# On reduced pairs of bounded closed convex sets


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Recibido: 12 de Septiembre de 2003
Aceptado: 23 de Febrero de 2002



## ABSTRACT

In this paper certain criteria for reduced pairs of bounded closed convex set are presented. Some examples of reduced and not reduced pairs are enclosed.



*2000 Mathematics Subject Classification:* 52A07, 26A27.
*Key words:* Convex analysis, pairs of convex sets


Let $X = (X, \tau)$ be a topological vector space over the field $\mathbb{R}$. Let $\mathcal{K}(X)\,[\mathcal{B}(X)]$ be a family of all nonempty compact [bounded closed ] convex subsets of $X$. For any $A, B \subset X$ the Minkowski sum is defined by $A + B = \{a + b \mid \ a \in A \ \text{ and } \ b \in B\}$. Since $A + B$ is not always closed [4],[9] we define $A \dot{+} B = \overline{A + B}$ for $A, B \in \mathcal{B}(X)$. It was showed in [9] that for $A, B, C \in \mathcal{B}(X)$ the inclusion $A \dot{+} B \subset B \dot{+} C$ implies $A \subset C$. From this it follows that $\mathcal{B}(X)$ together with " $\dot{+}$" is a semigroup satisfying the law of cancellation, i.e. $A \dot{+} B = B \dot{+} C$ implies $A = C$.

For $(A, B), (C, D) \in \mathcal{B}^2(X)$, let $(A, B) \sim (C, D)$ if and only if $A \subset C,\ B \subset D$ and $(A, B) \sim (C, D)$. The relation "$\sim$" is an equivalence relation in $\mathcal{B}^2(X)$ and "$\leq$" is an ordering in the equivalence class $[A, B]$ of any pair $(A, B)$. It should be mentioned that the space $\mathcal{K}(X)/_\sim$, $\mathcal{K}(X) = \{A \in \mathcal{B}(X) \mid A \text{ is compact}\}$, plays important role in quasidifferential calculus [2].

The set $A \in \mathcal{B}(X)$ is called a *polytope* if $A$ is convex hull of a finite set. If $A, B \in \mathcal{B}(X)$ then $A \vee B$ is the convex hull of $A \cup B$.

It was proved in [6] that if $A, B \in \mathcal{K}(X)$, then there exists minimal element $(C, D)$ in $[A, B]$ such that $(C, D) \leq (A, B)$. From [3], [8] we know that if $(A, B), (C, D) \in \mathcal{K}^2(X)$, are two minimal pairs in $[A, B]$ and $\dim X \leq 2$ then $C + x,\ D = B + x$.



ISSN: 1139-1138
http://dx.doi.org/10.5209/rev_REMA.2003.v16.n2.16838

Let $(A, B) \in \mathcal{B}^2(X)$. The pair $(A, B)$ is called *reduced* if for any $(C, D) \in [A, B]$ there exists $M \in \mathcal{B}(X)$ such that $C = A \dot{+} M$ and $D = B \dot{+} M$. Let us notice that every reduced pair is minimal. Every minimal pair is reduced in $X = \mathbb{R}$ (see, [6]).

Let $A \in \mathcal{K}(X)$, $f \in X^*$. Then $H_f A = \{x \in A \mid f(x) = \max_{y\in A} f(y)\}$.

The set $A \in \mathcal{B}(X)$ is called a *summand* of $B \in \mathcal{B}(X)$ if there exists $M \in \mathcal{B}(X)$ such that $B = A \dot{+} M$.

W. Weil has proved in [11] the following lemma.

**Lemma.** *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$ and $A$ be a convex polytope.Then $A$ is a summand of $B$ if an only if each one-dimensional face $H_f A$ is contained in a translate of the corresponding face $H_f B$.*

**Theorem 1.** *Let $A$, $B \in \mathcal{K}(\mathbb{R}^n)$ and $A$ be a convex polytope such that* $\operatorname{card} H_f B = 1$ *for each one-dimensional face $H_f A$. Then the pair $(A, B)$ is reduced.*

**Proof**. Let $(C, D \in [A, B]$. Then $A + D = B + C$. Let $f \in (\mathbb{R}^n)^*$ and $H_f A$ be one-dimensional face of $A$. Then, by virtue of the formula of the addition of faces, we have

$$H_f A + H_f D = H_f B + H_f C.$$

According to the assumption, $H_f B = \{b\}$ for some $b \in \mathbb{R}^n$. Then $H_f A \subset b-d+H_f C$, where $d \in H_f D$. Applying Lemma, we obtain that $C = A+M$ for some $M \in \mathcal{K}(\mathbb{R}^n)$. Hence, from the law of cancellation, it follows that $D = B + M$. ■

**Theorem 2.** *Let $A, B \in \mathcal{K}(\mathbb{R}^2)$ be a reduced pair. Then* $\operatorname{card} H_f B = 1$ *for each one-dimensional face $H_f A$.*

**Proof**. Let us assume that $\dim H_f B = \dim H_f A = 1$ for some $f \in (\mathbb{R}^2)^*$. Then there exists an interval $I$ and a triangle $T$ such that length of $I$ is not greater than both lengths of $H_f A$ and $H_f B$, and $H_{-f} T = I$. If $H_f T = \{b\}$ then $H_f(A + T) = H_f A+b$, $H_{-f}(A+T) = H_{-f}A+I$, $H_f(B+T) = H_f B+b$ and $H_{-f}(B+T) = H_{-f}B+I$. Hence $I$ is a summand of both $A+T$ and $B+T$, and $A+T = A'+I$, $B+T = B'+I$ for some $A', B' \in \mathcal{K}(\mathbb{R}^2)$. Then $A', B') \in [A, B]$, and since $H_f A$ is not a summand of $H_f A'$ then $A$ is not a summand of $A'$. Therefore, $(A, B)$ is not reduced. ■

**Proposition 1.** *Let $(A, B), (C, D), (E, F) \in \mathcal{B}^2(X)$ and $A = C \dot{+} E$, $B = D \dot{+} F$. If the pair $(A, B)$ is reduced then both $(C, D)$ and $(E, F)$ are reduced.*

**Proof**. Let $(C', D') \in [C, D]$. Then $C' \dot{+} D = C \dot{+} D'$, and we have

$$A \dot{+} D \dot{+} F \dot{+} D' = A \dot{+} B \dot{+} D' = C \dot{+} E \dot{+} B \dot{+} D' = E \dot{+} B \dot{+} C' \dot{+} D.$$

Hence $A \dot{+} F \dot{+} D' = B \dot{+} E \dot{+} C'$. From the assumption, it follows that $E \dot{+} C' = A \dot{+} M$ and $F \dot{+} D' = B \dot{+} M$ for some $M \in \mathcal{B}(X)$. Then $E \dot{+} C' = C \dot{+} E \dot{+} M$ and $F \dot{+} D' = D \dot{+} F \dot{+} M$. Hence $C' = C \dot{+} M$ and $D' = D \dot{+} M$. ■

**Proposition 2.** *Let $A, B \in \mathcal{B}(X)$. If the pair $(A \vee B, A+B)$ is reduced then $(A \vee B, B)$ is also reduced.*

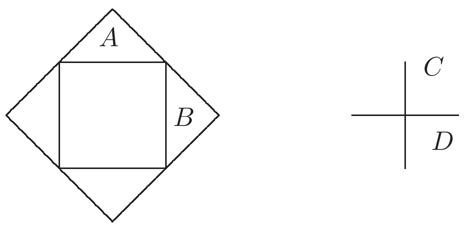
**Proof**. Since $(A \vee B, A+B) = (A \vee B, B) + (\{0\}, A)$ then applying Proposition 1 we obtain our Proposition. ■

Let $A, B \in \mathcal{B}(X)$. We call the pair $(A, B)$ *convex* if $A \cup B$ is convex. We call $(A, B)$ *convexly reduced* if for any convex pair $(C, D)$ in $[A, B]$ there exists $M \in \mathcal{B}(X)$ such that $C = A \dot{+} M$ and $D = B \dot{+} M$.

**Theorem 3.** *The convex pair $(A, B) \in \mathcal{B}^2(X)$ is convexly reduced if and only if $(A \cap B, A \cup B)$ is reduced.*

**Proof**. $\Rightarrow$) Let the pair $(A, B)$ be convexly reduced and $(F, G) \in [A \cap B, A \cup B]$. From [4],[10] it follows that there exists $(A_0, B_0) \in [A, B]$ such that $A_0 \cap B_0 = F$ and $A_0 \cup B_0 = G$. From the assumption, $A_0 = A \dot{+} M$ and $B_0 = B \dot{+} M$ for some $M \in \mathcal{B}(X)$. Then $F = A_0 \cap B_0 = A \cap B \dot{+} M$ and $G = A_0 \cup B_0 = A \cup B \dot{+} M$. Therefore, the pair $(A \cap B, A \cup B)$ is reduced.

$\Leftarrow$) Let $(A \cap B, A \cup B)$ be reduced, $(C, D) \in [A, B]$ and $C \cup D$ be convex. Then $A \dot{+} D = B \dot{+} C = A \cap B \dot{+} C \cup D = C \cap D \dot{+} A \cup B$, [see [10]]. Hence $C \cap D = A \cap B \dot{+} M$ and $C \cup D = A \cup B \dot{+} M$ for some $M \in \mathcal{B}(X)$. From the law of cancellation, we obtain $C = A \dot{+} M$ and $D = B \dot{+} M$. ■




The pair $(A, B)$ is convexly reduced and $(A, B) \sim (C, D)$.

**Theorem 4.** *Let $A, B \in \mathcal{B}(X)$. If $(A \vee B, B)$ is a reduced pair then the pair $(A, B)$ is reduced.*

**Proof**. Let $(C, D) \in [A, B]$. Then $A \dot{+} D = B \dot{+} C$. Therefore,

$$D \dot{+} A \vee B = (A \dot{+} D) \vee (B \dot{+} D) = (B \dot{+} C) \vee (B \dot{+} D) = B \dot{+} C \vee D.$$

Since the pair $(A \vee B, B)$ is reduced then $D = B \dot{+} M$ for some $M \in \mathcal{B}(X)$. From the law of cancellation ( [9] ) $C = A \dot{+} M$. ■

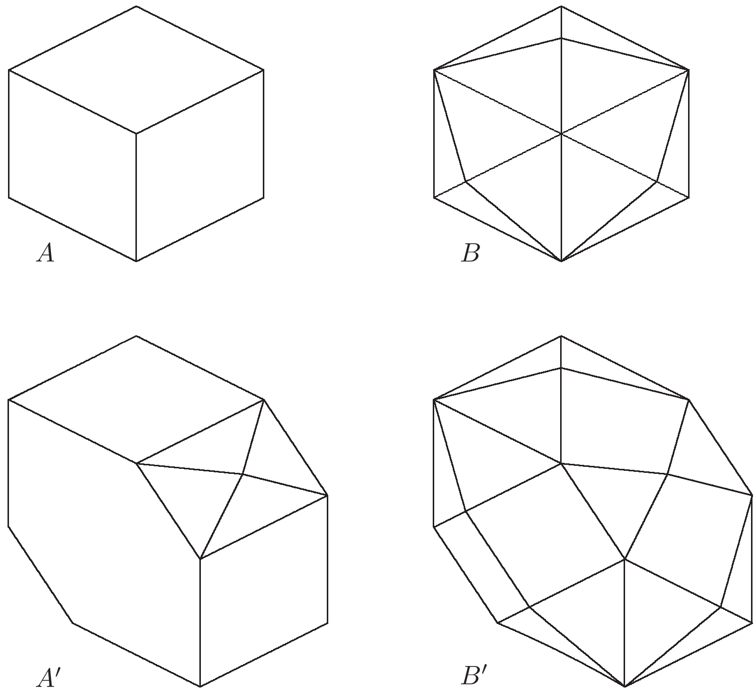
The pair $(A, B)$ is convexly reduced and $(A, B) \sim (C, D)$. The pair $(A, B)$ is also reduced and the class $[A, B]$ is convex, that is $C \cup D$ is convex for any $(C, D) \in [A, B]$ ([4]).

In [5] the following theorem was proved:

**Theorem 5.** *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$ and $A$ be a polytope with nonempty interior. Let* $\operatorname{card} H_f B = 1$ *for each face $H_f A$ such that* $\dim H_f A = n - 1$. *Then the pair $(A, B)$ is minimal.*

For $n = 2$, Theorem 1 and Theorem 5 have equivalent assumptions , hence Theorem 1 is stronger than Theorem 5. For $n = 3$, the assumption of Theorem 5 is weaker than the assumption of Theorem 1. The following example shows that generally we cannot replace the assumption in Theorem 1 with the assumption from Theorem 5.

**Example.** Let $A = [-1, 1]^3$ and
$B = A \vee (0, 0, 3/2) \vee (0, 0, -3/2) \vee (0, 3/2, 0) \vee (0, -3/2, 0) \vee (3/2, 0, 0) \vee (-3/2, 0, 0)$. Let us notice that if $\dim H_f A = 2$ then $\operatorname{card} H_f B = 1$. Let $I = (1, 0, 0) \vee (0, 1, 0)$. Let $A' = (A + I) \vee (5/3, 5/3, 0)$ and $B' = (B + I) \vee (5/3, 5/3, 0)$. We have $(A', B') \sim (A + I, B + I) \sim (A, B)$. Let us notice that $H_f A' = (5/3, 5/3, 0)$ and $H_f A = (1, 1, -1) \vee (1, 1, 1)$ for $f(x, y, z) = x + y$. Then $A$ is not a summand of $A'$. The pair $(A, B)$ is not reduced.

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