An Ostrowski like inequality for convex functions and applications

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Recibido: 23 de Julio de 2002
Aceptado: 17 de Febrero de 2003

ABSTRACT

In this paper we point out an Ostrowski type inequality for convex functions which complement in a sense the recent results for functions of bounded variation and absolutely continuous functions. Applications in connection with the Hermite-Hadamard inequality are also considered.

2000 Mathematics Subject Classification: Primary 26D15, 26D10.
Key words: Ostrowski type inequalities, Convex functions, Hermite-Hadamard type inequalities

1. Introduction

In 1938, A. Ostrowski [9] proved the following integral inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left\| f' \right\|_\infty \tag{1.1}$$

provided f is differentiable and $\left\| f' \right\|_\infty = \sup_{t\in(a,b)} \left| f'(t) \right| < \infty$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In the last 5 years, many authors have concentrated their efforts in generalising (1.1) and have applied the obtained results in different fields, including Numerical Integration, Probability Theory and Statistics, Information Theory, etc. For a comprehensive approach in the field, see the recent book [5] where many other references may be found.

ISSN: 1139-1138
http://dx.doi.org/10.5209/rev_REMA.2003.v16.n2.16807

One direction of generalising (1.1) was pointed out by the author in [2] – [4]. Let us recall here a couple of the main results obtained in the above papers.

Theorem 1. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, \dots, k + 1)$ be $k + 2$ points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \dots, k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \tag{1.2}$$
$$\leq \left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|,\ i = 0, \dots, k-1 \right\} \right] \bigvee_a^b (f),$$

where $\nu(h) := \max\{h_i | i = 0, \dots, k-1\}$, $h_i := x_{i+1} - x_i$ $(i = 0, \dots, k-1)$ and $\bigvee_a^b (f)$ is the total variation of f on $[a, b]$.

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

If one would assume more for the function f, for example, absolute continuity, then the following result holds.

Theorem 2. *Under the assumptions of Theorem 1 for I_k and α_i $(i = 0, \dots, k+1)$ and if $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then*

$$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \tag{1.3}$$
$$\leq \begin{cases} \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right] \right]^{\frac{1}{q}} \|f'\|_p & \text{if } f' \in L_p[a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|,\ i = 0, \dots, k-1 \right\} \right] \|f'\|_1, \end{cases}$$

where $\|\cdot\|_p$ $(p \in [1, \infty])$ are the Lebesgue norms, i.e.,

$$\|h\|_\infty := ess \sup_{t \in [a,b]} |h(t)|,$$
$$\|h\|_p := \left(\int_a^b |h(t)|^p\,dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(q+1)^{\frac{1}{q}}}$ *and* $\frac{1}{2}$ *are best in the sense mentioned above.*

In this paper, the case of convex functions $f : [a,b] \to \mathbb{R}$ is examined. Some particular cases in connection with the well known Hermite-Hadamard inequality for convex functions are also considered.

2. The Results

The following result holds.

Theorem 3. *Let* $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *be a division of the interval* $[a,b]$ *and* α_i $(i = 0, \ldots, k+1)$ *be* $k+2$ *points such that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = b$. *If* $f : [a,b] \to \mathbb{R}$ *is a convex function on* $[a,b]$, *then we have the inequality:*

$$\begin{aligned} &\frac{1}{2}\sum_{i=0}^{k-1}\left[(x_{i+1}-\alpha_{i+1})^2 f'_+(\alpha_{i+1}) - (\alpha_{i+1}-x_i)^2 f'_-(\alpha_{i+1})\right] \qquad (2.1)\\ \leq\ & \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i) f(x_{i+1}) - \int_a^b f(t)\,dt \\ \leq\ & \frac{1}{2}\sum_{i=0}^{k-1}\left[(x_{i+1}-\alpha_{i+1})^2 f'_-(x_{i+1}) - (\alpha_{i+1}-x_i)^2 f'_+(x_i)\right]. \end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*

Proof. Using the integration by parts formula, we may prove the equality (see for example [3]):

$$\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i) f(x_{i+1}) - \int_a^b f(t)\,dt = \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} (t-\alpha_{i+1}) f'(t)\,dt \qquad (2.2)$$

for any locally absolutely continuous function $f : (a,b) \to \mathbb{R}$.

Since f is convex, then it is locally Lipschitzian on (a,b) and thus the above equality holds. Also, we have

$$f'_+(x_i) \leq f'(t) \leq f'_-(\alpha_{i+1}) \quad \text{for a.e.} \quad t \in [x_i, \alpha_{i+1}] \qquad (2.3)$$

and

$$f'_+(\alpha_{i+1}) \leq f'(t) \leq f'_-(x_{i+1}) \quad \text{for a.e.} \quad t \in [\alpha_{i+1}, x_{i+1}]. \qquad (2.4)$$

Using (2.3) and (2.4), we may write that

$$\begin{aligned} f'_-(\alpha_{i+1})\int_{x_i}^{\alpha_{i+1}}(t-\alpha_{i+1})\,dt &\leq \int_{x_i}^{\alpha_{i+1}} f'(t)(t-\alpha_{i+1})\,dt \qquad (2.5)\\ &\leq f'_+(x_i)\int_{x_i}^{\alpha_{i+1}}(t-\alpha_{i+1})\,dt \end{aligned}$$

and

$$f'_+(\alpha_{i+1})\int_{\alpha_{i+1}}^{x_{i+1}}(t-\alpha_{i+1})\,dt \le \int_{\alpha_{i+1}}^{x_{i+1}} f'(t)(t-\alpha_{i+1})\,dt \le f'_-(x_{i+1})\int_{\alpha_{i+1}}^{x_{i+1}}(t-\alpha_{i+1})\,dt. \tag{2.6}$$

Adding (2.5) and (2.6) and taking into account that

$$\int_{x_i}^{\alpha_{i+1}}(t-\alpha_{i+1})\,dt = -\frac{1}{2}(\alpha_{i+1}-x_i)^2$$

and

$$\int_{\alpha_{i+1}}^{x_{i+1}}(t-\alpha_{i+1})\,dt = \frac{1}{2}(x_{i+1}-\alpha_{i+1})^2,$$

we get

$$\begin{aligned} &\frac{1}{2}\left[(x_{i+1}-\alpha_{i+1})^2 f'_+(\alpha_{i+1}) - (\alpha_{i+1}-x_i)^2 f'_-(\alpha_{i+1})\right] \\ \le\ & \int_{x_i}^{x_{i+1}}(t-\alpha_{i+1}) f'(t)\,dt \\ \le\ & \frac{1}{2}\left[(x_{i+1}-\alpha_{i+1})^2 f'_-(x_{i+1}) - (\alpha_{i+1}-x_i)^2 f'_+(x_i)\right] \end{aligned} \tag{2.7}$$

for any $i = 0, \dots, k-1$.

If we sum (2.7) over i from 0 to $k-1$ and use the identity (2.2), we deduce the desired result (2.1).

The sharpness will be proved in what follows for a particular case. ■

It is natural to consider the following particular case.

Corollary 1. *Let L_k and f be as in the above theorem. Then we have the inequality*

$$\begin{aligned} 0 &\le \frac{1}{8}\sum_{i=0}^{k-1}\left[f'_+\left(\frac{x_i+x_{i+1}}{2}\right) - f'_-\left(\frac{x_i+x_{i+1}}{2}\right)\right](x_{i+1}-x_i)^2 \\ &\le \frac{1}{2}\left[(x_1-a) f(a) + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1}) f(x_i) + (b-x_{k-1}) f(b)\right] \\ &\quad - \int_a^b f(t)\,dt \\ &\le \frac{1}{8}\sum_{i=0}^{k-1}\left[f'_-(x_{i+1}) - f'_+(x_i)\right](x_{i+1}-x_i)^2. \end{aligned} \tag{2.8}$$

The constant $\frac{1}{8}$ in both inequalities is sharp.

The proof follows by the above theorem choosing $\alpha_i = \frac{x_{i-1}+x_i}{2}$, $i = 1, \ldots, k$ and taking into account that (see also [2])

$$\begin{aligned} & \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \qquad (2.9) \\ = & \frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b)\right]. \end{aligned}$$

The following corollary for equidistant partitioning also holds.

Corollary 2. *Let*

$$I_k : x_i := a + (b-a) \cdot \frac{i}{k} \qquad (i = 0, \ldots, k)$$

be an equidistant partitioning of $[a, b]$. *If* $f : [a, b] \to \mathbb{R}$ *is convex on* $[a, b]$, *then we have the inequalities*

$$\begin{aligned} 0 \leq & \frac{(b-a)^2}{8n^2} \sum_{i=0}^{k-1} \left\{ f'_+ \left[a + \left(i + \frac{1}{2}\right) \frac{b-a}{n}\right] \right. \qquad (2.10) \\ & \left. - f'_- \left[a + \left(i + \frac{1}{2}\right) \frac{b-a}{n}\right] \right\} \\ \leq & \frac{1}{k} \cdot \frac{f(a) + f(b)}{2} (b-a) \\ & + \frac{b-a}{k} \sum_{i=1}^{k-1} f\left[\frac{(k-i) a + ib}{k}\right] - \int_a^b f(t)\, dt \\ \leq & \frac{(b-a)^2}{8n^2} \sum_{i=0}^{k-1} \left\{ f'_- \left[a + (i+1) \cdot \frac{b-a}{n}\right] - f'_+ \left[a + i \cdot \frac{b-a}{n}\right] \right\}. \end{aligned}$$

The following particular cases which hold when we assume differentiability conditions may be stated.

Corollary 3. *If* $\alpha_i \in (a, b)$ *for* $i = 1, \ldots, k$ *are points of differentiability for* f, *then we have the inequality*

$$\begin{aligned} & \sum_{i=0}^{k-1} (x_{i+1} - x_i) \left(\frac{x_i + x_{i+1}}{2} - \alpha_{i+1}\right) f'(\alpha_{i+1}) \qquad (2.11) \\ \leq & \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t)\, dt. \end{aligned}$$

If we denote by $\nu(I_n) := \max\{x_{i+1} - x_i | i = 0, \ldots, k-1\}$, then the following corollary also holds.

Corollary 4. *If x_i $(i = 1, \ldots, k-1)$ are points of differentiability for f then*

$$\begin{aligned} & \frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=0}^{k-1}(x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b)\right] - \int_a^b f(t)\,dt \\ \leq & \frac{1}{8}[\nu(I_n)]^2 \left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{2.12}$$

3. Some Particular Inequalities

(1) If we choose $x_0 = a$, $x_1 = b$, $\alpha_0 = a$, $\alpha_1 = x \in (a, b)$, $\alpha_2 = b$, then from (2.1) we deduce (see also [6])

$$\begin{aligned} & \frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] \\ \leq & (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt \\ \leq & \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_-(a)\right]. \end{aligned} \tag{3.1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities (see for example [6]).

If $x = \frac{a+b}{2}$, then by (3.1) one deduces (see also [6])

$$\begin{aligned} 0 \leq & \frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ \leq & \frac{f(a) + f(b)}{2} \cdot (b-a) - \int_a^b f(t)\,dt \\ \leq & \frac{1}{8}(b-a)^2\left[f'_-(b) - f'_+(a)\right] \end{aligned} \tag{3.2}$$

and the constant $\frac{1}{8}$ in both inequalities is sharp (see for example [6]).

If one would assume that $x \in (a, b)$ is a point of differentiability, then

$$(b-a)\left(\frac{a+b}{2} - x\right) f'(x) \leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt. \tag{3.3}$$

(2) If we choose $a = x_0 < x < x_2 = b$ and the numbers $\alpha_0 = a$, $\alpha \in (a, x]$, $\beta \in [x, b)$ and $\alpha_3 = b$, then by Theorem 3, we deduce

$$\begin{aligned} & \frac{1}{2}\left[(x-\alpha)^2 f'_+(\alpha) - (\alpha-a)^2 f'_-(\alpha) + (b-\beta)^2 f'_+(\beta) - (\beta-x)^2 f'_-(\beta)\right] \\ \leq\ & (\alpha-a) f(a) + (\beta-\alpha) f(x) + (b-\beta) f(b) - \int_a^b f(t)\,dt \qquad (3.4) \\ \leq\ & \frac{1}{2}\left[(x-\alpha)^2 f'_-(x) - (\alpha-a)^2 f'_+(a) + (b-\beta)^2 f'_-(b) - (\beta-x)^2 f'_+(x)\right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

(a) Note that if we let $\alpha \to a+$ and $\beta \to b-$, then from (3.4), by taking into account firstly that $(x-\alpha)^2 f'_+(a) \leq (x-\alpha)^2 f'_+(\alpha)$ and $-(\beta-x)^2 f'_-(b) \leq -(\beta-x)^2 f'_-(\beta)$, we may deduce the inequality obtained in [7]:

$$\begin{aligned} & \frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] \qquad (3.5) \\ \leq\ & \int_a^b f(t)\,dt - (b-a) f(x) \\ \leq\ & \frac{1}{2}\left[(\beta-x)^2 f'_-(b) + (x-a)^2 f'_+(a)\right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities (see for example [7]).

If in (3.5) we choose $x = \frac{a+b}{2}$, then (see also [7])

$$\begin{aligned} 0\ \leq\ & \frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \qquad (3.6) \\ \leq\ & \int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{1}{8}(b-a)^2\left[f'_-(b) - f'_+(a)\right] \end{aligned}$$

and the constant $\frac{1}{8}$ is sharp in both inequalities.

We may state now the following result for convex functions improving Hermite-Hadamard integral inequalities.

Proposition 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned} 0 &\leq \frac{1}{8}(b-a)\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right]. \end{aligned} \tag{3.7}$$

The constant $\frac{1}{8}$ is sharp in both parts.

If one would assume that $x\in(a,b)$ is a differentiability point for f, then we have the inequality [7]

$$(b-a)\left(\frac{a+b}{2}-x\right)f'(x)\leq\int_a^b f(t)\,dt-(b-a)f(x). \tag{3.8}$$

(b) If we choose $\alpha=\frac{a+x}{2}$ and $\beta=\frac{x+b}{2}$, then by (3.4) we have the three point inequality:

$$\begin{aligned} 0 &\leq \frac{1}{8}\left\{(x-a)^2\left[f'_+\left(\frac{a+x}{2}\right)-f'_-\left(\frac{a+x}{2}\right)\right]\right. \\ &\qquad \left.+(b-x)^2\left[f'_+\left(\frac{x+b}{2}\right)-f'_-\left(\frac{x+b}{2}\right)\right]\right\} \\ &\leq \frac{1}{2}\left[(x-a)f(a)+f(x)(b-a)+(b-x)f(b)\right]-\int_a^b f(t)\,dt \\ &\leq \frac{1}{8}\left\{(x-a)^2\left[f'_+(x)-f'_-(a)\right]+(b-x)^2\left[f'_-(b)-f'_+(x)\right]\right\} \end{aligned} \tag{3.9}$$

for any $x\in(a,b)$. The constant $\frac{1}{8}$ is sharp in both parts.

If in (3.9) we choose $x=\frac{a+b}{2}$, then we get

$$\begin{aligned} 0 &\leq \frac{1}{32}(b-a)^2\left[f'_+\left(\frac{3a+b}{4}\right)-f'_-\left(\frac{3a+b}{4}\right)\right. \\ &\qquad \left.+f'_+\left(\frac{a+3b}{4}\right)-f'_-\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right](b-a)-\int_a^b f(t)\,dt \\ &\leq \frac{1}{32}(b-a)^2\left[f'_-(b)-f'_+\left(\frac{a+b}{2}\right)+f'_-\left(\frac{a+b}{2}\right)-f'_+(a)\right] \end{aligned} \tag{3.10}$$

If one would assume that f is differentiable in $\frac{a+b}{2}$, then we get the following reverse of Bullen's inequality

$$\begin{aligned} 0 &\leq \frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2} \right) \right] (b-a) - \int_a^b f(t)\,dt \qquad (3.11) \\ &\leq \frac{1}{32} (b-a)^2 \left[f'_-(b) - f'_+(a) \right]. \end{aligned}$$

The constant $\frac{1}{32}$ is sharp.

(c) Now, if we choose $\alpha = \frac{5a+b}{6}$, $\beta = \frac{a+5b}{6}$ and $x \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$ in (3.4), then we have the inequalities

$$\begin{aligned} &\frac{1}{2} \left[\left(x - \frac{5a+b}{6} \right)^2 f'_+\left(\frac{5a+b}{6} \right) - \frac{(b-a)^2}{36} f'_-\left(\frac{5a+b}{6} \right) \right. \qquad (3.12) \\ &\left. + \frac{(b-a)^2}{36} f'_+\left(\frac{a+5b}{6} \right) - \left(\frac{a+5b}{6} - x \right)^2 f'_-\left(\frac{a+5b}{6} \right) \right] \\ \leq\ & \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f(x) \right] - \int_a^b f(t)\,dt \\ \leq\ & \frac{1}{2} \left[\left(x - \frac{5a+b}{6} \right)^2 f'_-(x) - \frac{(b-a)^2}{36} f'_+(a) \right. \\ &\left. + \frac{(b-a)^2}{36} f'_-(b) - \left(\frac{a+5b}{6} - x \right)^2 f'_+(x) \right]. \end{aligned}$$

If in (3.12) we choose $x = \frac{a+b}{2}$, then we get the Simpson's inequality

$$\begin{aligned} &\frac{1}{18} (b-a)^2 \left[f'_+\left(\frac{5a+b}{6} \right) - \frac{1}{4} f'_-\left(\frac{5a+b}{6} \right) \right. \qquad (3.13) \\ &\qquad \left. + \frac{1}{4} f'_+\left(\frac{a+5b}{6} \right) - f'_-\left(\frac{a+5b}{6} \right) \right] \\ \leq\ & \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] - \int_a^b f(t)\,dt \\ \leq\ & \frac{1}{18} (b-a)^2 \left[f'_-\left(\frac{a+b}{2} \right) - \frac{1}{4} f'_+(a) + \frac{1}{4} f'_-(b) - f'_+\left(\frac{a+b}{2} \right) \right]. \end{aligned}$$

If the function is differentiable on (a,b), then we get

$$\begin{aligned} & -\frac{1}{24}(b-a)^2\left[f'\left(\frac{a+5b}{6}\right)-f'\left(\frac{5a+b}{6}\right)\right] \\ \leq\ & \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt \\ \leq\ & \frac{1}{72}(b-a)^2\left[f'_-(b)-f'_+(a)\right]. \end{aligned} \tag{3.14}$$

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