free poset algebra.*

(ii) *B has a set H of generators with $1 \in H$ such that for every finite subsets $\{h_i : i < m\}$ and $\{k_j : j < n\}$ of H :*

  (a) $\prod_{i<m} h_i \ne 0$ *and*

  (b) if $\prod_{i<m} h_i \cdot \prod_{j<n} -k_j = 0$ *then there are i and j such that $h_i \le k_j$.*

Proof. (i) implies (ii). We may assume that $B = F(P)$. Let $H = \{V_q : q \in P\}$. By Lemma 1.1, (ii) holds.

(ii) implies (i). Let H be as in (ii). So H is a poset with a greatest element. To show that B is isomorphic to $F(H)$, let $f : H \to F(H)$ be defined by $f(h) = V_h$. By Lemma 1.1 (iv) and the hypothesis (ii)-(b), the following are equivalent:

1. $\prod_{i<m} h_i \cdot \prod_{j<n} -k_j = 0$
2. there is i and j such that $h_i \le k_j$, and
3. $\bigcap_{i<m} V_{h_i} \cap \bigcap_{j<n} -V_{k_j} = \emptyset$

By Sikorski's Criterion, f extends to an isomorphism $\hat{f}$ from $\mathrm{cl}_B(H)$ onto $\mathrm{Im}(f) \subseteq F(H)$. Since H generates B, and $\mathrm{Im}(f) = F(H)$, $\hat{f}$ is an isomorphism from B onto $F(H)$. □

2. Representation of $F(P)$

Let $\langle Q, \leq \rangle$. For $q \in Q$, put $b_q := \{u \in Q \,:\, u \geq q\}$. Next define the *Tail algebra*, $B(Q)$, as the subalgebra of the power set of Q generated by $\{b_q \,:\, q \in Q\}$. When $\langle T, \leq \rangle$ is an *upper semi-lattice* i. e., l. u. b. $\{x, y\} := x \vee y$ exists in $\langle T, \leq \rangle$ for $x, y \in T$; $B(T)$ is called the *upper semi-lattice* algebra over T. Notice that every member of $B(T)$ is a finite union of $\bigcap_{t\in\sigma} b_t \cap \bigcap_{s\in\tau} -b_s$ (where σ, τ are finite subsets of T).

In this section Theorem 2.3 shows that any $F(P)$ is isomorphic to an upper semi-lattice algebra. As for Theorem 2.4, a characterization of upper semi-lattice algebras that are isomorphic to an $F(P)$ is given using the idea of *prime elements* in the upper semi-lattice.

Before we state the following lemmas, notice that if Q is the poset $\{p, q, r, s\}$ with relations $p < r < q$, $p < s < q$ and r, s incomparable; then in the tail algebra $B(Q)$, $b_r \cap b_s = b_q$; nevertheless, in the free poset algebra $F(Q)$, $V_p \subset V_r \cap V_s$ and $V_r \cup V_s \subset V_q$.

Lemma 2.1. *Let $B(Q)$ be the tail algebra over Q. Let $p \in Q$ and τ be a finite subset of Q. The following properties are equivalent.*

(i) $b_p \subseteq \bigcup_{q\in\tau} b_q$.

(ii) *$q \in \tau$ such that $q \leq p$.*

Lemma 2.2. (i) *$\preceq$ is an ordering on* $\mathrm{Ant}(P)$.

(ii) *The greatest lower bound of σ and τ is* $\min(\sigma \cup \tau)$ *in* $\langle \mathrm{Ant}(P), \preceq \rangle$*, for* $\sigma, \tau \in \mathrm{Ant}(P)$*,*

(iii) $\langle \mathrm{Ant}(P), \succeq \rangle$ *is an upper semi-lattice, with a least element.*

Next theorem shows that the class of upper semi-lattice algebras contains the class of free poset algebras.

Theorem 2.3. *For every poset $(P, \leq)$, with a greatest element, $F(P)$ is isomorphic to the upper semi-lattice algebra $B(\langle \mathrm{Ant}(P), \succeq \rangle)$.*

Proof. Set $H = \{V_q \,:\, q \in P\}$. Recall that H generates $F(P)$. Now, for each $p \in P$, $\{p\} \in \mathrm{Ant}(P)$; and thus $b_{\{p\}} = \{\sigma \in \mathrm{Ant}(P) \,:\, \sigma \preceq \{p\}\}$.

Next, define $\varphi : H \to B(\langle \mathrm{Ant}(P), \succeq \rangle)$ by $\varphi(V_p) = b_{\{p\}}$. We claim that for every $\sigma, \tau \in \mathrm{Ant}(P)$:

$$\bigcap_{p\in\sigma} V_p \cap \bigcap_{q\in\tau} -V_q = \emptyset \quad \text{iff} \quad \bigcap_{p\in\sigma} b_{\{p\}} \cap \bigcap_{q\in\tau} -b_{\{q\}} = \emptyset. \tag{1}$$

By Lemma 1.1 (iv), we have:

$$\bigcap_{p\in\sigma} V_p \cap \bigcap_{q\in\tau} -V_q = \emptyset \quad \text{iff} \quad \text{“there are } p\in\sigma \text{ and } q\in\tau \text{ so that } p\leq q\text{”} \tag{2}$$

On the other hand, we have $\{\{p\} : p\in\sigma\} \subseteq \mathrm{Ant}(P)$ and $\min(\sigma)$ is its l.u.b. in $\langle \mathrm{Ant}(P), \succeq\rangle$. So $\bigcap_{p\in\sigma} b_{\{p\}} = b_{\min(\sigma)}$. So $\bigcap_{p\in\sigma} b_{\{p\}} \cap \bigcap_{q\in\tau} -b_{\{q\}} = \emptyset$; which means that $b_{\min(\sigma)} \subseteq \bigcup_{q\in\tau} b_{\{q\}}$. By Lemma 2.1, this is equivalent to the existence of $q\in\tau$ so that $\{q\}\succeq \min(\sigma)$.

Next $\{q\}\succeq\sigma$ iff there is $p\in\sigma$ so that $p\leq q$, that is (2). The proof of (1) is finished.

Next, by (1) and Sikorski's Criterion, φ extends to a monomorphism $\hat{\varphi}$ from $F(\langle P,\leq\rangle)$ into $B(\langle \mathrm{Ant}(P),\succeq\rangle)$. Now, since $\{V_p : p\in P\}$ generates $F(P)$, $\{b_{\{p\}} : p\in P\}$ generates $B(\mathrm{Ant}(P))$ and $\hat{\varphi}(V_p) = b_{\{p\}}$; the isomorphism is established and the proof of the theorem is finished. □

Recall that an element $p\in T$ is called a *prime* element of T whenever for every $u,v\in T$ so that $u\vee v$ exists in T and $p\leq u\vee v$, then $p\leq u$ or $p\leq v$. $\mathrm{Prim}(T)$ shall denote the set of all prime elements of T.

Next is a characterization of free poset algebras.

Theorem 2.4. *Let T be an upper semi-lattice. The following statement are equivalent.*

(i) *$B(T)$ is isomorphic to a free poset algebra.*

(ii) *There is a poset P, with a greatest element so that $B(T)\cong B(\langle \mathrm{Ant}(P),\succeq\rangle)$.*

(iii) *There is an upper semi-lattice T', with a least element, so that:*

(a) Every element of T' is a join of finitely many prime elements,

(b) $B(T)$ and $B(T')$ are isomorphic Boolean algebras , and

(c) $B(T') = \mathrm{cl}_{B(T')}(\{b_t : t\in \mathrm{Prim}(T')\})$.

Proof. (i) implies (ii) follows from Theorem 2.3.

(ii) implies (iii). Let $\langle T',\leq\rangle := \langle \mathrm{Ant}(P),\succeq\rangle$. By Lemma 2.2 (ii), T' is an upper semi-lattice. Next, it is straightforward to notice that $\sigma\in\mathrm{Ant}(P)$: σ is prime in $\langle \mathrm{Ant}(P),\succeq\rangle$ whenever σ is a singleton. Note that by the proof of Theorem 2.3, $\{b_{\{p\}} : p\in P\}$ generates $B(\mathrm{Ant}(P),\succeq)$.

(iii) implies (i). Let T' be as in (iii). Let t_0 be the least element of T'. Let $P=\mathrm{Prim}(T')$. (So $P\subseteq T'$.) Let $H' := \{b_t : t\in P\}$. By (iii)-(c)

$$B(T') = \mathrm{cl}_{B(T')}(H'). \tag{3}$$

Since $t_0\in P$:

$$1^{B(T')} = b_{t_0} \in H'. \tag{4}$$

For $\{t(1), \ldots, t(n)\} \subseteq P$, $\bigcap_{i=1}^{n} b_{t(i)} = b_{t(1)\vee\cdots\vee t(n)} \neq 0$, and thus:

$$H \text{ has the finite intersection property.} \tag{5}$$

For $\{t(1), \ldots, t(m)\} \subseteq P$ and $\{s(1), \ldots, s(n)\} \subseteq P$. we have:

$$\bigcap_{i=1}^{m} b_{t(i)} \cap \bigcap_{j=1}^{n} -b_{s(j)} = \emptyset \quad \text{whenever there are } i \text{ and } j \text{ so that } s(j) \leq t(i). \tag{6}$$

To see that (6) holds set $t = \bigvee_i t(i)$. So the left hand side of (6) is equivalent to $b_t \subseteq \bigcup_{j=1}^{n} b_{s(j)}$. In other words, $t \in \bigcup_{j=1}^{n} b_{s(j)}$. i. e., $t \geq s(j)$ for some j. Since $s(j) \in P = \mathrm{Prim}(T')$, this is equivalent to say that there is i so that $t(i) \geq s(j)$.

Next, let $H := \{V_p \,:\, p \in P\} \subseteq F(P)$ and $f : H' \to H$ defined by $f(b_p) = V_p$. By (6), Lemma 1.1 (iv) and using Sikorski's Criterion, f extends to a monomorphism $\hat{f}$ from $\mathrm{cl}_{B(T')}(H')$ into $F(P)$. By (3), $\mathrm{Dom}(f) = B(T')$, and since H generates $F(P)$, $\hat{f}$ is actually onto and thus, an isomorphism. □

The following proposition summarizes the relationship between T and P whenever $B(T) \cong F(P)$.

Proposition 2.5. (i) *If a poset $\langle P, \leq\rangle$ with a greatest element is so that $F(P) \cong B(T)$, then T may be chosen canonically to be $\langle T, \leq\rangle = \langle \mathrm{Ant}(P), \succeq\rangle$.*

(ii) *If an upper semi-lattice T is so that $B(T) \cong F(P)$, then an upper semi-lattice T' may be chosen so that $B(T) \cong B(T') = \mathrm{cl}_{B(T')}(\{b_t \,:\, t \in \mathrm{Prim}(T')\})$ and $P \cong \{b_t \,:\, t \in \mathrm{Prim}(T')\}$.*

3. Normal form of non-zero elements in the free poset algebra $F(P)$

In this section we show, by Lemma 3.4, that any non-zero element of $F(P)$ has a decomposition into normal form as it is the case in the class of pseudo-tree algebras see, for instance, [2] and [6, p. 51]. Unfortunately, this representation of elements, here in the free poset algebra $F(P)$, is not unique. Using symmetric difference $\triangle$, instead, we shall see that Theorem 3.5 gives uniqueness in normal form of non-zero elements.

Before we give next lemma, recall that if P is a poset then $\langle \mathrm{Fs}(P), \subseteq\rangle$ is a poset too. Thus, we consider the tail algebra over $\langle \mathrm{Fs}(P), \subseteq\rangle$. For notational purposes, we use small letters for elements of $\mathrm{Fs}(P)$, and $\leq$ denotes the inclusion relation. Thus, for $p \in P$, we set $h_p = P^{\geq p}$. Hence, in the tail algebra $B(\langle \mathrm{Fs}(P), \subseteq\rangle)$:

$$b_{h_p} = \{x \in \mathrm{Fs}(P) \,:\, x \geq h_p\}$$

The following Lemmas follow easily.

Lemma 3.1. *Let P be a poset and $q \in P$. We have $V_q = b_{h_q}$.*

Lemma 3.2. *Let P be a poset and $p, q \in P$.*

(i) $p \leq q$ *iff* $h_q \leq h_p$.

(ii) $p \parallel q$ *iff* $h_q \parallel h_p$.

(iii) *For each* $q \in P$, $h_q \in \mathrm{Prim}(\mathrm{Fs}(P))$.

(iv) $\bigcap_{i\in\sigma} V_{q(i)} = \bigcap_{i\in\sigma} b_{h_{q(i)}} = b_{\bigvee_{i\in\sigma} h_{q(i)}}$ *for each finite set* σ.

Let P be a poset. Let

$$T^P := \{ \bigvee_{i\in\sigma} h_{q(i)} : \{q(i) : i \in \sigma\} \in \mathrm{Ant}(P)\}.$$

Before we state next Lemma, we denote T^P by T.

Lemma 3.3. *Assume that $\langle P, \leq\rangle$ is a poset with a greatest element ∞. Then:*

(i) $V_\infty = \mathrm{Fs}(P)$;

(ii) $\langle T, \leq\rangle$ *is an upper semi-lattice with a least element* h_∞;

(iii) *For each* $f \in T$, $-b_f = b_{h_\infty} \cdot -b_f$ *and* $h_\infty \leq f$;

(iv) *For all* $f, g \in T$ *we have:*

(a) $f \leq g$ *iff* $b_g \cdot -b_f = \emptyset$.

(b) If $f \not\leq g$, *then* $b_g \cdot -b_f = b_g \cdot -b_{f\vee g}$.

Next, we prove the first lemma concerning the decomposition of non zero elements in the free Boolean algebra $F(P)$.

Lemma 3.4 (First normal form). *Every $b \in F(P) \setminus \{0\}$ can be written as $b = e_1 + \cdots + e_n$ where $e_i \cdot e_j = 0$ for $i \neq j$, and for every $i \in \{1, \cdots, n\}$, either $e_i = b_{h_i}$, or there is a finite anti-chain $\{f_1, \cdots, f_m\}$ in $\langle T, \leq\rangle$ such that $e_i = b_{h_i} \cdot -(b_{f_1} + \cdots + b_{f_m})$.*

Proof. Working out the proof, as in Proposition 4.4 in [6, p. 51], it suffices to show that each elementary product $\prod_{i=1}^n \varepsilon_i V_{q_i}$ can be written, in $F(P)$, under the form $b_h \cdot -(b_{f_1} + \cdots + b_{f_m})$ with $h, f_i \in T$, $h < f_i$, and $\{f_1, \cdots, f_m\}$ is an anti-chain.

Note that

$$\prod_{i=1}^n V_{q_i} = \bigcap_{i=1}^n V_{q_i} = \bigcap_{i=1}^n b_{h_{q_i}} = b_{\bigvee_{1\leq i\leq n} h_{q_i}}$$

and that, by Lemma 3.3 (iii),

$$\prod_{i=1}^n -V_{q_i} = \bigcap_{i=1}^n (-b_{h_{q_i}}) = \bigcap_{i=1}^n (b_{h_\infty} \cdot -b_{h_{q_i}}) = b_{h_\infty} \cdot -(b_{h_{q_1}} + \cdots + b_{h_{q_n}})$$

So we may assume that there are i, j so that $\varepsilon_i = 1$ and $\varepsilon_j = -1$.

Thus $\prod_{i=1}^{n} \varepsilon_i V_{q_i}$ can be written as,

$$\begin{aligned}\prod_{i=1}^{n} \varepsilon_i V_{q_i} &= V_{\alpha(1)} \cdot \cdots V_{\alpha(k)} \cdot -V_{\beta(1)} \cdot \cdots - V_{\beta(l)} \\ &= b_{h_{\alpha(1)}} \cdot \cdots b_{h_{\alpha(k)}} \cdot -b_{h_{\beta(1)}} \cdot \cdots - b_{h_{\beta(l)}} \\ &= b_{\vee_{1\le i\le k} h_{\alpha(i)}} \cdot -b_{h_{\beta(1)}} \cdot \cdots - b_{h_{\beta(l)}}\end{aligned}$$

Now, set $h = \vee_{1\le i\le k} h_i$. So,

$$\begin{aligned}\prod_{i=1}^{n} \varepsilon_i V_{q_i} &= b_h \cdot -b_{h_{\beta(1)}} \cdot \cdots - b_{h_{\beta(l)}} \\ &= (b_h \cdot -b_{h_{\beta(1)}}) \cdot (b_h \cdot -b_{h_{\beta(2)}}) \cdots (b_h \cdot -b_{h_{\beta(l)}})\end{aligned}$$

Since $\prod_{i=1}^{n} \varepsilon_i V_{q_i} \neq 0$, by Lemma 3.3 (iv)-(b), for each $j \in \{1, \cdots, l\}$ either $h < h_{\beta(j)}$ or $h \parallel h_{\beta(j)}$; moreover $b_h \cdot -b_{h_{\beta(j)}} = b_h \cdot -b_{f(j)}$ with $h < f(j)$ and $f(j) \in T$. So,

$$\prod_{i=1}^{n} \varepsilon_i V_{q_i} = (b_h \cdot -b_{f(1)}) \cdot \cdots \cdot (b_h \cdot -b_{f(l)}) = b_h \cdot -(b_{f(1)} + \cdots + b_{f(l)})$$

Now, by canceling some of $b_{f(i)}$'s, if necessary, we may write

$$\prod_{i=1}^{n} \varepsilon_i V_{q_i} = b_h \cdot -(b_{f(i(1))} + \cdots + b_{f(i(m))})$$

where $\{f(i(1)), \ldots, f(i(m))\}$ is an anti-chain and $h < f(i(k))$ for each k. This finishes up the proof of Lemma 3.4. □

Remark. Notice that normal form of non zero elements of $F(P)$, given by Lemma 3.4, may not be unique as shown by the following counterexample.
Let $p, q \in P$ with $p \parallel q$ and set $b = b_{h_p} + b_{h_q}$. we have

$$b = \underbrace{b_{h_p} \cdot -b_{h_p \vee h_q}}_{e_1} + \underbrace{b_{h_q}}_{e_2} = \underbrace{b_{h_p}}_{e'_1} + \underbrace{b_{h_q} \cdot -b_{h_p \vee h_q}}_{e'_2}$$

Indeed, by Lemma 3.3 (iv)-(b), $b_{h_p} \cdot -b_{h_p \vee h_q} = b_{h_p} \cdot -b_{h_q}$. Thus,

$$b_{h_p} \cdot -b_{h_p \vee h_q} + b_{h_q} = b_{h_p} \cdot -b_{h_q} + b_{h_q} = b_{h_p} + b_{h_q} = b.$$

Next, before we state the main theorem in this section, recall that whenever P is a poset, $T := \{\bigvee_{i\in\sigma} h_{q(i)} \ : \ \{q(i) \ : \ i \in \sigma\} \in \mathrm{Ant}(P)\}$ will be the upper semi-lattice that is going to be referred to in the next theorem.

Theorem 3.5 (Normal form of non-zero elements in $F(P)$). *Every $b \in F(P) \setminus \{0\}$ has a unique decomposition as $b = b_{g_1} \triangle \cdots \triangle b_{g_n}$, where $g_i \in T$, and $g_i \neq g_j$ for $i \neq j$.*

The proof of this theorem uses the following lemma.

Lemma 3.6. (i) *For all $f, h \in T$, $\quad b_h \cdot -b_f = b_h \triangle b_{f \vee h}$,*

(ii) *If $b = b_{f_1} \triangle \cdots \triangle b_{f_n}$ then $b \cdot -b_f = \triangle_{i=1}^m b_{g_i}$ where $g_i \in T$.*

(iii) *If $h < f_i$ (for every i), then $b_h \cdot -(b_{f_1} + \cdots + b_{f_m}) = \triangle_{i=1}^m b_{g_i}$ —where $g_i \in T$.*

(iv) *Set $\min\{f_1, \cdots, f_n\} = \{f_{i(1)}, \cdots, f_{i(p)}\}$. Then,*

$$b_{f_1} \triangle \cdots \triangle b_{f_n} \subseteq b_{f_{i(1)}} \cup \cdots \cup b_{f_{i(p)}} \quad \textit{and} \quad f_{i(k)} \in b_{f_1} \triangle \cdots \triangle b_{f_n}.$$

(v) *If $b_{f_1} \triangle \cdots \triangle b_{f_n} = b_{g_1} \triangle \cdots \triangle b_{g_m} \neq 0$, then there are i, j so that $b_{f_i} = b_{g_j}$.*

Proof. (i) If $f \leq h$ then $b_h \cdot -b_f = \emptyset = b_h \triangle b_h$. If $f \not\leq h$, by Lemma 3.3 (iv)-(b) we have, $b_h \cdot -b_f = b_h \cdot -b_{f \vee h}$ and,

$$b_h \triangle b_{f \vee h} \quad = \quad b_h \cdot -b_{f \vee h} + \underbrace{b_{f \vee h} \cdot -b_h}_{=\emptyset} \quad = \quad b_h \cdot -b_{f \vee h}$$

Thus, $b_h \cdot -b_f = b_h \triangle b_{f \vee h}$.

(ii) Let

$$\begin{aligned} b &= b_{f_1} \triangle \cdots \triangle b_{f_n} \\ b \cdot -b_f &= (b_{f_1} \triangle \cdots \triangle b_{f_n}) \cdot -b_f \\ &= (b_{f_1} \cdot -b_f) \triangle \cdots \triangle (b_{f_n} \cdot -b_f) \\ &= (b_{f_1} \triangle b_{f \vee f_1}) \triangle \cdots \triangle (b_{f_n} \triangle b_{f \vee f_n}) \end{aligned}$$

(iii) Let $h < f_i$ and show by induction on n that:

$$b_h \cdot -\sum_{i=1}^{n} b_{f_i} \quad = \quad \triangle_{i=1}^m b_{g_i}$$

Suppose that we have shown what we wanted up to $n-1$. Let $b = b_h \cdot -(b_{f_1} + \cdots + b_{f_{n-1}} + b_{f_n})$ and set $b' = b_h \cdot -(b_{f_1} + \cdots + b_{f_{n-1}})$. So, $b = b_h \cdot -b_{f_1} \cdot \cdots \cdot -b_{f_{n-1}} \cdot -b_{f_n} = b' \cdot -b_{f_n}$. Now, by induction hypothesis $b' = \triangle_{i=1}^k b_{g_i}$. So, by (ii), $b = b' \cdot -b_{f_n} = (\triangle_{i=1}^k b_{g_i}) \cdot -b_{f_n} = \triangle_{j=1}^m b_{h_j}$.

(iv) If $f < g$ then $b_g \subseteq b_f$. So,

$$b_{f_1} \triangle \cdots \triangle b_{f_n} \subseteq b_{f_1} \cup \cdots \cup b_{f_n} \subseteq b_{f_{i(1)}} \cup \cdots \cup b_{f_{i(p)}}.$$

For each $k \in \{1, \cdots, p\}$, $f_{i(k)} \in b_{f_1} \triangle \cdots \triangle b_{f_n}$. Indeed, $f_{i(k)} \in b_{f_{i(k)}}$ and for $j \neq i(k)$, $f_{i(k)} \notin b_{f_j}$ (if not $f_j < f_{i(k)}$). So,

$$f_{i(k)} \in b_{f_{i(k)}} \triangle (\triangle_{j \neq i(k)} b_{f_j}) = b_{f_1} \triangle \cdots \triangle b_{f_n}.$$

(v) Suppose $b := b_{f_1} \triangle \cdots \triangle b_{f_n} = b_{g_1} \triangle \cdots \triangle b_{g_m}$, where $f_i \neq f_j$ and $g_k \neq g_l$ for $i \neq j$ and $k \neq l$. Let

$$\min\{f_1, \cdots, f_n\} = \{f_{i(1)}, \ldots, f_{i(p)}\}, \qquad \min\{g_1, \ldots, g_m\} = \{g_{j(1)}, \cdots, g_{j(q)}\}$$

By (iv) we have:

$$b \subseteq b_{f_{i(1)}} \cup \cdots \cup b_{f_{i(p)}} \tag{7}$$

$$b \subseteq b_{g_{j(1)}} \cup \cdots \cup b_{g_{j(q)}} \tag{8}$$

$$\text{for all } k \text{ and } \ell, \quad f_{i(k)} \in b \text{ and } g_{j(\ell)} \in b. \tag{9}$$

Let k be given. So, by (9) $f_{i(k)} \in b$. By (8), let $\ell(k)$ be such that $f_{i(k)} \geq g_{j(\ell(k))}$. Similarly (using (7) instead of (8)), let k' be such that $g_{j(\ell(k))} \geq f_{i(k')}$. Hence $f_{i(k)} \geq f_{i(k')}$, and thus $f_{i(k)} = f_{i(k')} = g_{j(\ell(k))}$. □

Now we prove Theorem 3.5.

Proof of Theorem 3.5. We prove first the existence. Let $b \in F(P)$. By Lemma 3.4, $b = \sum_{i=1}^{n} e_i$ where $e_i \cdot e_j = \emptyset$ for $i \neq j$ and thus $b = \triangle_{i=1}^{n} e_i$. In addition, by Lemma 3.4 again, either $e_i = b_{h_i}$, or $e_i = b_{h_i} \cdot -(\sum_{j=1}^{n} b_{f_j})$, and by Lemma 3.6 (iii) $e_i = \triangle_{i=1}^{m} b_{g_i}$ ($g_i \in T$).

We prove the uniqueness. Let $b = \triangle_{i=1}^{n} b_{f_i} = \triangle_{j=1}^{m} b_{g_j} \neq 0$. By Lemma 3.6(v) there are i, j so that $b_{f_i} = b_{g_j}$. Without loss of generality, $i = 1 = j$. So $f_1 = g_1 := h$. We have $b_h \triangle b = \triangle_{i=2}^{n} b_{f_i} = \triangle_{j=2}^{m} b_{g_j}$. The uniqueness follows. □

4. Examples of free poset algebras $F(P)$

The following proposition characterizes atoms in $F(P)$. To this end, let $Atom(F(P))$ denotes the set of atoms of $F(P)$, and recall that, in $F(P) = \langle V_q : q \in P \rangle$, each element of $F(P)$ is a finite union of $\bigcap_{p \in \sigma} V_p \cap \bigcap_{q \in \tau} -V_q$. The proof of the next result is obvious.

Proposition 4.1. *Let P be a poset. The following properties are equivalent.*

(i) $b \in Atom(F(P))$.

(ii) *There are finite subsets σ and τ of P such that $b = \bigcap_{p \in \sigma} V_p \cap \bigcap_{q \in \tau} -V_q$ and for every $s \in P$, either there is $p \in \sigma$ such that $s \geq p$ or there is $q \in \tau$ such that $s \leq q$.*

Let us give some examples.

1. Let C_1, C_2 be two chains with least elements α, β respectively. Then $B(C_1 \times C_2) \cong F(P)$, for some poset P. Indeed,

$$\text{Prim}(C_1 \times C_2) = \{(\alpha, q) : q \in C_2\} \cup \{(p, \beta) : p \in C_1\}.$$

Moreover $b_{(s,t)} = b_{(\alpha,t)} \vee b_{(s,\beta)}$. Thus $B(C_1 \times C_2) = \mathrm{cl}(\{b_{(s,t)} : (s,t) \in \mathrm{Prim}(C_1 \times C_2)\})$. Now by Theorem 2.4 (iii), $B(C_1 \times C_2) \cong F(P)$, for some poset P.

2. For every set I, there is a poset $\langle P, \leq \rangle$ with a greatest element so that the free Boolean algebra over I is isomorphic to $F(P)$. (Consider I as an anti-chain and $P = I \cup \{\infty\}$ with $p < \infty$ for every $p \in I$.)

The following proposition shows that free poset algebras is a proper subclass of upper semi-lattice algebras.

Proposition 4.2. *Let T be an anti-chain of size $\aleph_1$ so that $x \vee y =_{def} \infty$ for all $x, y \in T$. Then $B(T \cup \{\infty\})$ is an upper semi-lattice algebra that is not a free poset algebra.*

Proof. Suppose the contrary and pick P so that $\mathcal{I}d(T \cup \{\infty\}) = \mathcal{U}lt(B(T \cup \{\infty\}))$ and $\mathrm{Fs}(P)$ are homeomorphic spaces. It follows that $\mathrm{Fs}(P)$ is a scattered topological space and thus $| \mathrm{Fs}(P) | = | P | = \aleph_1$, see [5] and P has no infinite anti-chains see [5]. Next, by Ben Dushnik-Miller theorem, see [4], either there is an infinite set of incomparable elements in $(P, \leq)$ or there is a chain of size $\aleph_1$ in $(P, \leq)$. Now since all antichains in $(P, \leq)$ are finite, it follows that there is a chain C in $(P, \leq)$ of size $\aleph_1$. Thus, since C is scattered, ω_1 or ω_1^* embeds in $(C, \leq)$ see [7]. Therefore there are at least two limit points in $\mathrm{Fs}(P)$ which is a contradiction since the set of limit points of $\mathcal{I}d(T \cup \{\infty\})$ is reduced to $\{\infty\}$. □

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