Reaction-diffusion-convection problems in unbounded cylinders

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ABSTRACT

The work is devoted to reaction-diffusion-convection problems in unbounded cylinders. We study the Fredholm property and properness of the corresponding elliptic operators and define the topological degree. Together with analysis of the spectrum of the linearized operators it allows us to study bifurcations of solutions, to prove existence of convective waves, and to make some conclusions about their stability.

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1. Introduction

Propagation of chemical waves can be accompanied by various instabilities. If the density of the medium depends on the temperature or on the depth of conversion, then the gravity can lead to appearance of natural convection. Convective instability of reaction fronts is studied experimentally [5], [17], [20], [21], [22] and theoretically [9], [11], [12], [18], [19], with the use of approximate analytical and asymptotical methods.

In this work we study elliptic operators of the reaction-diffusion-convection (RDC) type in unbounded cylinders. We define the topological degree and use it to study bifurcations of convective fronts.

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Consider a reaction-diffusion system coupled with the Navier-Stokes equations:

$$\frac{\partial T}{\partial t} + v \cdot \nabla T = \kappa \Delta T + F_0(T, A) \tag{11}$$

$$\frac{\partial A_j}{\partial t} + v \cdot \nabla A_j = d_j \Delta A_j + F_j(T, A), \;\; j = 1, ..., m \tag{12}$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p + \nu \Delta v + \beta g (T - T^*) \tau \tag{13}$$

$$div\ v = 0 \tag{14}$$

Here T is the temperature, A_j are concentrations of reagents, $v = (v_1, v_2)$ is the velocity of the medium, p the pressure, κ the coefficient of thermal diffusivity, d_j are the coefficients of mass diffusion, ρ is the density, β the coefficient of thermal expansion, g the gravity acceleration, T^* a characteristic constant temperature, $\tau = (0, 1)$ the unit vector in the vertical direction, $F_j, \; j = 0, ..., m$ are nonlinear terms describing the rates of chemical reactions. Their specific form determined by equations of chemical kinetics will not be used.

The system of equations (11)-(14) is considered in the strip $0 < x_1 < l$, $-\infty < x_2 < \infty$ with the no-flux boundary conditions for the temperature and for the concentrations:

$$x_1 = 0, l : \frac{\partial T}{\partial x_1} = 0, \; \frac{\partial A_j}{\partial x_1} = 0, \tag{15}$$

and with the free surface boundary conditions for the velocity:

$$x_1 = 0, l : v_1 = 0, \; \frac{\partial v_2}{\partial x_1} = 0. \tag{16}$$

We will see below that these boundary conditions for the velocity allow some simplifications and, what is more important, in this case we can obtain a more complete information about the spectrum of the linearized problem.

The Navier-Stokes equations are written above under the so-called Boussinesq approximation where the medium is considered as incompressible and the density is everywhere constant except for the buoyancy term, which describes the action of gravity and appears as a result of linearization of the density. Existence of solutions of the RDC problems and some of their properties are studied in [1], [4], [15], [16].

If the medium is unmovable, then we have a reaction-diffusion system (11), (12) with $v = 0$. Under certain conditions there exists a travelling wave solution of this system, i.e., a solution of the form

$$T(x_1, x_2, t) = \tilde{T}(x_1, x_2 - ct), \; A_j(x_1, x_2, t) = \tilde{A}_j(x_1, x_2 - ct)$$

(see [29]), and this solution may not depend on the variable x_1 (see Section 2). Here c is the wave velocity. Obviously this is also a solution of the complete system (11)-(14) with $v = 0$, since the pressure can take into account the term for the temperature in the Navier-Stokes equations.

Suppose that this solution is stable as a solution of the reaction-diffusion system (11), (12). However it can be unstable as a solution of (11)-(14). The physical interpretation of this effect is related to natural convection. If the wave propagates upwards ($c > 0$) and the reaction is exothermic, then under some conditions convection will appear. There is an analogy with convection in a layer of a liquid heated from below. If the Rayleigh number is sufficiently large, then a stationary temperature distribution loses its stability, and a convective structure appears. In the case of upwards propagating reacting fronts, an exothermic chemical reaction plays a role of heating from below, and can lead to appearance of convective travelling waves. Under some additional conditions on the functions F_j, the principal eigenvalue of the linearized problem can be found in the form of a minimax representation [32]. This allows us to find the stability boundary.

If an eigenvalue of the linearized problem passes through zero, then we can expect a bifurcation of new solutions. To justify this conclusion, our approach consists in defining a topological degree for the corresponding operators. Since we consider a problem in unbounded domains, the Leray-Schauder theory cannot be applied. However the degree for elliptic operators in unbounded domains can be defined [7], [8], [27], [28], [33]. One of the approaches is based on the theory of Fredholm operators [10], [14], [25]. We employ it in this work to define the degree for the RDC operators. We show first that the corresponding linearized operators are Fredholm and study their index (Section 4). In Section 5 we prove that nonlinear operators are proper. (An operator $A : E_1 \to E_2$ is called proper if for any compact set $\mathcal{D} \subset E_2$ and for any ball B in E_1, the set $A^{-1}(\mathcal{D}) \cap B$ is compact.) The degree construction is discussed in Section 6. In Section 7 we analyse the discrete spectrum of the operators linearized about the one-dimensional wave and study bifurcations of convective waves. Finally in Section 8 we analyze the spectrum of the operators linearized about the bifurcating waves.

2. Main results

To formulate the main results of this work we suppose in this section that the reaction-diffusion system (11)-(12) with $v = 0$ consists only of one equation. The results remain valid for a system, if

$$\kappa = d_1 = ... = d_m = D$$

and if chemical kinetics allow the reduction to the so-called monotone system [29]. This means that there exists a linear change of variables

$$\theta_i = a_{i0} T + \sum_{j=1}^{m} a_{ij} A_j, \quad i = 0, ...m,$$

such that the system

$$\frac{\partial \theta}{\partial t} = D \Delta \theta + F(\theta), \tag{21}$$

where $\theta = (\theta_0, \cdots \theta_m)$, $F = (F_0, \cdots F_m)$, satisfies the condition

$$\frac{\partial F_i}{\partial \theta_j} > 0, \quad i, j = 0, ...m, \ i \neq j. \tag{22}$$

This condition is essential for the bifurcation analysis, because we need to reduce the reaction-diffusion system to a monotone system. In the general case we cannot study the eigenvalues with the method presented here. However, the equality of the diffusion coefficients is not necessary in Sections 3 to 6, where we study some qualitative properties of the operators and construct the topological degree. Hence this condition will not be assumed in those sections.

Remarks.

1. We point out that, due to the change of variables, the nonlinear function F appearing here is not the same function as in equations (11)-(12).

2. To simplify the presentation, in the following we view θ_0 as the temperature, and $\theta_1, \cdots \theta_m$ as dimensionless concentrations. In the analysis of eigenvalues, we consider θ as a scalar variable.

In this case, existence and stability of travelling waves for reaction diffusion systems is well-studied. We use below the following existence result [28]:

Let $F(\theta^+) = F(\theta^-) = 0$, where $\theta^+ < \theta^-$ and the inequality between two vectors is understood componentwise. Suppose that there exists a finite number of zeros $\theta^{(1)}$, ..., $\theta^{(s)}$, of the function F in the interval $\theta^+ < \theta < \theta^-$, that the matrices $F'(\theta^\pm)$ have all eigenvalues in the left half-plane, and each of the matrices $F'(\theta^{(i)})$, $i = 1, \cdots s$, has at least one eigenvalue in the right half-plane. Then there exists a monotone travelling wave solution of (21), i.e. a solution

$$\theta(x, t) = \Theta(x_2 - ct),$$

where $\Theta(\pm\infty) = \theta^\pm$, and all the components of the vector-valued function $\Theta(x_2)$ are monotonically decreasing. Since this solution does not depend on the horizontal variable x_1, we call it a "one-dimensional wave". It is a non-convective wave, in the sense that the velocity field is zero.

We note that the inequality (22) may be nonstrict if we suppose that the matrices $F'(\theta^\pm)$ are irreducible.

Throughout this section, we also make the following assumptions:

1. $F \in C^{2+\alpha}, F(\theta^\pm) = 0$.

2. All eigenvalues of the matrices $F'(\theta^\pm)$ have a negative real part.

3. Equation (21) has a monotonically decreasing travelling wave solution $\Theta(x_2)$.

A travelling wave solution of the system (21) is at the same time a solution with $v = 0$ of the elliptic problem

$$\begin{aligned} -c\frac{\partial \theta}{\partial x_2} + v \cdot \nabla \theta &= D\Delta\theta + F(\theta) && (23)\\ -c\frac{\partial v}{\partial x_2} + (v\cdot\nabla)v &= -\nabla p + P\Delta v + PR(\theta_0 - \theta_0^*)\tau && (24)\\ \operatorname{div} v &= 0 && (25)\\ x_1 = 0, l : \frac{\partial \theta}{\partial x_1} &= 0,\ v_1 = 0,\ \frac{\partial v_2}{\partial x_1} = 0, && (26) \end{aligned}$$

where θ_0^* is constant and corresponds to the characteristic temperature T^*. The Navier-Stokes equations are written in the dimensionless form with P being the Prandtl number and R the Rayleigh number. As customary, R will be used as a bifurcation parameter.

We introduce the stream function

$$v_1 = \frac{\partial \psi}{\partial x_2}, \quad v_2 = -\frac{\partial \psi}{\partial x_1}$$

and the vorticity

$$\omega = -\Delta\psi.$$

Then we can write the problem (23)-(26) in the form

$$\begin{aligned} D\Delta\theta - \frac{\partial \psi}{\partial x_2}\frac{\partial \theta}{\partial x_1} + \left(c + \frac{\partial \psi}{\partial x_1}\right)\frac{\partial \theta}{\partial x_2} + F(\theta) &= 0 && (27)\\ P\Delta\omega - \frac{\partial \psi}{\partial x_2}\frac{\partial \omega}{\partial x_1} + \left(c + \frac{\partial \psi}{\partial x_1}\right)\frac{\partial \omega}{\partial x_2} + PR\frac{\partial \theta_0}{\partial x_1} &= 0 && (28)\\ \Delta\psi + \omega &= 0 && (29) \end{aligned}$$

$$x_1 = 0, l : \frac{\partial \theta}{\partial x_1} = 0,\ \psi = 0,\ \omega = 0. \qquad (210)$$

Solutions of (27)-(210) with $\omega \not\equiv 0$, $\psi \not\equiv 0$ will be called “convective waves”.

In the case of no-slip boundary conditions for the velocity we cannot write the boundary condition in terms of vorticity. This is an advantage of the free surface boundary condition, which simplifies the analysis of the RDC operators.

In the remaining part of Section 2, we consider the case where θ is a scalar variable. We linearize the problem (27)-(210) about $\Theta(x_2)$, $\omega = 0$, $\psi = 0$, and we consider the eigenvalue problem

$$D\Delta\theta + c\,\frac{\partial\theta}{\partial x_2} + F'(\Theta(x_2))\theta + \Theta'(x_2)\,\frac{\partial\psi}{\partial x_1} = \lambda\theta \tag{211}$$

$$P\Delta\omega + c\,\frac{\partial\omega}{\partial x_2} + PR\frac{\partial\theta}{\partial x_1} = \lambda\omega \tag{212}$$

$$\Delta\psi + \omega = \lambda\psi \tag{213}$$

$$x_1 = 0, l : \frac{\partial\theta}{\partial x_1} = 0,\ \psi = 0,\ \omega = 0. \tag{214}$$

Since its coefficients do not depend on the variable x_2, it is convenient to study the Fourier modes,

$$\theta(x_1, x_2) = \theta_k(x_2)\cos(kx_1),$$
$$\omega(x_1, x_2) = \omega_k(x_2)\sin(kx_1),$$
$$\psi(x_1, x_2) = \psi_k(x_2)\sin(kx_1).$$

Here we assume that $l = \pi$. Thus we obtain the one-dimensional eigenvalue problem

$$\begin{aligned} D\theta_k^{''} + c\,\theta_k' + (F'(\Theta(x_2)) - Dk^2)\theta_k + k\Theta'(x_2)\psi_k &= \lambda\theta_k \\ P\omega_k^{''} + c\,\omega_k' - Pk^2\omega_k - PRk\theta_k &= \lambda\omega_k \\ \psi_k^{''} - k^2\psi_k + \omega_k &= \lambda\psi_k. \end{aligned} \tag{215}$$

The main results of the work are given by Theorems 2.1 and 2.2. More detailed statements and proofs of these theorems will be found in Sections 7 and 8.

Theorem 2.1. *For each $k = -1, -2, \cdots$ there exists a unique critical value $R_c(k)$ such that the principal eigenvalue of the problem (215) is negative for $R < R_c(k)$ and positive for $R > R_c(k)$. All critical values $R_c(k)$ are positive, and when k varies in $(-\mathbb{N}^*)$ there exists a minimal value $R_c(k_0)$. If $R_c(k) \neq R_c(k_0)$ for all $k \neq k_0$, then $R_c(k_0)$ is a bifurcation point, i.e. in each its neighbourhood there exists a value R such that the problem (27)-(210) admits a solution (θ, ω, ψ) with $(\psi, \omega) \not\equiv 0$ (convective solution).*

Definition 2.1. *We will call the solutions arising in Theorem 2.1 "bifurcating waves".*

Remarks.

1. As usual in the problems describing travelling wave solutions, the constant c in (27)-(210) is not given. It should be found together with the solution (θ, ω, ψ). For the bifurcating waves $c = c(R)$ also changes. When we consider an operator formulation of the problem, we introduce a functionalization of the parameter c. This means that instead of the unknown constant c we consider a given functional $c(\theta, \omega, \psi)$. It removes the zero eigenvalue of the linearized operator and allows the construction of the topological degree.

2. The second remark concerns the stability of the one-dimensional solution. It is determined by the eigenvalue problem

$$D\Delta\theta + c\,\frac{\partial\theta}{\partial x_2} + F'(\Theta(x_2))\theta + \Theta'(x_2)\,\frac{\partial\psi}{\partial x_1} = \lambda\theta \tag{216}$$

$$P\Delta\omega + c\,\frac{\partial\omega}{\partial x_2} + PR\frac{\partial\theta}{\partial x_1} = \lambda\omega \tag{217}$$

$$\Delta\psi + \omega = 0 \tag{218}$$

$$x_1 = 0, l : \frac{\partial\theta}{\partial x_1} = 0,\ \psi = 0,\ \omega = 0. \tag{219}$$

which differs from (211)-(214). In Section 7 we show that at $R = R_c(k_0)$ a simple eigenvalue of this problem passes from the left half-plane to the right half-plane through zero. We also prove that for $R \leq R_c(k_0)$ there are no real eigenvalues greater than this one. However we cannot prove that there are no complex conjugate eigenvalues with a greater real part.

3. The physically important eigenvalue problem is the problem (216)-(219), with no λ in the last equation. However the problem (211)-(214) is interesting for two reasons: first, it arises naturally in the bifurcation analysis when we compute the index of the one-dimensional solution. The second reason is that we use its properties to study the problem (216)-(219).

For the stability of the bifurcating solutions, we have the same restriction as in the second remark. We show in Section 8 the following result:

Theorem 2.2. *If convective solutions exist only for $R > R_c(k_0)$ (supercritical bifurcation), then among them there are solutions $(\theta_2, \omega_2, \psi_2)$ for which the problem*

$$\begin{aligned} D\Delta\theta + \left(c + \frac{\partial\psi_2}{\partial x_1}\right)\frac{\partial\theta}{\partial x_2} + F'(\theta_2(x_2))\theta + \theta_2'(x_2)\,\frac{\partial\psi}{\partial x_1} &= \lambda\theta \\ P\Delta\omega + \left(c + \frac{\partial\psi_2}{\partial x_1}\right)\frac{\partial\omega}{\partial x_2} + \frac{\partial\omega_2}{\partial x_2}\frac{\partial\psi}{\partial x_1} + PR\frac{\partial\theta}{\partial x_1} &= \lambda\omega \\ \Delta\psi + \omega &= 0 \end{aligned}$$

$$x_1 = 0, l : \frac{\partial\theta}{\partial x_1} = 0,\ \psi = 0,\ \omega = 0.$$

has no positive eigenvalue, for R sufficiently close to $R_c(k_0)$ (see fig. 1b). If convective solutions exist only for $R < R_c(k_0)$ (subcritical bifurcation), then among them there are solutions for which this problem has a nonnegative eigenvalue, for R sufficiently close to $R_c(k_0)$ (see fig. 1a).

Remark. This result is not exhaustive, for example we can have a convective solution for $R < R_c$ and another one for $R > R_c$, as described in fig. 1c. Note that the degree arguments do not allow us to single out the type of the bifurcating wave.

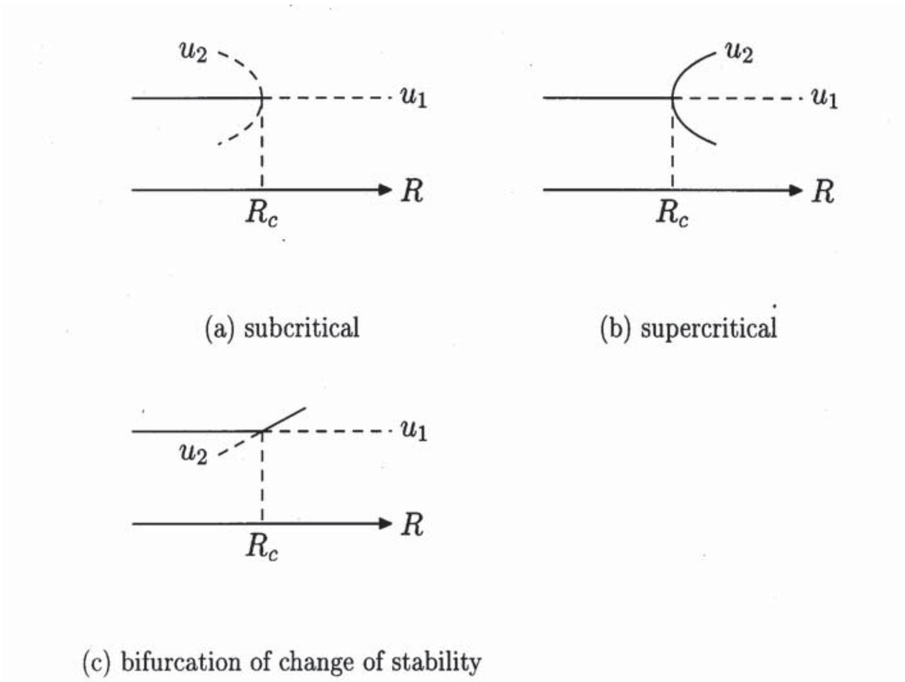

Figure 1: Different types of bifurcations

3. Operators and spaces

For the sake of generality we consider in this section reaction-diffusion-convection problems in unbounded cylinders in $\mathbb{R}^n$, $n \geq 2$. The problem (27)-(210) is a particular case of the problem

$$a(x)\Delta w + \sum_{i=1}^{n} b_i(x)\frac{\partial w}{\partial x_i} + B(x, \nabla w) + G(x, w) = 0, \tag{31}$$

$$x \in \partial\Omega : w_i = 0,\ i = 1, ..., k,\ \ \frac{\partial w_i}{\partial n} = 0,\ i = k+1, ..., p. \tag{32}$$

Here $x = (x_1, ..., x_n) = (x', x_n)$, $w = (w_1, ..., w_p)$, $G = (G_1, ..., G_p)$, $\Omega = \Omega' \times \mathbb{R}$ is an unbounded cylinder with bounded section $\Omega' \subset \mathbb{R}^{n-1}$ and with the axis along the direction x_n, $\partial/\partial n$ is the normal derivative, a and b_i are $p \times p$ matrices, $B(x, \nabla w)$ is a vector with the components

$$B_k(x, \nabla w) = \sum_{i,j=1}^{n} \left(b_{ij}^{(k)}(x)\frac{\partial w}{\partial x_i}, \frac{\partial w}{\partial x_j} \right),\ k = 1, ..., p,$$

where $b_{ij}^{(k)}(x)$ are $p \times p$ matrices, and $(\ ,\ )$ denotes the scalar product in $\mathbb{R}^p$.

We suppose that a is a symmetric positive-definite matrix,

$$a \in C^{2+\alpha}(\bar{\Omega}), b_i \in C^{1+\alpha}(\bar{\Omega}), b_{ij}^{(k)} \in C^{\alpha}(\bar{\Omega}),\ 0 < \alpha < 1,$$

$$G(x, w) \in C^{2+\alpha}(\bar{\Omega} \times \mathbb{R}^p),$$

and that the boundary $\partial\Omega$ is of the class $C^{2+\alpha}$. We assume moreover that there exist the limits

$$a^{\pm}(x') = \lim_{x_n \to \pm\infty} a(x),\ \ b_i^{\pm}(x') = \lim_{x_n \to \pm\infty} b_i(x),\ \ b_{ij}^{(k)\pm}(x') = \lim_{x_n \to \pm\infty} b_{ij}^{(k)}(x),$$

$$G^{\pm}(x', w) = \lim_{x_n \to \pm\infty} G(x, w).$$

Here $x' = (x_1, ..., x_{n-1})$.

Consider the limiting problems

$$a^{\pm}(x')\Delta' w + \sum_{i=1}^{n-1} b_i^{\pm}(x') \frac{\partial w}{\partial x_i} + B^{\pm}(x', \nabla w) + G^{\pm}(x', w) = 0, \tag{33}$$

$$x' \in \partial\Omega' : w_i = 0,\ i = 1, ..., k,\ \ \frac{\partial w_i}{\partial n} = 0,\ i = k+1, ..., p \tag{34}$$

in the section Ω' of the cylinder. Suppose that they have solutions $w^+(x')$ and $w^-(x')$ in $C^{2+\alpha}(\bar{\Omega}')$. We look for solutions of the problem (31), (32) having the limits at infinity

$$w^{\pm}(x') = \lim_{x_n \to \pm\infty} w(x). \tag{35}$$

Let

$$\phi(x) = s(x_n) w^+(x') + (1 - s(x_n)) w^-(x'), \tag{36}$$

where $s(x_n)$ is a sufficiently smooth function equal to 1 for $x_n \geq 1$ and to 0 for $x_n \leq 0$. We represent $w(x)$ in the form $w(x) = \phi(x) + u(x)$, where $u(x)$ is a solution of the problem

$$a(x)\Delta(u + \phi) + \sum_{i=1}^{n} b_i(x) \frac{\partial(u + \phi)}{\partial x_i} + B(x, \nabla(u + \phi)) + G(x, u + \phi) = 0, \tag{37}$$

$$x \in \partial\Omega : u_i = 0,\ i = 1, ..., k,\ \ \frac{\partial u_i}{\partial n} = 0,\ i = k+1, ..., p. \tag{38}$$

We introduce weighted Hölder spaces $C_{\mu}^{j+\alpha}(\bar{\Omega}) = \{u : u\mu \in C^{j+\alpha}(\bar{\Omega})\}$. As a weight function we take $\mu(x_n) = 1 + x_n^2$. We note that functions decreasing exponentially at infinity belong to the weighted space and that

$$\frac{\mu'(x_n)}{\mu(x_n)} \to 0,\ \frac{\mu''(x_n)}{\mu(x_n)} \to 0,\ \ x_n \to \pm\infty. \tag{39}$$

This property will be used below. Finally, multiplication by the weight function does not change the boundary conditions.

We put

$$E_1 = \left\{ u \in C^{2+\alpha}_{\mu}(\bar{\Omega}),\ u_{i|\partial\Omega} = 0,\ i = 1, ..., k,\ \frac{\partial u_i}{\partial n}_{|\partial\Omega} = 0,\ i = k+1, ..., p \right\},$$

$$E_1^0 = \left\{ u \in C^{2+\alpha}_{\mu_0}(\bar{\Omega}),\ u_{i|\partial\Omega} = 0,\ i = 1, ..., k,\ \frac{\partial u_i}{\partial n}_{|\partial\Omega} = 0,\ i = k+1, ..., p \right\},$$

$$E_2 = C^{\alpha}_{\mu}(\bar{\Omega}),\ E_2^0 = C^{\alpha}_{\mu_0}(\bar{\Omega}),$$

where $\mu_0(x_n)$ is identically equal to 1, and we consider the operator

$$A(u) = a(x)\Delta(u+\phi) + \sum_{i=1}^{n} b_i(x)\frac{\partial(u+\phi)}{\partial x_i} + B(x, \nabla(u+\phi)) + G(x, u+\phi) \quad (310)$$

acting from E_1 to E_2 or from E_1^0 to E_2^0. The choice of function spaces is important for what follows. In spaces without weight, the operators are not proper and the topological degree cannot be defined [3].

Let $u_0 \in E_1$. We consider the linearized operator

$$Lu = a(x)\Delta u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u + \mathcal{B}u,$$

acting from E_1 to E_2 or from E_1^0 to E_2^0. Here

$$c(x) = G'_u(x, u_0(x) + \phi(x)),$$

where G'_u denotes the derivative of G with respect to the second argument, and $\mathcal{B}u$ is a linear operator, with the components

$$(\mathcal{B}u)_k = \sum_{i,j=1}^{n} \left(b_{ij}^{(k)}(x)\frac{\partial(u_0+\phi)}{\partial x_i}, \frac{\partial u}{\partial x_j} \right) + \sum_{i,j=1}^{n} \left(b_{ij}^{(k)}(x)\frac{\partial u}{\partial x_i}, \frac{\partial(u_0+\phi)}{\partial x_j} \right),$$

$$k = 1, ..., p.$$

Denote

$$c^{\pm}(x') = \lim_{x_n \to \pm\infty} c(x) = {G^{\pm}}'_u(x', w^{\pm}(x')).$$

In the next section we study the Fredholm property of the operator L, in Section 5 the properness of the operator A, and in Section 6 the topological degree. In Section 7 we apply these results to study the bifurcations, and in Section 8 we analyze the spectrum of the operator linearized about the bifurcating solutions.

4. Fredholm property and index

Consider the operator $L : E_1 \to E_2$ introduced in the previous section. L is said to be normally solvable if its range is closed. It is said to be Fredholm if its kernel has finite dimension α, if its range is closed and has finite co-dimension β. In this case, its index κ is defined by the equality $\kappa = \beta - \alpha$.

All the following results concerning the Fredholm property can be obtained in spaces without weight, but the weight is important for the construction of the topological degree in Section 6. We define the limiting operators $L^{\pm} : E_1^0 \to E_2^0$ by

$$L^{\pm}u = a^{\pm}(x')\Delta u + \sum_{i=1}^{n} b_i^{\pm}(x')\frac{\partial u}{\partial x_i} + c^{\pm}(x')u + \mathcal{B}^{\pm}u,$$

where

$$(\mathcal{B}^{\pm}u)_k = \sum_{i,j=1}^{n}\left(b_{ij}^{(k)\pm}(x')\frac{\partial w^{\pm}(x')}{\partial x_i}, \frac{\partial u}{\partial x_j}\right) + \sum_{i,j=1}^{n}\left(b_{ij}^{(k)\pm}(x')\frac{\partial u}{\partial x_i}, \frac{\partial w^{\pm}(x')}{\partial x_j}\right),$$

$$k = 1, ..., p.$$

Condition 1. Equations

$$L^{\pm}u = 0$$

do not have nonzero solutions in E_1^0.

Condition 2. Equations

$$L^{\pm}u - \lambda u = 0$$

do not have nonzero solutions in E_1^0 for any $\lambda \geq 0$.

Theorem 4.1. *The operator L is normally solvable with a finite dimensional kernel if and only if Condition 1 is satisfied. If Condition 2 is satisfied, the operator L is Fredholm with index* 0.

The proof of this theorem is given in [33], in the case of the boundary conditions of the Dirichlet or of the Neumann type. In this work we consider the case where some components satisfy the Dirichlet boundary condition and the others satisfy the Neumann condition. The theorem remains valid. We recall that the index is a homotopy invariant in the class of Fredholm operators [13]. If L is normally solvable with a finite dimensional kernel, and if its range has infinite co-dimension, then L is said to be semi-Fredholm. If we consider a homotopy L_τ, such that all operators L_τ satisfy Condition 1, and one of them is semi-Fredholm, then they are all semi-Fredholm. These properties are used implicitly below.

Suppose that the solutions $w^{\pm}(x')$ of the limiting problems (33), (34) are constant vectors, $w^{\pm}(x') \equiv w^{\pm}$. Then the limiting operators $\mathcal{B}^{\pm}u$ vanish.

Lemma 4.2. *Assume that Condition 1 is satisfied. Suppose that $w^{\pm}(x')$ are constant vectors and that the codimension of the image of the operator*

$$L^{(0)}u = a(x)\Delta u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

is finite. Then L and $L^{(0)}$ are Fredholm, and the index of the operator L equals the index of the operator $L^{(0)}$.

Proof. The proof of the lemma directly follows from the homotopy invariance of the index in the class of Fredholm operators [13]. Indeed, the homotopy

$$L_\tau = \tau L + (1-\tau)L^{(0)}$$

preserves Condition 1. The lemma is proved. ■

We apply now this lemma for the problem (27)-(210). In this section, we need not assume that D is a scalar matrix, and that the system can be reduced to a monotone system. Hence θ will be considered here as a vector. We look for solutions having the limits at infinity

$$\theta^{\pm} = \lim_{x_2\to\pm\infty} \theta(x), \quad \lim_{x_2\to\pm\infty} \omega(x) = 0, \quad \lim_{x_2\to\pm\infty} \psi(x) = 0,$$

where $\theta^{\pm}$ are constants. Hence we define the function ϕ by (36) where $w^{\pm} = (\theta^{\pm}, 0, 0)$ and write the unknown as $w = u + \phi$, where $u = (\theta, \omega, \psi) \in E_1$.

We consider the operator A_R corresponding to the system (27)-(210) and denote the operator linearized about some $(\bar{\theta}(x), \bar{\omega}(x), \bar{\psi}(x))$ by L_R. As above, L_R is considered as acting from E_1 to E_2. The limiting problems $L_R^{\pm}u = 0$, $u \in E_1^0$, have the form

$$D\Delta\theta + c\,\frac{\partial\theta}{\partial x_2} + F'(\theta^{\pm})\theta = 0 \tag{41}$$

$$P\Delta\omega + c\,\frac{\partial\omega}{\partial x_2} + PR\frac{\partial\theta_0}{\partial x_1} = 0 \tag{42}$$

$$\Delta\psi + \omega = 0 \tag{43}$$

$$x_1 = 0, l : \frac{\partial\theta}{\partial x_1} = 0, \ \psi = 0, \ \omega = 0. \tag{44}$$

Condition 3. The problems

$$D\Delta\theta + c\,\frac{\partial\theta}{\partial x_2} + F'(\theta^{\pm})\theta = 0, \quad \frac{\partial\theta}{\partial x_1}|_{\partial\Omega} = 0 \tag{45}$$

do not have nonzero solutions in $C^{2+\alpha}(\bar{\Omega})$.

Lemma 4.3. *If Condition 3 is satisfied, then the operator L_R is normally solvable with a finite dimensional kernel.*

Proof. To prove the lemma it is sufficient to note that the problem (42)-(44) with $\theta \equiv 0$ has only the zero solution in $C^{2+\alpha}(\bar{\Omega})$. Hence Condition 1 is satisfied, and one can use Theorem 4.1 to complete the proof. ■

Consider the operator

$$L^1\theta = D\Delta\theta + c\,\frac{\partial\theta}{\partial x_2} + F'(\bar{\theta})\theta$$

acting from the space F_1 to the space $F_2 = C^{\alpha}_{\mu}(\bar{\Omega})$, where

$$F_1 = \left\{\theta \in C^{2+\alpha}_{\mu}(\bar{\Omega}),\ \frac{\partial\theta}{\partial x_1}|_{\partial\Omega} = 0\right\}$$

Theorem 4.4. *Suppose that*

$$\bar{\theta}(x) \to \theta^{\pm},\ \bar{\psi}(x) \to 0,\ \bar{\omega}(x) \to 0,\ \ x_2 \to \pm\infty.$$

If Condition 3 is satisfied and the co-dimension of the image of the operator $L^1 : F_1 \to F_2$ is finite, then the operator $L_R : E_1 \to E_2$ is Fredholm and has the same index.

Proof. The result is clearly true for $R = 0$, because the operator

$$(\omega, \psi) \mapsto \left(P\Delta\omega + c\frac{\partial\omega}{\partial x_2},\ \Delta\psi + \omega\right),$$

acting from the space $\{(\omega,\psi) \in (C^{2+\alpha}_{\mu}(\bar{\Omega}))^2, \omega_{|\partial\Omega} = \psi_{|\partial\Omega} = 0\}$ into the space $(C^{\alpha}_{\mu}(\bar{\Omega}))^2$ is invertible. For $R \neq 0$ the homotopy $L_{\tau R}, \tau \in [0,1]$ proves that L_R is Fredholm. The homotopy invariance of index in the class of Fredholm operators completes the proof. ■

This theorem shows that the index of the RDC operators is the same as the index of the reaction-diffusion operators. The Navier-Stokes equations do not change it. The index of the reaction-diffusion operators is computed in [6].

We will see in Section 6 that the construction of the topological degree requires Condition 2. The similar condition for the operator L^1 is as follows.

Condition 4. The problems

$$D\Delta\theta + c\,\frac{\partial\theta}{\partial x_2} + F'(\theta^{\pm})\theta = \lambda\theta,\ \ \frac{\partial\theta}{\partial x_1}|_{\partial\Omega} = 0 \qquad (46)$$

do not have nonzero solutions in $C^{2+\alpha}(\bar{\Omega})$ for any $\lambda \geq 0$.

Remark. This condition is satisfied if all eigenvalues of $F'(\theta^{\pm})$ have negative real part.

Thus we have the theorem:

Theorem 4.5. *If Condition 4 is satisfied, then the operator L_R is Fredholm with zero index.*

Proof. It is sufficient to show that Condition 2 is satisfied for L_R, and to use Theorem 4.1. To prove Condition 2, we fix $\lambda \geq 0$, and we consider a solution $u = (\theta, \omega, \psi) \in E_1^0$ of the equation

$$L_R^+ u - \lambda u = 0.$$

Then θ is a solution of (46), and Condition 4 implies that $\theta = 0$. Besides (ω, ψ) satisfies the problem

$$\begin{cases} P\Delta\omega + c\dfrac{\partial \omega}{\partial x_2} = \lambda\omega \\ \Delta\psi = \lambda\psi \\ x_1 = 0, l: \; \omega = 0, \; \psi = 0. \end{cases}$$

We first note that $\omega = 0$, hence ψ is a solution of the problem

$$\Delta\psi = \lambda\psi, \quad \psi|_{\partial\Omega} = 0.$$

This problem has no nontrivial solution in $C^{2+\alpha}(\bar{\Omega})$ if $\lambda \geq 0$. Condition 2 is proved, and the theorem follows. ■

The problems (46) have constant coefficients and they can be reduced to algebraic problems. Applying formally the Fourier transform with respect to the x_2-variable, we obtain the eigenvalue problems on the interval $0 \leq x_1 \leq l$:

$$D\theta'' + (-D\xi^2 + ci\xi E_p + F'(\theta^{\pm}))\theta = \lambda\theta, \;\; \theta'(0) = \theta'(l) = 0. \tag{47}$$

Here prime denotes the derivative with respect to x_1.

It is shown in [33] that Condition 4 is equivalent to the condition that the problems (47) do not have nonzero solutions for any real ξ and nonnegative λ.

Without loss of generality we can put $l = \pi$. Then the eigenvalues of the problems (47) can be found from the equalities

$$det(-(k^2 + \xi^2)D + ci\xi E_p + F'(\theta^{\pm}) - \lambda E_p) = 0 \tag{48}$$

for all real ξ and all integer k. Here E_p is the identity matrix.

If the matrix $F'(\theta^{\pm})$ has a real positive eigenvalue, then obviously (48) has a solution for a positive λ, and Condition 4 is not satisfied.

Suppose now that all eigenvalues of the matrix $F'(\theta^{\pm})$ are in the left half-plane. In Section 2 we have made the assumption that the matrix D is scalar, i.e., $D = dE_p$ where d is a constant. Hence all eigenvalues of the matrix

$$T(\xi, k) = -(k^2 + \xi^2)D + ci\xi + F'(\theta^{\pm})$$

are also in the left half-plane, and Condition 4 is satisfied. Note however that if D is not scalar, Condition 4 may be not satisfied. If the diffusion coefficients strongly differ from each other, eigenvalues of the matrix $T(\xi, k)$ can be in the right half-plane. This is similar to the Turing instability where a homogeneous state is stable without diffusion, and it may become unstable with diffusion.

5. Properness

Consider the operator $A(u) : E_1 \to E_2$ defined in Section 3. It is known that elliptic operators in unbounded domains may be not proper in spaces without weight [3]. Properness of the operators corresponding to reaction-diffusion problems is proved in [33]. Here we consider the RDC operators.

This property is essentially used for the construction of the topological degree. Moreover we need to prove the properness of operators depending on parameters. We consider the operator $A(u, \tau) : E_1 \to E_2$ depending on a parameter $\tau \in [0, 1]$:

$$\begin{aligned} A(u,\tau) &= a(x,\tau)\Delta(u+\phi) + \sum_{i=1}^{n} b_i(x,\tau)\frac{\partial(u+\phi)}{\partial x_i} + \\ &\quad B(x, \nabla(u+\phi), \tau) + G(x, u+\phi, \tau), \end{aligned}$$

where

$$B_k(x, \nabla w, \tau) = \sum_{i,j=1}^{n} \left(b_{ij}^{(k)}(x,\tau)\frac{\partial w}{\partial x_i}, \frac{\partial w}{\partial x_j} \right), \ k = 1, ..., p,$$

$a(x,\tau), b_i(x,\tau), b_{ij}^{(k)}(x,\tau) \in C^\alpha(\bar{\Omega})$, $G(x, u+\phi, \tau) \in C^2(\bar{\Omega} \times \mathbb{R}^p)$ for each τ.

We assume that the following conditions are satisfied:

1. For each τ there exist the limits

$$a^\pm(x',\tau) = \lim_{x_n \to \pm\infty} a(x,\tau), \ \ b_i^\pm(x',\tau) = \lim_{x_n \to \pm\infty} b_i(x,\tau),$$

$$b_{ij}^{(k)\pm}(x',\tau) = \lim_{x_n \to \pm\infty} b_{ij}^{(k)}(x,\tau),$$

$$c^\pm(x',\tau) = \lim_{x_n \to \pm\infty, w \to w^\pm} G'_u(x, w, \tau), \tag{51}$$

2. The following convergence takes place

$$\|a(x,\tau) - a(x,\tau_0)\|_{C^\alpha(\bar{\Omega})} \to 0, \ \ \|b_i(x,\tau) - b_i(x,\tau_0)\|_{C^\alpha(\bar{\Omega})} \to 0,$$

$$\left\| b_{ij}^{(k)}(x,\tau) - b_{ij}^{(k)}(x,\tau_0) \right\|_{C^\alpha(\bar{\Omega})} \to 0$$

as $\tau \to \tau_0$, and

$$\| \left(G(x,u(x)+\phi(x),\tau) - G(x,u(x)+\phi(x),\tau_0)\right) \mu\|_{C^\alpha(\bar\Omega)} \to 0$$

uniformly in $u(x)$ from any bounded set in E_1.

To simplify the presentation we assume that the limiting functions $w^\pm(x')$ and $\phi(x)$ are independent of τ.

Theorem 5.1. *If Condition 1 is satisfied for each $\tau \in [0,1]$, then the operator $A(u,\tau)$: $E_1 \times [0,1] \to E_2$ is proper.*

Proof. Let (u_m) be a bounded sequence in E_1 and (τ_m) be a bounded sequence in $\mathbb{R}$ such that the following convergence takes place in E_2:

$$A(u_m,\tau_m) = f_m \to f_0,\ m \to +\infty. \tag{52}$$

We should show that we can choose a converging subsequence from the sequence (u_m). Without loss of generality we can assume that $\tau_m \to \tau_0$. Since (u_m) is bounded in $C^{2+\alpha}(\bar\Omega)$ we can assume (to within a subsequence (u_{m_k})) that it converges to some limiting function $u_0 \in C^{2+\alpha}(\bar\Omega)$ in C^2 uniformly on every bounded subset of $\bar\Omega$ [33]. From the boundedness of u_m, u_0 in the weighted space, it follows that the convergence $u_m \to u_0$ is uniform in $C^2(\bar\Omega)$. We should show that this is a convergence in the norm of E_1.

Passing to the limit in (52), we obtain

$$A(u_0,\tau_0) = f_0. \tag{53}$$

Denote

$$v_m = \mu u_m,\ v_0 = \mu u_0,\ w_m = v_m - v_0,\ g_m = \mu f_m,\ g_0 = \mu f_0.$$

Taking the difference between (52) and (53) and multiplying it by μ, we obtain

$$(A(u_m,\tau_m) - A(u_m,\tau_0))\mu + (A(u_m,\tau_0) - A(u_0,\tau_0))\mu = g_m - g_0. \tag{54}$$

We denote

$$r_m = (A(u_m,\tau_m) - A(u_m,\tau_0))\mu$$

and note that

$$\|r_m\|_{C^\alpha(\bar\Omega)} \to 0,\ m \to \infty.$$

We can represent (54) in the form

$$\begin{aligned} a(x,\tau_0)\Delta w_m - b_n(x,\tau_0) w_m \frac{\mu'}{\mu} + \sum_{j=1}^n b_j(x,\tau_0)\frac{\partial w_m}{\partial x_j} - \\ a(x,\tau_0)\left(-w_m \frac{\mu''}{\mu} - 2\frac{\partial w_m}{\partial x_n}\frac{\mu'}{\mu} + w_m\left(\frac{\mu'}{\mu}\right)^2\right) + \\ (B(x,\nabla(u_m+\phi),\tau_0) - B(x,\nabla(u_0+\phi),\tau_0))\mu + \\ (G(x,u_m+\phi,\tau_0) - G(x,u_0+\phi,\tau_0))\mu = g_m - g_0 - r_m. \end{aligned} \tag{55}$$

We have

$$(G(x,u_m+\phi,\tau_0)-G(x,u_0+\phi,\tau_0))\mu=\int_0^1 G'_u(x,u_0+\phi+t(u_m-u_0),\tau_0)dt\,w_m,$$

Put

$$\beta_{ijk}\equiv\mu\times\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial(u_m+\phi)}{\partial x_i},\frac{\partial(u_m+\phi)}{\partial x_j}\right)\\ -\mu\times\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial(u_0+\phi)}{\partial x_i},\frac{\partial(u_0+\phi)}{\partial x_j}\right).$$

Then for $i,j\neq n$ we have

$$\beta_{ijk}=\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial w_m}{\partial x_i},\frac{\partial(u_m+\phi)}{\partial x_j}\right)+\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial(u_m+\phi)}{\partial x_i},\frac{\partial w_m}{\partial x_j}\right),$$

and

$$\beta_{ink}=\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial w_m}{\partial x_i},\frac{\partial(u_m+\phi)}{\partial x_n}\right)\\ +\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial(u_m+\phi)}{\partial x_i},\frac{\partial w_m}{\partial x_n}-w_m\,\frac{\mu'}{\mu}\right),$$

$$\beta_{njk}=\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial w_m}{\partial x_n}-w_m\,\frac{\mu'}{\mu},\frac{\partial(u_m+\phi)}{\partial x_j}\right)\\ +\left(b_{ij}^{(k)}(x,\tau_0)\frac{\partial(u_m+\phi)}{\partial x_n},\frac{\partial w_m}{\partial x_j}\right).$$

We show first of all that the sequence (w_m) converges to zero uniformly in $C(\bar\Omega)$. Suppose that it is not so. Then there exists a sequence $x^{(m)}$ such that $|w_m(x^{(m)})|\geq\epsilon>0$. Moreover the coordinate $x_n^{(m)}$ of the points $x^{(m)}$ is unbounded. Without loss of generality we can assume that $x_n^{(m)}\to\pm\infty$. Let us introduce the shifted functions

$$\tilde w_m(x)=w_m(x+x^{(m)}).$$

Since

$$\|\tilde w_m\|_{C^{2+\alpha}(\bar\Omega)}\leq M,$$

where the constant M is independent of m, then from the sequence $\tilde w_m(x)$ we can choose a subsequence $\tilde w_{m_k}(x)$ converging to some limiting function $\tilde w_0\in C^{2+\alpha}(\bar\Omega)$ in C^2 uniformly on every bounded set in $\bar\Omega$. We have $|\tilde w_0(0)|\geq\epsilon>0$.

The functions $\tilde{w}_{m_k}(x)$ satisfy the equation (55) with the shifted coefficients. Since

$$\frac{\mu'}{\mu} \to 0, \ \frac{\mu''}{\mu} \to 0,$$

as $x_n \to \infty$, and

$$\left| \frac{\partial(u_m + \phi)}{\partial x_i}(x + x^{(m)}) \right| \le \frac{M}{\mu(x_n + x_n^{(m)})},$$

then the terms $\beta_{ijk}(x + x^{(m)})$ tend to zero as m tends to infinity, for any $i, j = 1, ...n$ and $k = 1, ...p$. Besides the term

$$\int_0^1 G'_u(x + x^{(m)}, u_0 + \phi + t(u_m - u_0), \tau_0)dt$$

tends to $c^{\pm}(x')$ thanks to the condition (51) and the inequalities

$$\begin{aligned} |(u_m + \phi)(x + x^{(m)}) - w^{\pm}| &\le \frac{M}{\mu(x_n + x_n^{(m)})}, \\ |(u_0 + \phi)(x + x^{(m)}) - w^{\pm}| &\le \frac{M}{\mu(x_n + x_n^{(m)})}. \end{aligned}$$

Hence

$$L^{\pm}\tilde{w}_0 = 0.$$

This contradicts Condition 1.

Thus we have proved that the convergence $w_m \to 0$ is uniform in $C(\bar{\Omega})$. Since the functions w_m are uniformly bounded in $C^{2+\alpha}(\bar{\Omega})$, then this convergence is in $C^2(\bar{\Omega})$. From this convergence and from the Schauder estimates it follows that $w_m \to w_0$ in $C^{2+\alpha}(\bar{\Omega})$. The theorem is proved. ■

6. Topological degree

6.1. Elliptic operators in unbounded domains

If we consider elliptic operators in unbounded domains, then the Leray-Schauder degree cannot be applied, because we cannot write the operators as compact perturbations of identity. However the degree can be defined. One of the approaches to construct it is based on the theory of Fredholm operators. In [10] the degree is constructed for bounded Fredholm and proper operators acting in a Banach space. In [33] these results are used to construct the degree for elliptic operators in unbounded domains. In this work we generalize the results of [33] for the RDC operators.

We define the class Φ of operators $A(u) : E_1 \to E_2$ in the form (310) such that

1. Operator $A(u)$ has second-order Fréchet derivatives. The functions $a^{\pm}(x')$, $b_i^{\pm}(x')$, $b_{ij}^{(k)\pm}(x')$, $c^{\pm}(x')$, and $\partial a^{\pm}(x')/\partial x_i$, $i = 1, ..., n$ are continuous,

2. Condition 2 of Section 4 is satisfied.

We define also the class H of operators $A(u,\tau) : E_1 \times [0,1] \to E_2$ such that

3. $A(u,\tau)$ has second-order Fréchet derivatives with respect to u and τ. The functions $a^{\pm}(x',\tau)$, $b_i^{\pm}(x',\tau)$, $b_{ij}^{k\pm}(x',\tau)$, $c^{\pm}(x',\tau)$, and $\partial a^{\pm}(x',\tau)/\partial x_i$, $i = 1, ..., n$ are continuous in x and τ,

4. Condition 2 of Section 4 is satisfied, for any $\tau \in [0,1]$.

Theorem 6.1. *There exists a unique topological degree for the classes Φ and H.*

The proof of this theorem is given in [33] for reaction-diffusion operators. It can be easily generalized to the RDC operators.

We recall that if Condition 2 is satisfied, then the index of the linearized operator is 0. An interesting and still open question is whether the degree can be constructed in the case of a nonzero index. In the end of this section we consider an example which shows how the degree and index are related.

One more remark concerns the index of a stationary point equal, by definition, to the topological degree taken with respect to a small neighbourhood containing this point. As usual, its value is $(-1)^{\nu}$, where ν is the number of positive eigenvalues of the linearized operator taken with their multiplicities.

6.2. Travelling waves

When we look for travelling wave solutions, we obtain a particular case of the problem (31), (32) where the coefficients do not depend on the variable x_n along the axis of the cylinder. The appearance of an x_n-derivative of w takes into account the displacement of the front at constant velocity c:

$$a(x')\Delta w + c\frac{\partial w}{\partial x_n} + \sum_{i=1}^{n} b_i(x')\frac{\partial w}{\partial x_i} + B(x', \nabla w) + G(x', w) = 0, \tag{61}$$

$$w(x', \pm\infty) = w^{\pm}(x') \tag{62}$$

$$x \in \partial\Omega : w_i = 0,\ i = 1, ..., k,\ \ \frac{\partial w_i}{\partial n} = 0,\ i = k+1, ..., p, \tag{63}$$

We recall that c here is an unknown constant that should be found along with the function $w(x)$.

Solutions of the problem (61) - (63) are invariant with respect to translation in the direction x_n. It means that if there exists a solution $w(x)$ of this problem, then there exists also the whole family of solutions $w(x', x_n + h)$ for $h \in \mathbb{R}$. The topological degree is constructed in the previous section in the weighted Hölder space and, generically, it cannot be constructed in spaces without weight [3]. The weighted norm $\|w(x', x_n+h)\mu(x_n)\|$ tends to infinity as $h \to \pm\infty$. Therefore, a bounded domain D in the function space E_1 does not contain solutions or the branch of solutions intersects its boundary. So the degree cannot be used.

To avoid this problem and to apply the topological degree in the neighbourhood of a particular travelling wave w_1 we get rid of the invariance of this solution with respect to translations. One of the ways to do it is to introduce a functionalization of the parameter [28], [29], [33]. It means that instead of the unknown constant c we introduce a functional $c(w) = c(u + \phi)$ which satisfies the following properties:

1. $c(u + \phi)$ satisfies a Lipschitz condition on every bounded set in E_1 and has a Fréchet derivative $c'(u + \phi)$, which is continuous with respect to u;

2. The function $\tilde{c}(h) = c(w_1(x', x_n + h))$ is a decreasing function of h, $\tilde{c}(-\infty) = +\infty, \tilde{c}(+\infty) = -\infty$;

3. The solution w_1 of the problem (61) - (63) satisfies

$$\left\langle c'(w_1), \frac{\partial w_1}{\partial x_n} \right\rangle \neq 0,$$

where $\langle , \rangle$ denotes the duality between E_1 and E_1^*.

There are various ways to construct a functional satisfying these conditions. Following [28] we can take it in the form

$$c(w) = \ln \int_\Omega |w(x) - w^+(x')|^2 \sigma(x_n) dx,$$

where $\sigma(x_n)$ is an increasing function, $\sigma(-\infty) = 0, \sigma(+\infty) = 1$,

$$\int_{-\infty}^0 \sigma(x_n) dx_n < \infty.$$

We note that

$$\tilde{c}(h) = \ln \int_\Omega |w_1(x) - w^+(x')|^2 \sigma(x_n - h) dx.$$

Differentiation with respect to h shows that it is a decreasing function.

The properties 2 and 3 are satisfied if, for example, the solution $w_1(x)$ is monotone with respect to x_n. The first condition on the functional can be easily verified.

Thus instead of the equation (61) we have the equation

$$a(x')\Delta w + c(w)\frac{\partial w}{\partial x_n} + \sum_{i=1}^n b_i(x')\frac{\partial w}{\partial x_i} + B(x', \nabla w) + G(x', w) = 0. \tag{64}$$

If the problem (62) - (64) has a solution $w(x)$, then the problem (61) - (63) has a solution with the value $c = c(w)$.

Linearizing the equation (64) about the solution $w_1(x)$, we obtain

$$L_1 u = -\frac{\partial w_1}{\partial x_n} \langle c'(w_1), u \rangle, \tag{65}$$

Here $L_1 u \equiv Lu + c(w_1)\dfrac{\partial u}{\partial x_n}$, where L is the operator defined in Section 3.

From the invariance of solutions with respect to translation it follows that if w_1 is in $C^{3+\alpha}$ the operator L_1 has the zero eigenvalue and the corresponding eigenfunction is $u_0 = \partial w_1/\partial x_n$. We will show that, if this zero eigenvalue is simple, then the functionalization of the parameter removes it (cf. [28]), and that the other eigenvalues remain unchanged.

Lemma 6.2. *Let us assume that zero is a simple eigenvalue of L_1, and consider the operator L_2 defined by:*

$$L_2 u = L_1 u + < c'(w_1), u > \frac{\partial w_1}{\partial x_n}.$$

Then

1. *L_2 has no zero eigenvalue, but it has a negative eigenvalue*

$$\Lambda = < c'(w_1), u_0) >$$

with the eigenfunction u_0.

2. *All other eigenvalues of L_2 are also eigenvalues of L_1.*

Proof. First we prove 1. It is clear that Λ is an eigenvalue of L_2. It is negative, because of the second and third conditions in the definition of the functional c.

To show that zero is not an eigenvalue of L_2, we first make the following observation. If the coefficients a and b are sufficiently regular, e.g.

$$a \in C^{2+\alpha}(\bar{\Omega}), b_i \in C^{1+\alpha}(\bar{\Omega}),$$

the formally adjoint operator L_1^* can be defined. Applying the formal Fourier transform it can be shown that it satisfies Condition 2, hence it is also Fredholm with index 0.

Suppose that L_2 has a zero eigenvalue. In other terms, there exists a nonzero solution u of (65) with the corresponding boundary conditions. Denote v the eigenfunction corresponding to the zero eigenvalue of the formally adjoint operator L_1^*. We multiply (65) by v and integrate over Ω. Then we obtain

$$\langle c'(w_1), u\rangle \int_\Omega (u_0, v) dx = 0.$$

Since the zero eigenvalue is simple, then the integral in the last equality is different from zero. Hence $\langle c'(w_1), u\rangle = 0$, and from (65) we conclude that u and u_0 are proportional. The equality $\langle c'(w_1), u_0\rangle = 0$ contradicts the last condition imposed on the functional $c(w)$.

We now turn to the proof of the second point. Let λ be an eigenvalue of L_2, and v an eigenfunction of L_2^* corresponding to the eigenvalue $\bar{\lambda}$. Then

$$L_2^* v = L_1^* v + c'(w_1) < v, u_0 >= \bar{\lambda} v. \tag{66}$$

We multiply by u_0 and integrate:

$$< c'(w_1), u_0 >< v, u_0 >= \bar{\lambda} < v, u_0 > .$$

If the product $< v, u_0 >$ equals zero then from (66) we find that $\bar{\lambda}$ is an eigenvalue of L_1^*, hence λ is an eigenvalue of L_1. If it is not zero then $\lambda =< c'(w_1), u_0 >$. Hence all eigenvalues of L_2 are eigenvalues of L_1 except the eigenvalue $< c'(w_1), u_0 >$ which replaces the zero eigenvalue of L_1. ■

The construction of the topological degree for the operators with the functionalization of the parameter remains practically the same (see [33]).

6.3. Example

Consider the scalar one-dimensional reaction-diffusion problem

$$w'' + cw' + F(w) = 0, \;\; w(\pm\infty) = w^{\pm}, \; w^+ < w^-. \tag{67}$$

where $F(w^+) = F(w^-) = 0$ and $w \in C^{2+\alpha}(\mathbb{R})$. The corresponding operator linearized about a solution w_0, considered from $C_\mu^{2+\alpha}(\mathbb{R})$ to $C_\mu^{\alpha}(\mathbb{R})$, is

$$Lu = u'' + cu' + F'(w_0(x))u.$$

As well as in Section 4, it is Fredholm if the curves

$$\lambda(\xi) = -\xi^2 + ci\xi + F'(w^{\pm}),$$

which determine the continuous spectrum, do not pass through zero when ξ changes from $-\infty$ to ∞. This condition is, obviously, satisfied if $F'(w^{\pm}) \neq 0$ and $c \neq 0$.

Let us now replace the constant c by the functional $c(w)$:

$$w'' + c(w)w' + F(w) = 0, \;\; w(\pm\infty) = w^{\pm}. \tag{68}$$

The linearized operator becomes

$$L_1 u = u'' + c(w_0)u' + F'(w_0(x))u + \langle c'(w_0), u \rangle w_0'.$$

Since $w_0'(x) \to 0$ as $x \to \pm\infty$, then the continuous spectrum and the conditions for the operator to be Fredholm remain the same. Moreover, the linear homotopy between the operators L and L_1 shows that they have the same index.

If $F'(w^+) < 0$ and $F'(w^-) < 0$, then the continuous spectrum of the linearized operator lies in the left half-plane, and Condition 2 is satisfied. The topological degree can be defined. We note that the scalar equation is a particular case of monotone systems for which existence of waves is studied by the Leray-Schauder method [28], [29]. Existence of waves for the scalar equation can be proved by a much simpler phase plane analysis. If for example there exists only one zero of the function $F(w)$ in the interval (w^+, w^-), then the wave exists for a unique value of c.

If $F'(w^+) > 0$ and $F'(w^-) < 0$, then a part of the continuous spectrum lies in the right half-plane, and Condition 2 is not satisfied. The index of the linearized operator equals 1 for $c > 0$ and -1 for $c < 0$ [6]. The construction of the degree is not applicable in this case.

Suppose for simplicity that $F(w) > 0$ for $w^+ < w < w^-$ and that $F(w)$ is negative outside of this interval. Using the phase plane analysis, we can easily prove existence of waves for all values of $c > 0$. We emphasize that the waves exist for all positive velocities and not for all velocities greater or equal to some minimal velocity $c_0 > 0$ as it is usually presented. In the last case only monotone in x waves are considered.

If we introduce the functionalization of the parameter to remove the zero eigenvalue, then there exists a one-parameter family of solutions with all $c > 0$. The dimension of the manifold of solutions equals the index of the linearized operator [24]. This family of solutions is not bounded in the weighted norm. Therefore, any bounded domain in the space E_1 does not have solutions inside or the family of solutions intersects its boundary. This situation recalls the situation discussed in Section 6.2 where the one-parameter family of solutions was determined by the invariance of solutions with respect to translation in space. To get rid of it we have introduced the functionalization of parameter. Here the existence of the one-parameter family of solutions is connected with the positive index of the linearized operator. We can introduce an exponential weight to move the continuous spectrum in the left half-plane. This approach is well known in the analysis of stability of travelling waves [23]. If it can be done, then Condition 2 will be satisfied in the weighted space, and the degree can be defined. As for the family of solutions, all of them except a finite number do not belong to the space. So we have a usual situation with a discrete set of solutions and the zero index of the linearized operator.

The case $F'(w^+) < 0$ and $F'(w^-) > 0$ can be reduced to the previous one by a change of variable which changes also the sign of the velocity.

Consider finally the case $F'(w^+) > 0$ and $F'(w^-) > 0$. Condition 2 is not satisfied, and the degree construction is not applicable. The index of the linearized operator equals zero [6]. It is not clear whether the degree can be defined in this case. The phase plane analysis shows that solutions of the problem (67) do not exist for any c.

7. Spectrum and bifurcations

In this section we consider the problem (27)-(210) in the case where θ is a scalar variable, with the conditions at infinity

$$\theta(\pm\infty) = \theta^{\pm},\, \theta^+ < \theta^-, \quad \omega(\pm\infty) = 0, \quad \psi(\pm\infty) = 0.$$

All results remain valid for the monotone systems of equations characterized by the additional condition

$$\frac{\partial F_i}{\partial \theta_j} \geq 0, \; i \neq j$$

describing chemical waves with complex kinetics in the case of equality of transport coefficients [29], [30], [31].

If $\psi = \omega = 0$, we obtain, up to notations, the problem (67) for the unknown θ. We assume that $F'(\theta^{\pm}) < 0$, so that Condition 4 of Section 4 is satisfied, and we also assume that there exists a monotonically decreasing solution $\Theta(x_2)$. We linearize (27)-(210) about the solution $\theta = \Theta(x_2), \psi = \omega = 0$ and obtain the following eigenvalue problem (which is identical to (211)-(214)):

$$D\Delta\theta + c\,\frac{\partial \theta}{\partial x_2} + F'(\Theta(x_2))\theta + \Theta'(x_2)\,\frac{\partial \psi}{\partial x_1} = \lambda\theta \tag{71}$$

$$P\Delta\omega + c\,\frac{\partial \omega}{\partial x_2} + PR\frac{\partial \theta}{\partial x_1} = \lambda\omega \tag{72}$$

$$\Delta\psi + \omega = \lambda\psi \tag{73}$$

$$x_1 = 0, l : \frac{\partial \theta}{\partial x_1} = 0, \; \psi = 0, \; \omega = 0. \tag{74}$$

Without loss of generality we assume that $l = \pi$. We look for the solution of this problem in the form

$$\theta(x) = \tilde{\theta}(x_2)\cos kx_1, \; \psi(x) = \tilde{\psi}(x_2)\sin kx_1, \; \omega(x) = \tilde{\omega}(x_2)\sin kx_1, \tag{75}$$

where k is an integer. We obtain

$$D\tilde{\theta}^{''} + c\,\tilde{\theta}' + (F'(\Theta(x_2)) - Dk^2)\tilde{\theta} + k\Theta'(x_2)\tilde{\psi} = \lambda\tilde{\theta} \tag{76}$$

$$P\tilde{\omega}^{''} + c\,\tilde{\omega}' - Pk^2\tilde{\omega} - PRk\tilde{\theta} = \lambda\tilde{\omega} \tag{77}$$

$$\tilde{\psi}^{''} - k^2\tilde{\psi} + \tilde{\omega} = \lambda\tilde{\psi}, \tag{78}$$

which is the same system as (215). Since $\Theta'(x_2) < 0$, we take $k \leq 0$. Then we obtain the system

$$au^{''} + bu' + c(x_2)u = \lambda u,$$

where $u = (\tilde{\theta}, \tilde{\omega}, \tilde{\psi})$, a is a constant diagonal matrix with positive diagonal elements $a_1 = D, a_2 = P, a_3 = 1$, b is a constant diagonal matrix with the diagonal elements $b_1 = b_2 = c, b_3 = 0$, and $c(x_2)$ is a matrix with nonnegative off-diagonal elements

$$c_{13} = k\Theta'(x_2), \;\; c_{21} = -PRk, \;\; c_{32} = 1.$$

Hence (76)-(78) is a so-called monotone system of equations, and its principal eigenvalue satisfies properties which we will describe below [34].

For $k = 0$ the eigenvalues are independent of R, and they have negative real part, except the eigenvalue $\lambda = 0$. Consider now a negative integer k. The continuous spectrum of the problem (76)-(78) lies in the left half-plane. More precisely all points $\sigma \in \mathbb{C}$ of the continuous spectrum satisfy

$$Re\ \sigma \leq \min(F'(\Theta^{\pm}) - Dk^2, -Pk^2, -k^2) = m < 0.$$

Suppose that the real part of the principal eigenvalue $\lambda_0(R,k)$ is greater than the upper bound m of the continuous spectrum. Then it is real, simple, the corresponding eigenfunction is positive, and the following minimax representation holds [34]:

$$\begin{aligned}\lambda_0(R,k) &= \sup_{\rho} \inf_{i,x_2} \frac{a_i\rho_i^{''} + b_i\rho_i' + \sum_{j=1}^{3} c_{ij}(x_2)\rho_j}{\rho_i} \\ &= \inf_{\rho} \sup_{i,x_2} \frac{a_i\rho_i^{''} + b_i\rho_i' + \sum_{j=1}^{3} c_{ij}(x_2)\rho_j}{\rho_i}.\end{aligned}$$

Here $\rho = (\rho_1, \rho_2, \rho_3)$ is an arbitrary vector-function with positive components bounded and continuous together with their second derivatives.

The principal eigenvalue λ_0 is an increasing function of the parameter R. There exists a critical value $R = R_c(k)$ such that

$$\lambda_0(R,k) < 0,\ R < R_c(k),\ \ \lambda_0(R,k) > 0,\ R > R_c(k).$$

It can be shown by estimating the principal eigenvalue by the principal eigenvalue of a problem with constant coefficients considered on a bounded interval.

We call a critical value $R = R_c(k_0)$ *simple* if $R_c(k_0) \neq R_c(k)$ for any k different from k_0.

Lemma 7.1. *(i) Denote by V_k the set of eigenvalues of the problem (76)-(78) for each integer $k < 0$ fixed and by V_0 the set of nonzero eigenvalues of (76)-(78) for $k = 0$. Put*

$$V = \bigcup_{k \leq 0} V_k.$$

Then $V \cup \{0\}$ is the set of eigenvalues of the problem (71)-(74).

(ii) Let $\lambda_0(R)$ (respectively $\lambda_0(R,k)$) be the eigenvalue with the maximal real part in V (respectively V_k, for all integer $k \leq 0$). Then $\lambda_0(R) = \max_{k<0} \lambda_0(R,k)$, it is real and the corresponding eigenfunction is positive.

Remark. In V_0 we do not consider the simple zero eigenvalue arising for $k = 0$, because we have shown that it is removed by the functionalization of the parameter. Here we are interested only in the eigenvalue with maximal real part among the other eigenvalues. Note however that for $k < 0$, zero can be an eigenvalue of (76)-(78), and in this case it must be taken into account.

Proof. Consider a nonzero eigenvalue λ of the problem (71)-(74), and the corresponding eigenfunction (θ,ω,ψ). We multiply the first equation by $\cos kx_1$, two other equations by $\sin kx_1$ and integrate with respect to x_1. The functions

$$\tilde{\theta}_k(x_2) = \int_0^\pi \theta(x)\cos kx_1 dx_1,\ \tilde{\omega}_k(x_2) = \int_0^\pi \omega(x)\sin kx_1 dx_1,$$
$$\tilde{\psi}_k(x_2) = \int_0^\pi \psi(x)\sin kx_1 dx_1 \quad (79)$$

satisfy the problem (76)-(78) and there exists at least one value of k for which one of the functions $\tilde{\theta}_k,\tilde{\omega}_k,\tilde{\psi}_k$ is not identically zero. Indeed, for each x_2 fixed we can consider these functions as coefficients of the Fourier series. If for all x_2 and all k these coefficients equal zero, then the functions $\theta(x),\omega(x),\psi(x)$ indentically equal zero. Thus $\lambda \in V_k$.

Conversely if $\lambda \in V_k, k < 0$ and $(\tilde{\theta},\tilde{\omega},\tilde{\psi})$ is a corresponding eigenfunction of (76)-(78) then (75) is an eigenfunction of the problem (71)-(74) for the eigenvalue λ. This is also true for $\lambda \in V_0$ because the corresponding eigenfunction has the form $(\theta,0,0)$ where $\theta \not\equiv 0$. (i) is proved. (ii) follows directly from (i) and from the properties of monotone problems [34]. ■

Consider $R_c(k)$ for all negative k. Let $R_c(k_0)$ be the minimal value. We will show below that this minimum exists. Then the principal eigenvalue of the problem (71)-(74) equals zero at $R = R_c(k_0)$ and becomes positive for $R > R_c(k_0)$.

We note that the sign of the principal eigenvalue of the problem (71)-(74) does not allow us to make a direct conclusion about the stability of solutions of the problem (27)-(210) with respect to the problem

$$\frac{\partial\theta}{\partial t} = D\Delta\theta - \frac{\partial\psi}{\partial x_2}\frac{\partial\theta}{\partial x_1} + \left(c + \frac{\partial\psi}{\partial x_1}\right)\frac{\partial\theta}{\partial x_2} + F(\theta) \quad (710)$$
$$\frac{\partial\omega}{\partial t} = P\Delta\omega - \frac{\partial\psi}{\partial x_2}\frac{\partial\omega}{\partial x_1} + \left(c + \frac{\partial\psi}{\partial x_1}\right)\frac{\partial\omega}{\partial x_2} + PR\frac{\partial\theta}{\partial x_1} \quad (711)$$
$$0 = \Delta\psi + \omega \quad (712)$$

$$x_1 = 0, l : \frac{\partial\theta}{\partial x_1} = 0,\ \psi = 0,\ \omega = 0, \quad (713)$$

which we obtain introducing the stream function and the vorticity in the parabolic problem. The corresponding eigenvalue problem is

$$D\Delta\theta + c\,\frac{\partial\theta}{\partial x_2} + F'(\Theta(x_2))\theta + \Theta'(x_2)\,\frac{\partial\psi}{\partial x_1} = \lambda\theta \quad (714)$$
$$P\Delta\omega + c\,\frac{\partial\omega}{\partial x_2} + PR\frac{\partial\theta}{\partial x_1} = \lambda\omega \quad (715)$$
$$\Delta\psi + \omega = 0 \quad (716)$$

$$x_1 = 0, l : \frac{\partial \theta}{\partial x_1} = 0,\ \psi = 0,\ \omega = 0. \tag{717}$$

This system is the same as (216)-(219). The difference with (71)-(74) is in the equation (716) where there is no $\lambda\psi$ in the right-hand side. In the following we call a complex number λ eigenvalue of (714)-(717) if the problem (714)-(717) has a nonzero solution. As above, these eigenvalues can be found as eigenvalues of the problem

$$D\tilde{\theta}'' + c\,\tilde{\theta}' + (F'(\Theta(x_2)) - Dk^2)\tilde{\theta} + k\Theta'(x_2)\tilde{\psi} = \lambda\tilde{\theta} \tag{718}$$

$$P\tilde{\omega}'' + c\,\tilde{\omega}' - Pk^2\tilde{\omega} - PRk\tilde{\theta} = \lambda\tilde{\omega} \tag{719}$$

$$\tilde{\psi}'' - k^2\tilde{\psi} + \tilde{\omega} = 0. \tag{720}$$

We will analyze the principal eigenvalues of the problems (714)-(717) and (718)-(720). Before that we prove the following lemma:

Lemma 7.2. *There is a positive, continuous, and monotonically increasing function $M(R) > 0$ independent of the integer $k < 0$ such that for Re $\lambda > M(R)$ the problems (718)-(720) have no nonzero solution.*

Proof. Consider $\lambda \in \mathbb{C}$ with a positive real part such that the problem (718)-(720) has a nonzero complex solution (θ, ω, ψ). (For simplicity we omit the tildas). Define

$$M_1 = \sup_{x_2 \in R} |F'(\Theta(x_2))|, \quad M_2 = \sup_{x_2 \in R} |\Theta'(x_2)|.$$

We multiply (718) by $\bar{\theta}$, (719) by $\bar{\omega}$ and (720) by $\bar{\psi}$ and we integrate with respect to $x_2 \in \mathbb{R}$. Here the bars denote the complex conjugate functions. Using the limits at infinity

$$\theta(x),\ \omega(x),\ \psi(x),\ \theta'(x),\ \omega'(x),\ \psi'(x) \longrightarrow 0, \quad \text{as } x_2 \to \pm\infty$$

and taking the real part of the sum of the integrated equations we obtain

$$\begin{aligned}\int F'(\Theta)|\theta|^2 + k\int Re\ (\Theta'\psi\bar{\theta}) - PRk\int Re(\theta\bar{\omega}) + \int Re(\omega\bar{\psi}) \\ = (Re\ \lambda + Dk^2)\int |\theta|^2 + D\int |\theta'|^2 \\ + (Re\ \lambda + Pk^2)\int |\omega|^2 + P\int |\omega'|^2 + k^2\int |\psi|^2 + \int |\psi|'^2.\end{aligned}$$

Therefore

$$\begin{aligned}\int F'(\Theta)|\theta|^2 + k\int Re\ (\Theta'\psi\bar{\theta}) - PRk\int Re(\theta\bar{\omega}) + \int Re(\omega\bar{\psi}) \\ \geq (Re\ \lambda + Pk^2)\int |\omega|^2 + (Re\ \lambda + Dk^2)\int |\theta|^2 + k^2\int |\psi|^2.\end{aligned}$$

For any $\epsilon > 0$ the integrals from the left-hand side can be estimated by

$$\left| \int F'(\Theta)|\theta|^2 \right| \le M_1 \int |\theta|^2,$$

$$\left| \int Re(\Theta' \psi \bar{\theta}) \right| \le M_2 \int |\psi \bar{\theta}| \le M_2 \left(\epsilon \int |\psi|^2 + \frac{1}{4\epsilon} \int |\theta|^2 \right),$$

$$\left| \int Re\ (\omega \bar{\theta}) \right| \le \epsilon \int |\omega|^2 + \frac{1}{4\epsilon} \int |\theta|^2, \quad \left| \int Re\ (\psi \bar{\omega}) \right| \le \epsilon \int |\psi|^2 + \frac{1}{4\epsilon} \int |\omega|^2.$$

Hence

$$(k^2 - \epsilon\ (1 - kM_2)) \int |\psi|^2 \le \left(M_1 - \frac{kM_2}{4\epsilon} - \frac{PRk}{4\epsilon} - Dk^2 - Re\ \lambda \right) \int |\theta|^2 + \left(\frac{1}{4\epsilon} - PRk\epsilon - Pk^2 - Re\ \lambda \right) \int |\omega|^2.$$

Choose ϵ such that $\epsilon(1 - kM_2) < k^2$. It can be chosen independently of $k < 0$. Since (θ, ω, ψ) is nonzero and $k < 0$ we obtain that

$$Re\ \lambda \le M(R,k) := \max \left(\frac{1}{4\epsilon} - PRk\epsilon - Pk^2, M_1 - \frac{kM_2}{4\epsilon} - \frac{PRk}{4\epsilon} - Dk^2 \right).$$

Both expressions are bounded from above with respect to $k < 0$, $k \in \mathbb{Z}$. The lemma is proved. ■

Let $\lambda_1(R,k)$ be the eigenvalue with the maximal real part of the problem (718)-(720) and $\lambda_1(R)$ be an eigenvalue with the maximal real part among the $\lambda_1(R,k)$. By virtue of the preceding lemma they are well-defined, but we do not know if they are real.

Lemma 7.3. *If $\lambda_0(R,k) < 0$, then the problem (718)-(720) does not have nonzero solutions for any real nonnegative λ.*

Proof. Suppose that the problem (718)-(720) has a nonzero solution for a nonnegative λ. Then $\sigma = 0$ is an eigenvalue of the monotone problem

$$\begin{aligned} D\tilde{\theta}'' + c\,\tilde{\theta}' + (F'(\Theta(x_2)) - Dk^2)\tilde{\theta} + k\Theta'(x_2)\tilde{\psi} - \lambda\tilde{\theta} &= \sigma\tilde{\theta} \\ P\tilde{\omega}'' + c\,\tilde{\omega}' - Pk^2\tilde{\omega} - PRk\tilde{\theta} - \lambda\tilde{\omega} &= \sigma\tilde{\omega} \\ \tilde{\psi}'' - k^2\tilde{\psi} + \tilde{\omega} &= \sigma\tilde{\psi} \end{aligned}$$

and its principal eigenvalue $\sigma_0(\lambda)$ is nonnegative. Since $\lambda \ge 0$ the continuous spectrum of this problem lies in the left half-plane, and the results about monotone systems can be applied. Therefore $\sigma_0(0) \ge \sigma_0(\lambda) \ge 0$. On the other hand, $\sigma_0(0) = \lambda_0(R,k) < 0$. This contradiction proves the lemma. ■

It is shown in [32] that if the problem (718)-(720) has a positive solution u for some λ, then λ is the principal eigenvalue of this problem. This result will be used repeatedly in the proofs of Lemmas 7.4, 7.5, and 8.7.

Lemma 7.4. *If $\lambda_0(R,k)=0$, then $\lambda_1(R,k)=0$.*

Proof. The problems (76)-(78) and (718)-(720) coincide if $\lambda=0$. The solution of the problem (76)-(78) is positive. Hence there exists a positive solution of (718)-(720) with $\lambda=0$. Therefore all other eigenvalues of the problem (718)-(720) lie in the left half-plane [32]. The lemma is proved. ■

Lemma 7.5. *The following assertions hold:*
(i) If $\lambda_0(R)=0$, then $\lambda_1(R)=0$;
(ii) if $\lambda_0(R)>0$, then $\lambda_1(R)>0$;

Proof. First we prove assertion (i). Suppose that $\lambda_0(R)=0$. Then according to Lemma 7.1, for all $k<0$, $\lambda_0(R,k)\leq 0$ and $\lambda_0(R,k_0)=0$ for some k_0. By virtue of Lemma 7.4, $\lambda_1(R,k_0)=0$, hence $\lambda_1(R)$ is nonnegative. Besides there exists k_1 such that $\lambda_1(R)=\lambda_1(R,k_1)\geq 0$. By virtue of Lemma 7.3, $\lambda_0(R,k_1)$ cannot be negative, so $\lambda_0(R,k_1)=0$. Applying Lemma 7.4 once more we obtain that $\lambda_1(R,k_1)=\lambda_1(R)=0$.

Now we prove (ii). Consider $k<0$ such that $\lambda_0(R,k)>0$. We put $R^*=R_c(k)$. Then $\lambda_0(R^*,k)=0$, hence $\lambda_1(R^*,k)=0$. We want to prove that $\lambda_1(R,k)>0$.

Let $\mu\in\mathbb{R}$ and consider the problem

$$\begin{aligned} D\theta'' + c\,\theta' + (F'(\Theta(y)) - Dk^2)\theta + k\Theta'(y)\psi &= \lambda\theta \\ P\omega'' + c\,\omega' - Pk^2\omega - PRk\theta &= \lambda\omega \\ \psi'' - k^2\psi + \omega + \mu\psi &= \lambda\psi. \end{aligned}$$

Denote by $\lambda_R(\mu)$ its principal eigenvalue. Then λ_R is an increasing and continuous function of R and μ, and we have $\lambda_{R^*}(0)=0$.

We prove that for any $\mu>0$, $\lambda_{R^*}(\mu)<\mu$. Suppose that it is not true. Then for some $\mu>0$ we have $\lambda_{R^*}(\mu)\geq\mu$. By virtue of the definition of $\lambda_{R^*}(\mu)$ there exists a nonzero solution of the problem

$$\begin{aligned} D\theta'' + c\,\theta' + (F'(\Theta(x_2)) - Dk^2)\theta + k\Theta'(x_2)\psi - \lambda_{R^*}(\mu)\theta &= 0 \\ P\omega'' + c\,\omega' - Pk^2\omega - PR^*k\theta - \lambda_{R^*}(\mu)\omega &= 0 \\ \psi'' - k^2\psi + \omega + (\mu - \lambda_{R^*}(\mu))\psi &= 0. \end{aligned}$$

But zero is the principal eigenvalue of the problem

$$\begin{aligned} D\theta'' + c\,\theta' + (F'(\Theta(x_2)) - Dk^2)\theta + k\Theta'(x_2)\psi &= 0 \\ P\omega'' + c\,\omega' - Pk^2\omega - PR^*k\theta &= 0 \\ \psi'' - k^2\psi + \omega &= 0. \end{aligned}$$

This contradicts the fact that the principal eigenvalue is an increasing function of the coefficients. We note that this result is applicable here, because for every $\mu \geq 0$, the condition that $\lambda_R(\mu) > 0$ ensures that the continuous spectrum lies in the left half-plane.

Then for $R > R^*$, with $R - R^*$ sufficiently small, $\lambda_R(0) > 0$, $\lambda_R(\mu) < \mu$ for μ sufficiently large. Since $\lambda_R(\mu)$ is a continuous function of μ then the equation $\lambda_R(\mu) = \mu, \mu > 0$ has a solution. We show that this is true for any $R > R^*$. Suppose that it is not the case and put

$$R_0 = \inf\{R > R^*, \forall \mu > 0, \lambda_R(\mu) \neq \mu\}.$$

If $R < R_0$ there is $\mu > 0$ such that $\lambda_R(\mu) = \mu$. Then μ is an eigenvalue of (718)-(720). Hence $\mu \leq M(R) \leq M(R_0)$, where $M(R)$ is like in Lemma 7.2. Thus for $\mu > M(R_0)$ and $R < R_0$ we have $\lambda_R(\mu) \neq \mu$. As well as previously we can show that $\lambda_R(\mu) < \mu$ (see Figure 2).

We fix $\mu_0 > M(R_0)$ and we pass to the limit in the previous inequality as R tends to R_0. We obtain that $\lambda_{R_0}(\mu_0) \leq \mu_0$. Besides there is a decreasing sequence (R_n) with the limit R_0 such that the equations $\lambda_{R_n}(\mu) = \mu$ have no positive solution. Since $\lambda_{R_n}(0) > \lambda_{R^*}(0) = 0$ we have $\lambda_{R_n}(\mu) > \mu$ for any positive μ. Passing to the limit as $n \to +\infty$ we obtain $\lambda_{R_0}(\mu) \geq \mu$. Hence $\lambda_{R_0}(\mu_0) = \mu_0$, and μ_0 is an eigenvalue of (718)-(720). But this contradicts the fact that $\mu_0 > M(R_0)$. We have shown that for any $R > R^*$ the equation $\lambda_R(\mu) = \mu$ has a positive solution μ. This solution is an eigenvalue of the problem (718)-(720) and the associated eigenfunction is positive. Indeed it corresponds to the principal eigenvalue of the problem

$$\begin{aligned} D\theta'' + c\,\theta' + (F'(\Theta(x_2)) - Dk^2)\theta + k\Theta'(x_2)\psi &= \lambda_R(\mu)\theta \\ P\omega'' + c\,\omega' - Pk^2\omega - PRk\theta &= \lambda_R(\mu)\omega \\ \psi'' - k^2\psi + \omega + \mu\psi &= \lambda_R(\mu)\psi. \end{aligned}$$

Therefore μ is the principal eigenvalue of the problem (718)-(720).

We have proved that $\lambda_1(R,k) > 0$. Hence $\lambda_1(R) \geq \lambda_1(R,k) > 0$, and (ii) is proved. ■

Lemma 7.6. $\lambda_1(R) = 0$ *if and only if* $\lambda_0(R) = 0$, *and* $\lambda_1(R) > 0$ *if and only if* $\lambda_0(R) > 0$.

Proof. It is clear that if $\lambda_1(R) = 0$ then $\lambda_0(R) \geq 0$. Indeed if $\lambda_1(R,k_1) = 0$ then the problem (718)-(720) has a solution for $k = k_1$ and $\lambda = 0$. But this problem coincides with (76)-(78). Hence

$$\lambda_0(R) \geq \lambda_0(R,k_1) \geq 0.$$

Together with Lemma 7.5 it proves the first equivalence. It remains to show that if $\lambda_1(R) > 0$ then $\lambda_0(R) > 0$, which is a consequence of Lemmas 7.3 and 7.5. The lemma is proved. ■

Theorem 7.7.

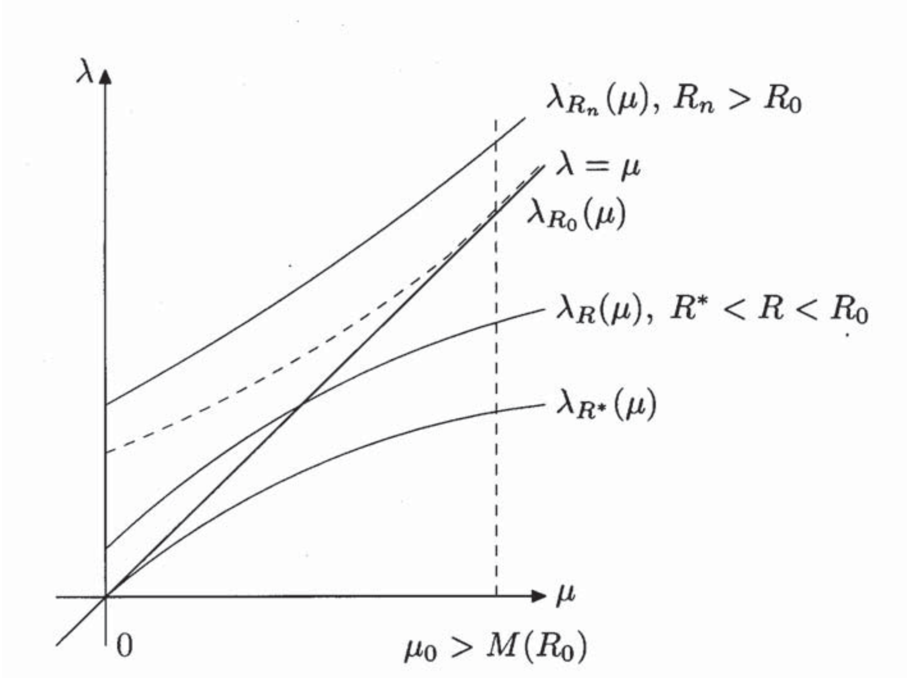

Figure 2: Graphical resolution of $\lambda_R(\mu) = \mu$

1. *For each negative integer k there exists a critical value $R = R_c(k)$ such that the problem (71)-(74) has a zero eigenvalue.*
2. *There exists k_0 such that $R_c(k_0) \leq R_c(k)$, $\forall k \in (-\mathbb{N}^*)$.*
3. *If the inequality is strict for all $k \neq k_0$, then $R_c(k_0)$ is a bifurcation point, i.e., in each its neighbourhood there exist values of R such that the problem (27)-(210) has solutions (θ, ω, ψ) in a neighbourhood of the one-dimensional wave, with $(\omega, \psi) \not\equiv 0$.*

Proof. The first assertion follows directly from the above lemmas.

In order to prove the second assertion, we first note that for all $k < 0$, $R_c(k) \geq 0$. Indeed for $R = 0$ the problem (76)-(78) does not have nonnegative eigenvalues. Now we show that $R_c(k) \to +\infty$ as $k \to -\infty$. It is equivalent to prove that for fixed R, and for $|k|$ sufficiently large, $\lambda_0(R, k)$ is negative. The estimate of Lemma 7.2 proves that $Re\ \lambda_1(R, k)$ is negative. But we have just shown that if $\lambda_0(R, k)$ were nonnegative, so would be $\lambda_1(R, k)$.

It remains to prove the bifurcation result. When R passes through the value $R_c(k_0)$, an eigenvalue of the problem (71)-(74) passes through zero. It is simple with respect to the problem (76)-(78). We show first of all that it is also simple as eigenvalue of the problem (71)-(74).

Suppose that there exists a nonzero solution $\theta(x), \omega(x), \psi(x)$ of the problem (71)-(74). $R_c(k_0)$ is simple and is the minimal value, hence for $k \neq k_0$ we have $R_c(k) > R_c(k_0)$, and $\lambda_0(R,k) < 0$. Thus $\tilde{\theta}_k$, $\tilde{\omega}_k$, and $\tilde{\psi}_k$ defined by (79) are identically zero.

If we suppose that there exists another eigenfunction corresponding to the same eigenvalue, then for some x_2 its Fourier expansion contains also terms with $k \neq k_0$. In this case the problem (76)-(78) would have a zero eigenvalue with a different value of k, which contradicts the fact that $\lambda_0(R,k) < 0$ for $k \neq k_0$.

Suppose finally that there exists $v \neq 0$, $m \geq 2$ such that $L_R^m v = 0$, $L_R^{m-1} v \neq 0$, where L_R is the operator acting from E_1 to E_2 corresponding to the linearization of the system (27)-(210). Put $w = L_R^{m-2} v$, then up to a nonzero factor $L_R w = u$. Multiplying the first components by $\cos k_0 x_1$, the second and third components by $\sin k_0 x_1$ and integrating with respect to x_1 we obtain a contradiction with the fact that zero is a simple eigenvalue of (76)-(78) for $k = k_0$. This contradiction proves that the zero eigenvalue of the problem (71)-(74) is simple. Hence the functionalization of the front velocity removes it and does not change the other eigenvalues.

Now by virtue of Condition 4, and of Theorems 4.4, 5.1 and 6.1, the topological degree can be defined, and we can use its properties. The index of the stationary point $\theta = \Theta, \omega = \psi = 0$ equals $(-1)^\nu$ where ν is the number of positive eigenvalues of the problem (71)-(74). As R passes through $R_c(k)$, ν changes by one. The index changes from 1 to -1 or vice versa. Since the topological degree is a homotopy invariant, and it is equal to the sum of indices of all stationary points, then at least two other solutions appear for the problem with functionalization. They are also solutions of (27)-(210) with adequate velocity c.

It remains to note that $(\omega, \psi) \not\equiv 0$ for the bifurcating solutions, because the solution with $\omega \equiv 0$, $\psi \equiv 0$ is unique. The theorem is proved. ■

Remarks.

1. The question of the linear stability of the one-dimensional wave in the neighbourhood of the bifurcation point $R_c(k_0)$ is not completely solved. Indeed we have not shown that the principal eigenvalue of the problem (714)-(717) is real. For $R > R_c(k_0)$ we know that this problem has a positive eigenvalue, hence the one-dimensional wave can be said to be unstable. But for $R < R_c(k_0)$ we only know that this problem has no real positive eigenvalues. However, it might have nonreal eigenvalues with positive real part, so an oscillatory instability may exist.

2. In Section 6 we have shown that the topological degree cannot be used directly for travelling waves, and we have introduced the functionalization of the parameter c to solve this difficulty. The results concerning the spectrum are valid with the functionalization of c, because all eigenvalues remain unchanged, except for the zero eigenvalue (see Lemma 6.2).

8. Further results about the spectrum

In the previous section we analyzed the spectrum of the operator linearized about the one-dimensional solution to find conditions for its stability. The stability of the convective solutions can be studied with the same arguments. Throughout this section we suppose that the conditions of Section 7 are satisfied, and that the minimum critical Rayleigh number $R_c(k_0)$ is simple.

For convenience we introduce the following notations:

- $R_c = R_c(k_0)$;
- A_R denotes the operator corresponding to the problem (27)-(29), acting from E_1 to E_2;
- $u_1 \in E_1$ denotes the one-dimensional wave, $A_R(u_1) = 0, \ \forall R$;
- $u_2 \in E_1$ denotes a two-dimensional wave in the neighbourhood of u_1, such that $A_R(u_2) = 0$ for some R in the neighbourhood of R_c. (For simplicity we omit the dependance of u_2 with respect to R);
- $L^1_R = A'_R(u_1)$, $L^2_R = A'_R(u_2)$;
- C is the matrix $C = diag(1, 1, 0)$,
- γ is the topological degree constructed in Section 6.

We will denote by $\lambda_0(L^i_R)$, $i = 1, 2$, the principal eigenvalue of $L^i_R u = \lambda u$, which we will call the classical eigenvalue problem, and $\lambda_C(L^i_R)$ the principal eigenvalue of $L^i_R u = \lambda C u$. We will call this last problem the C-eigenvalue problem for L^i_R.

The results about the spectrum of the operators L^2_R are based on two remarks: first, the homotopy invariance of the degree gives informations about the index of two-dimensional waves, and hence about the number of positive eigenvalues of the linearized operator. The second remark is that if the solutions u_1 and u_2 are close, the eigenvalues of the linearized operators are also close, in a sense that will be made clearer below. As well as in Section 7, the method of the topological degree makes it necessary to functionalize the velocity c. In the case of the one-dimensional wave we have shown that the functionalization does not change the eigenvalues of the linearized operator, except the zero eigenvalue which is replaced by

$$\left\langle c'(w_1), \frac{\partial w_1}{\partial x_n} \right\rangle < 0, \quad w_1 = u_1 + \phi.$$

The same is true for the two-dimensional wave, except that the new eigenvalue

$$\left\langle c'(w_2), \frac{\partial w_2}{\partial x_n} \right\rangle$$

can be zero if the wave w_2 is not monotone with respect to x_n. However it is clear that for $\|u_1 - u_2\|_{C^{2+\alpha}}$ sufficiently small, this eigenvalue is negative. Hence to analyze existence of eigenvalues with nonnegative real part it is sufficient to study the spectrum of the operators without functionalization. This is what we do in the following.

First we recall the result that was proved in Section 7 regarding local bifurcations:

Lemma 8.1. *For any $\eta > 0$ there is $R > 0$, $|R - R_c| < \eta$ such that the equation $A_R(u) = 0$ has a solution $u_2 \neq u_1$, with $\|u_1 - u_2\|_{C^{2+\alpha}} < \eta$.*

8.1. The classical eigenvalue problem

To study the stability of the bifurcating solutions we have to compare the eigenvalues of $L^1_{R_c}$ and of L^2_R. This comparison is based on the properties of Fredholm operators [13]. In the following we consider the operators L^1_R and L^2_R as acting from C^α to C^α, with domain E_1. Throughout this subsection we use the notation $\|.\|$ instead of $\|.\|_{C^\alpha}$. Note that these operators are now unbounded. Hence to compare their eigenvalues we have to establish the following estimate:

Lemma 8.2. *For any $k > 0$ there exists $\eta > 0$ such that, for any R, $|R - R_c| < \eta$, for any $u_2 \in E_1$, $\|u_1 - u_2\|_{C^{2+\alpha}} < \eta$, and for any $u \in E_1$, $u \neq 0$,*

$$\|L^1_{R_c} u - L^2_R u\| < k\left(\|u\| + \|L^1_{R_c} u\|\right).$$

Proof. The operator in the left-hand side has the form

$$L^1_{R_c} u - L^2_R u = b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + cu,$$

where b_1, b_2 and c are matrices such that

$$\|b_i\| \leq \|\nabla u_1 - \nabla u_2\| + P|R - R_c|,\ i = 1, 2, \quad \|c\| \leq \|F'(u_1) - F'(u_2)\|.$$

Since $F \in C^2$ there is a constant $K > 0$ such that if $\|u_1 - u_2\|_{C^{2+\alpha}} < \eta$, then $\|F'(u_1) - F'(u_2)\| < K\eta$. If we also have $|R - R_c| < \eta$ then

$$\|L^1_{R_c} u - L^2_R u\| < K\eta(\|u\| + \|\nabla u\|).$$

By virtue of the Schauder estimate,

$$\|\nabla u\| \leq \|u\|_{C^{2+\alpha}} \leq M(\|L^1_{R_c} u\| + \|u\|).$$

The lemma is proved. ∎

In the following lemma we say that a complex number λ is a regular point for an operator L if $L - \lambda I$ is continuously invertible, i.e. there exists a bounded operator R_λ, defined on all of C^α, such that

$$R_\lambda(L - \lambda I) = (L - \lambda I)R_\lambda = I.$$

Lemma 8.3. *Consider $\epsilon > 0$ such that each complex number λ, with $0 < |\lambda| \leq \epsilon$, is a regular point for $L^1_{R_c}$. There exists $\eta > 0$ such that for $|R - R_c| < \eta$, $\|u_1 - u_2\|_{C^{2+\alpha}} < \eta$, the operator L^2_R has exactly one simple eigenvalue $\lambda_0 \in B(0, \epsilon)$, and all other points of $\bar{\mathcal{D}} \cup \bar{B}(0, \epsilon)$ are regular for L^2_R, where*

$$\mathcal{D} = \{\lambda \in \mathbb{C}, Re\lambda > 0, |\lambda| > \epsilon\}.$$

Proof. The proof consists of three steps:

1. L^2_R has a simple eigenvalue in the open ball, and all other points of the closed ball are regular;
2. L^2_R has only regular points outside of an angle $C_{M,\alpha}$;
3. L^2_R has only regular points in $\bar{C}_{M,\alpha} \cap \bar{\mathcal{D}}$.

Step 1. It is based on a remark of [13]: the sum of multiplicities of eigenvalues inside a given contour is preserved under a small change of the operator. More precisely, there exists a positive constant k_1 such that for any operator $A : C^\alpha \to C^\alpha$ satisfying the estimate

$$\|Au\| < k_1(\|u\| + \|L^1_{R_c} u\|), \; u \in E_1, u \neq 0,$$

$L^1_{R_c} + A$ has a single simple eigenvalue in the ball $B(0, \epsilon)$, and all other points of the closed ball are regular. Lemma 8.2 completes this first step.

Step 2. For any $\alpha \in (0, \pi/2)$ we can find M depending only on Ω, α, on the principal coefficients of L^2_R and on the boundary conditions such that all points outside of the angle $C_{M,\alpha}$ are regular. We note that M neither depends on R nor on u_2.

Step 3. It is identical to Step 1, but here we consider the contour delimiting the region $C_{M,\alpha} \cap \mathcal{D}$ (see Figure 3), where $L^1_{R_c}$ only has regular points. The lemma is proved. ■

It is easy to obtain the following corollary:

Corollary 8.4. *If ϵ, η, u_2 and R are like in Lemma 8.3, then L^2_R has at most one eigenvalue with nonnegative real part. Moreover this eigenvalue is real.*

Indeed, if it were nonreal, then its conjugate would also be an eigenvalue, which contradicts the uniqueness. ■

With this corollary it is easy to prove the following theorem:

Theorem 8.5. *If convective solutions exist only for $R > R_c$ (supercritical bifurcation), then among them there are solutions u_2 for which $\lambda_0(L^2_R) \leq 0$, for R sufficiently close to R_c (see fig. 1b).*

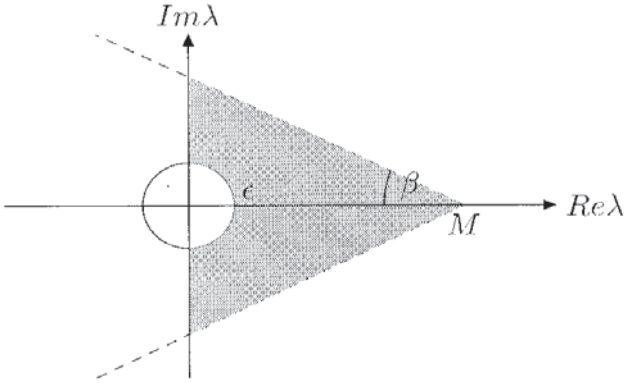

Figure 3: The contour in Lemma 8.3, Step 3.

If convective solutions exist only for $R < R_c$ (subcritical bifurcation), then among them there are solutions for which $\lambda_0(L^2_R) \geq 0$, for R sufficiently close to R_c (see fig. 1a).

Proof. Consider the first case. The proof of the second case is similar. Suppose that there is no convective solution for $R \leq R_c$. Let ϵ and η be like in Lemma 8.3, and let V be the ball of center u_1 and of radius η in E_1. Then A_{R_c} has no zero on the boundary of V. Since this operator is proper and continuous, we can choose $R_0 \in (R_c - \eta, R_c)$, $R_1 \in (R_c, R_c + \eta)$ such that

$$\forall R \in [R_0, R_1],\ \forall u \in \partial V,\ \|A_R(u)\| \neq 0.$$

Indeed suppose the countrary. Then for all sufficiently large integer n,

$$\exists R_n \in \left[R_c - \frac{1}{n}, R_c + \frac{1}{n}\right], \exists u_n \in \partial V, A_{R_n}(u_n) = 0.$$

The sequence (R_n, u_n) lies in the compact set $A^{-1}(0) \cap ([R_0, R_1] \times \partial V)$. Hence we can choose a converging subsequence, with the limit (R_c, u), $u \in \partial V$. By virtue of the continuity of A with respect to R and u, we find that $A_{R_c}(u) = 0$, $u \in \partial V$, which contradicts the hypothesis.

Then we can apply the homotopy invariance of the degree on V, and we have: $\gamma(A_{R_0}, V) = \gamma(A_{R_1}, V)$. For $R = R_0$ there is only the one-dimensional solution, and its index equals 1. Hence $\gamma(A_{R_0}, V) = 1$.

On the other hand, suppose that all convective solutions u_2 are such that $\lambda_0(L^2_R) > 0$. Then by virtue of Corollary 8.4, the positive eigenvalue is simple, and all other eigenvalues of L^2_R have negative real part. Hence the index of u_2 equals -1, and $\gamma(A_{R_1}, V) \leq -1$. Note that the indices can be used because none of the possible solutions has a zero eigenvalue. This contradiction proves the theorem. ■

Remark. As mentioned in Section 2, this result is not exhaustive. Other situations are possible, for example the situation of fig. 1c, where

$$\gamma(A_{R_0}, V) = \gamma(A_{R_1}, V) = 0.$$

8.2. The C-eigenvalue problem

The question of the stability of a bifurcating solution u_2 with respect to the nonstationary problem (710)-(713) leads to the C-eigenvalue problem

$$L_R^2 u = \lambda C u. \tag{81}$$

As in Section 7 we introduce the auxiliary problem

$$L_R^2 u + \mu P u = \lambda u, \tag{82}$$

where $P = E_3 - C$ and μ is a real parameter. We recall that E_3 is the identity matrix. For $\mu = \lambda$ both problems are equivalent.

Lemma 8.6. *There exists some constant $\eta > 0$ such that for $|R - R_c| < \eta$, $|u_2 - u_1| < \eta$, if $\lambda_0(L_R^2) \geq 0$ then $\lambda_C(L_R^2) \geq 0$.*

Proof. Suppose that the problem $L_R^2 u = \lambda u$ has an eigenvalue with nonnegative real part. By virtue of Corollary 8.4 it is real. Besides it is also an eigenvalue of the problem (82) with $\mu = 0$. In other words, $\lambda_R^2(0) \geq 0$, where $\lambda_R^2(\mu)$ is the principal eigenvalue of the problem (82). If $\lambda_R^1(\mu)$ denotes the principal eigenvalue of the problem

$$L_R^1 u + \mu P u = \lambda u, \tag{83}$$

then for any nonnegative μ, $\lambda_{R_c}^1(\mu) = \lambda_{R_c}^2(\mu)$. Indeed for $R = R_c$, u_1 and u_2 coincide. $\lambda_R^2(\mu)$ is continuous with respect to μ and R. Hence

$$\lambda_R^2(\mu) \to \lambda_{R_c}^1(\mu), \quad R \to R_c. \tag{84}$$

We have seen in the proof of Lemma 7.5 that

$$\forall \mu > 0, \lambda_{R_c}^1(\mu) < \mu. \tag{85}$$

Fix $\mu > 0$. By virtue of (84) and (85) there exists $\eta > 0$ such that

$$|R - R_c| < \eta \Rightarrow \lambda_R^2(\mu) < \mu.$$

Hence for $|R - R_c| < \eta$ the equation $\lambda_R^2(\mu) = \mu$ has at least one solution $\mu \geq 0$. This solution is an eigenvalue of (81). The lemma is proved. ∎

The same argument can be used to prove that if $\lambda_0(L_R^2) < 0$, then the problem (81) has a negative eigenvalue (this result is established in Lemma 8.7). But of course this is not sufficient to prove that $\lambda_C(L^2_R) < 0$. We need to show that all eigenvalues of (81) have negative real part. This can be done by comparing (81) with the similar problem for $L^1_{R_c}$, as was done in Subsection 8.1. This comparison yields Lemma 8.8 which, together with Lemma 8.7, establishes the link between the classical eigenvalue problem and the C-eigenvalue problem for bifurcating solutions, if $|R-R_c|$ and $\|u_1 - u_2\|_{C^{2+\alpha}}$ are small.

Lemma 8.7. *For any $\epsilon > 0$, there exists some constant $\eta > 0$ such that for $|R-R_c| < \eta$, $\|u_2 - u_1\|_{C^{2+\alpha}} < \eta$, if $\lambda_0(L^2_R) < 0$ then the problem (81) has a real eigenvalue λ, $-\epsilon < \lambda < 0$.*

Proof. The proof is similar to the proof of Lemma 8.6. All we have to establish here is that there exists $\epsilon > 0$ such that

$$\forall \mu,\ -\epsilon < \mu < 0,\ \lambda^1_{R_c}(\mu) > \mu.$$

First we take ϵ such that the essential spectrum lies in the half-plane $\{Re\lambda < -\epsilon\}$. Hence the methods of Section 7 can be applied.

We know that all eigenvalues of (83) are also eigenvalues of

$$\begin{aligned} D\tilde{\theta}'' + c\,\tilde{\theta}' + (F'(\Theta(x_2)) - Dk^2)\tilde{\theta} + k\Theta'(x_2)\tilde{\psi} &= \lambda\tilde{\theta} \\ P\tilde{\omega}'' + c\,\tilde{\omega}' - Pk^2\tilde{\omega} - PRk\tilde{\theta} &= \lambda\tilde{\omega} \\ \tilde{\psi}'' - k^2\tilde{\psi} + \tilde{\omega} + \mu\tilde{\psi} &= \lambda\tilde{\psi} \end{aligned}$$

for some negative integer k. Call $\lambda^1_{R,k}(\mu)$ the principal eigenvalue of this problem. Then it is real and

$$\lambda^1_R(\mu) = \sup_{k\in\mathbb{Z},k<0} \lambda^1_{R,k}(\mu) \geq \lambda^1_{R_c,k_0}(\mu).$$

Hence it is sufficient to prove that

$$\forall \mu,\ -\epsilon < \mu < 0,\ \lambda^1_{R_c,k_0}(\mu) > \mu. \tag{86}$$

Before establishing (86), we remark that for any R and k, the equation $\lambda^1_{R,k}(\mu) = \mu$ cannot have more than one real solution μ. Indeed such a solution is the principal eigenvalue of the monotone problem

$$\begin{aligned} D\theta'' + c\theta' + (F'(\Theta(x_2)) - Dk^2)\theta + k\Theta'(x_2)\psi &= \lambda^1_{R,k}(\mu)\theta \\ P\omega'' + c\omega' - Pk^2\omega - PRk\theta &= \lambda^1_{R,k}(\mu)\omega \\ \psi'' - k^2\psi + \omega + \mu\psi &= \lambda^1_{R,k}(\mu)\psi. \end{aligned}$$

Hence the corresponding eigenfunction is positive. But μ is also an eigenvalue of the problem (718)-(720), with the same positive eigenfunction. Therefore μ must be the principal eigenvalue of this problem [32], and the uniqueness follows.

Now we prove (86). At first we know that $\lambda^1_{R_c,k_0}(0) = 0$. From the preceding observation we can deduce that

$$\forall \mu,\ -\epsilon < \mu < 0,\ \lambda^1_{R_c,k_0}(\mu) \neq \mu.$$

But $\lambda^1_{R,k}(\mu)$ is continuous with respect to μ. Hence if (86) does not hold, then

$$\forall \mu,\ -\epsilon < \mu < 0,\ \lambda^1_{R_c,k_0}(\mu) < \mu.$$

We use a result of strict monotonicity of the principal eigenvalue of a monotone problem with respect to the coefficients [35]. For $R > R_c$ and for any μ, $-\epsilon < \mu < 0$, $\lambda^1_{R,k_0}(\mu) > \lambda^1_{R_c,k_0}(\mu)$.

Take R sufficiently close to R_c in order that the following inequalities are satisfied, as in Figure 4:

$$\lambda^1_{R,k_0}(-\epsilon) < -\epsilon, \quad \lambda^1_{R,k_0}(1) < 1.$$

We also have $\lambda^1_{R,k_0}(0) > 0$. Hence the equation $\lambda^1_{R,k_0}(\mu) = \mu$ has at least two real solutions, which gives a contradiction. (86) is proved, and the end of the proof of Lemma 8.7 is identical to the proof of Lemma 8.6. ■

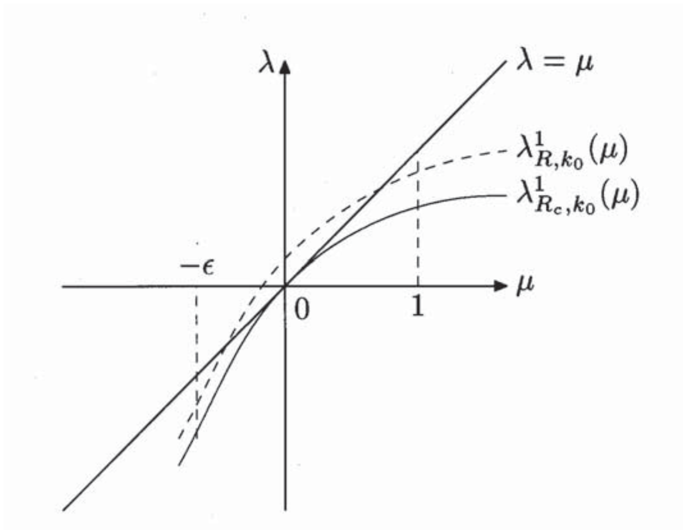

Figure 4: Graphical resolution of $\lambda^1_{R,k_0}(\mu) = \mu$

It remains to prove that the problem (81) has no eigenvalue with positive real part. We will show that there is $\epsilon > 0$ such that it has only one eigenvalue in $\bar{\mathcal{D}} \cup \bar{B}(0, \epsilon)$.

Lemma 8.8. *There exists $\epsilon > 0$ such that the problems (87), $i = 1, 2$, have exactly one simple eigenvalue in $B(0, \epsilon)$, and all other points of the closed ball are regular.*

Proof. The idea of the proof is the same as for Lemma 8.3. The difficulty here is that there is no λ for the third component. Therefore we cannot use directly the result that multiplicity of eigenvalues is preserved under a small perturbation of the operator.

We change the problem to a usual eigenvalue problem. We write $u_i = (\theta_i, \omega_i, \psi_i)$, $i = 1, 2$, $(\omega_1 = \psi_1 = 0)$ and we note that for $i = 1, 2$, the problem

$$L_R^i u = \lambda C u, \quad u \in E_1 \tag{87}$$

is equivalent to the problem

$$\begin{aligned} D\Delta\theta + b(\psi_i, \theta_i, \psi, \theta) + F'(\theta_i)\theta &= \lambda\theta \\ -P\Delta\Delta\psi + b(\psi_i, -\Delta\psi_i, \psi, -\Delta\psi) + PR\frac{\partial\theta}{\partial x_1} &= -\lambda\Delta\psi, \\ \frac{\partial\theta}{\partial x_1} = 0, \psi = \Delta\psi = 0, & \qquad x \in \partial\Omega, \end{aligned}$$

where

$$b(w_i, v_i, w, v) = -\frac{\partial w_i}{\partial x_2}\frac{\partial v}{\partial x_1} - \frac{\partial w}{\partial x_2}\frac{\partial v_i}{\partial x_1} + \left(c + \frac{\partial w_i}{\partial x_1}\right)\frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_1}\frac{\partial v_i}{\partial x_2}.$$

Consider the operator $\Delta^{-1} : C^\alpha \to \{u \in C^{2+\alpha}, u_{|\partial\Omega} = 0\}$, which associates to a given function $f \in C^\alpha$ the unique function $\psi \in C^{2+\alpha}, \psi_{|\partial\Omega} = 0, \Delta\psi = f$. It is well defined and continuous.

Define the spaces

$$\begin{aligned} F_1 &= \left\{\theta \in C^{2+\alpha}(\bar\Omega), \frac{\partial\theta}{\partial x_1} = 0, x \in \partial\Omega\right\}, \\ F_2 &= \{\psi \in C^{4+\alpha}(\bar\Omega), \psi = \Delta\psi = 0, x \in \partial\Omega\}, \\ F_3 &= C^\alpha(\bar\Omega), \\ F_4 &= \{g \in C^{2+\alpha}(\bar\Omega), g = 0, x \in \partial\Omega\}, \end{aligned}$$

and the operators $\mathcal{L}^1, \mathcal{L}^2 : F_1 \times F_2 \to F_3 \times F_4$ by $\mathcal{L}^i(\theta, \psi) = (f_i, g_i)$, where

$$\begin{aligned} f_i &= D\Delta\theta + b(\psi_i, \theta_i, \psi, \theta) + F'(\theta_i)\theta, \\ g_i &= -\Delta^{-1}\left(P\Delta\Delta\psi + b(\psi_i, -\Delta\psi_i, \psi, -\Delta\psi) + PR\frac{\partial\theta}{\partial x_1}\right), \quad i = 1, 2. \end{aligned}$$

With these notations, the problems (87) are equivalent to

$$\bar u = (\theta, \psi) \in F_1 \times F_2, \quad \mathcal{L}^i \bar u = \lambda \bar u.$$

It remains to show that

1. for any $\lambda \geq 0$, $\mathcal{L}^i - \lambda I$ are Fredholm with zero index, $i = 1, 2$;
2. if $|R - R_c| < \eta$, $\|u_1 - u_2\|_{C^{2+\alpha}} < \eta$, then

$$\forall \bar{u} \in F = F_1 \times F_2,\ \|\mathcal{L}^1 \bar{u} - \mathcal{L}^2 \bar{u}\|_E < k(\|\mathcal{L}^1 \bar{u}\|_E + \|\bar{u}\|_E),$$

where $E = F_3 \times F_4$.

The first point is clear, because Condition 2 for $\mathcal{L}^i$ is equivalent to the fact that the problem

$$L_R^{i\ \pm} u = \lambda C u$$

has no nontrivial solution for $\lambda \geq 0$. This is ensured by the assumption $F'(\theta^\pm) < 0$. As a consequence, we know that the eigenvalues of $\mathcal{L}^i$ are isolated. For $R = R_c$, $\mathcal{L}^1$ has a zero simple eigenvalue, thus there exists $\epsilon > 0$ such that all complex numbers λ with $0 < |\lambda| < \epsilon$ are regular for $\mathcal{L}^1$, hence they are also regular for the problem $L^1_{R_c} u = \lambda C u, u \in E_1$.

To prove the second point we use the Schauder estimates and the continuity of Δ^{-1} as operator acting from F_4 to F_2, and we first conclude that

$$\forall \bar{u} \in F,\ \|\bar{u}\|_F \leq K(\|\mathcal{L}^1 \bar{u}\|_E + \|\bar{u}\|_E). \tag{88}$$

Now for $\bar{u} = (\theta, \psi) \in F$,

$$(\mathcal{L}^2 - \mathcal{L}^1)\bar{u} = (b(\psi_2 - \psi_1, \theta_2 - \theta_1, \psi, \theta), b(\psi_2 - \psi_1, \Delta\psi_2 - \Delta\psi_1, \psi, \Delta\psi))$$

We can find $\eta > 0$ such that if $\|u_1 - u_2\|_{C^2 + \alpha} < \eta$, then

$$\|b(\psi_2 - \psi_1, \theta_2 - \theta_1, \psi, \theta)\|_{C^\alpha} \leq \epsilon$$

and

$$\|b(\psi_2 - \psi_1, \Delta\psi_2 - \Delta\psi_1, \psi, \Delta\psi)\|_{C^\alpha} \leq \epsilon.$$

Hence

$$\forall \bar{u} \in F_1 \times F_2,\ \|\mathcal{L}^1 \bar{u} - \mathcal{L}^2 \bar{u}\|_E \leq \epsilon K\left(\|\theta\|_{C^{1+\alpha}} + \|\psi\|_{C^{3+\alpha}}\right) \leq \epsilon K \|\bar{u}\|_F. \tag{89}$$

The conclusion follows from (88) and (89). The end of the proof is the same as the first step of Lemma 8.3. ■

Corollary 8.9. *There exists some constant $\eta > 0$ such that for $|R - R_c| < \eta$, $\|u_2 - u_1\|_{C^{2+\alpha}} < \eta$, if $\lambda_0(L_R^2) \leq 0$ and if $\lambda_C(L_R^2)$ is real then $\lambda_C(L_R^2) \leq 0$. If the problem $L_R^1 u = \lambda C u$ has eigenvalues with positive real part, then the sum of their multiplicities equals the sum of the multiplicities of eigenvalues with positive real part of the problem $L_R^2 u = \lambda C u$.*

Proof. Consider a solution u_2 with $\lambda_0(L_R^2) \leq 0$. From Lemma 8.7 we know that the problem (81) has a real eigenvalue $\lambda \in (-\epsilon, 0)$. By virtue of Lemma 8.8, it has no eigenvalue with positive real part in the ball $\bar{B}(0,\epsilon)$. It remains to show that the sum of multiplicities of all eigenvalues in the domain $\bar{\mathcal{D}}$ defined in Subsection 8.1 is the same as for the problem $L^1_R u = \lambda C u$. This follows exactly from the same arguments as in the proof of Lemma 8.3. ■

Hence we have proved the following theorem, which is the counterpart to Theorem 8.5.

Theorem 8.10. *If convective solutions exist only for $R > R_c$ (supercritical bifurcation), then among them there are solutions u_2 for which the problem (81) has no real positive eigenvalue, for R sufficiently close to R_c.*

If convective solutions exist only for $R < R_c$ (subcritical bifurcation), then among them there are solutions for which the problem (81) has a nonnegative eigenvalue, for R sufficiently close to R_c.

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