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ON THE ASYMPTOTIC BEHAVIOR FOR CONVECTION-DIFFUSION EQUATIONS ASSOCIATED TO HIGHER ORDER ELLIPTIC OPERATORS IN DIVERGENCE FORM

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Abstract

We consider the linear convection-diffusion equation associated to higher order elliptic operators

$$\begin{cases} u_t + \mathcal{L}_t u = a\nabla u & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(0) = u_0 \in L^1(\mathbb{R}^n), \end{cases} \tag{1}$$

where a is a constant vector in $\mathbb{R}^n$, $m \in \mathbb{N}^*$, $n \geq 1$ and $\mathcal{L}_0$ belongs to a class of higher order elliptic operators in divergence form associated to non-smooth bounded measurable coefficients on $\mathbb{R}^n$. The aim of this paper is to study the asymptotic behavior, in L^p $(1 \leq p \leq \infty)$, of the derivatives $D^\gamma u(t)$ of the solution of (1) when t tends to ∞.

1 Introduction

In this paper, we deal with the large time behavior of solutions of the convection-diffusion equation (1), where

$$\mathcal{L}_t = L_0^* A L_0$$

with $A(x,t) = A(x+at) \in L^\infty$ is positive and

$$L_0 = (-1)^m \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha+\beta}.$$

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($\mathcal{L}_0$ will correspond to $\mathcal{L}_t$ when $t = 0$).

The positive coefficients $a_{\alpha\beta}$ are assumed to be constants. More details on these operators with general coefficients can be found, for example, in [1, 3, 5, 6].

This problem is a generalization to the case of higher order elliptic operators of the problem studied by Escobedo and Zuazua [4] but for the case where $q = 1$ in their notations. Our goal is to obtain information on the distribution of the constant mass in space.

Before describing our results, we will briefly recall the known results on the heat equation. If w solves the linear heat equation $w_t - \Delta w = 0$, with initial data w_0, then $w = G(., t) * w_0(.)$, where G is the fundamental solution of the heat equation:

$$G(x, t) = (4\pi t)^{-n/2} exp(-|x|^2/4t).$$

It is then easy to see that when $\int_{\mathbb{R}^n} w_0 = M$, then

$$t^{n(1-1/p)/2} ||w(t) - MG(., t)||_p \to 0,$$

as $t \to \infty$ for every $1 \leq p \leq +\infty$; this means that the large time profile of solutions is given by the fundamental solution with the appropriate mass. For the proof of this result see, for instance, Escobedo and Zuazua [4].

Let us assume now that w solves the equation $w_t - \Delta w = a.\nabla w$ with $a \in \mathbb{R}^n$, then $\tilde{w} = w(x - at, t)$ solves the linear equation $\tilde{w} - \Delta\tilde{w} = 0$. The previous result applies to $\tilde{w}$ to lead to the asymptotic behavior of w,

$$t^{n(1-1/p)/2} ||w(t) - MG(. + at, t)||_p \to 0, \qquad \text{as } t \to \infty.$$

Note that the asymptotic behavior is that of the Gaussian MG but, in this case, the mass center moves with spead a as t increases.

In this paper, we prove similar results (see formula (4)) but for the class of higher order operators $\mathcal{L}_t$ with non-smooth coefficients. Regarding the elliptic operators of the type

$$\tilde{L}_t = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta} D^\beta),$$

where $A(x,t) = \big(a_{\alpha\beta}(x,t)\big) = A(x+at)$ ($\tilde{L}_0$ will correspond to $\tilde{L}_t$ when $t=0$), we can obtain the same results for the solution of the equation

$$\begin{cases} u_t + \tilde{L}_t u = a\nabla u \quad \text{on } \mathbb{R}^n \times (0,\infty) \\ u(0) = u_0 \in L^1(\mathbb{R}^n), \end{cases} \tag{2}$$

when the positive coefficients $a_{\alpha\beta}$ are bounded uniformly continuous or in $L^\infty \cap VMO$ (see section 5).

The paper is organized as follows. In section 2, we supply notations and introduce the class of elliptic operators. In sections 3 and 4 we respectively state and prove the main result. Finally, we conclude with a few remarks in section 5.

2 Preliminaries

2.1 Notations

The following notations will be used throughout this paper. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$,

$$D_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \ .$$

By $\nabla^m u$ and $|\nabla^m u|$, we denote respectively the vector $(D^\alpha u)_{|\alpha|=m}$ and its length

$$|\nabla^m u| = \Big(\sum_{|\alpha|=m} |D^\alpha u|^2 \Big)^{\frac{1}{2}} \ .$$

We shall use the classical definition for the Sobolev space $W^{m,p}$, $m \in \mathbb{Z}$ and $1 \le p \le \infty$. In particular, the notation H^m stands for $W^{m,2}$. Norms in L^p-spaces will be denoted by $\| \ \|_p$. We shall also use the weighted space

$$L^1(\mathbb{R}^n; 1+|x|) = \{ f \in L^1(\mathbb{R}^n), \int_{\mathbb{R}^n} |f(x)|(1+|x|)dx < \infty \}$$

with the norm

$$||f||_{L^1(\mathbb{R}^n;|x|)} = \int_{\mathbb{R}^n} |f(x)||x|dx.$$

Eventually, by $E(\mu)$ and $\star$ we denote respectively the entire part of $\mu \in \mathbb{R}$ and the symbol of convolution with respect to the space variable x.

Now, let us define the class of operators used here and mention some of their properties. Further details can be found in [1, 2, 3].

2.2 The class of elliptic operators

Let $m \in \mathbb{N}^*$. Let $a_{\alpha\beta}(x)$ be bounded measurable functions on $\mathbb{R}^n$ where α, β are multi-indices such that $| \alpha |=| \beta |= m$. Set

$$\mathcal{Q}(u,v) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta u(x) D^\alpha v(x) dx$$

for all $u, v \in H^m(\mathbb{R}^n)$. The form $\mathcal{Q}$ is continuous on $H^m(\mathbb{R}^n)$ and if one defines $M = \big\| \|(a_{\alpha\beta}(x))\| \big\|_\infty$, where $\|(a_{\alpha\beta}(x))\|$ is the norm of the matrix $(a_{\alpha\beta}(x))$, then

$$|\mathcal{Q}(u,v)| \ \leq M \ \|\nabla^m u\|_2 \ \|\nabla^m v\|_2$$

for all $u, v \in H^m(\mathbb{R}^n)$.

Under these assumptions, by a variation on the Lax-Milgram lemma, there exists a unique operator in divergence form $L : H^m(\mathbb{R}^n) \longrightarrow H^{-m}(\mathbb{R}^n)$, linear and continuous, such that for all $u, v \in H^m(\mathbb{R}^n)$,

$$\langle Lu, v \rangle \ = \ \mathcal{Q}(u,v) \ .$$

We write this operator as $(-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{\alpha\beta} D^\beta)$ and we say it is associated with the coefficients $a_{\alpha\beta}$. Note that $\langle \ , \rangle$ stands for the usual scalar product on L^2.

We suppose that the class of operators L is elliptic in the sense of the Gårding inequality: there exists a constant $\delta > 0$ such that for all $u \in H^m(\mathbb{R}^n)$,

$$\mathcal{Q}(u,u) \geq \delta \|\nabla^m u\|_2^2 \ . \tag{3}$$

Set

$$\mathcal{D}(L) = \{u \in H^m(\mathbb{R}^n) \ , \ Lu \in L^2(\mathbb{R}^n)\} \ .$$

As a consequence of (3), the operator L, restricted to $\mathcal{D}(L)$, is maximal accretive of type $\omega < \frac{\pi}{2}$ and $(-L)$ is the generator of a contraction semigroup e^{-tL} on $L^2(\mathbb{R}^n)$.

Remark 1. We can relax the ellipticity condition by replacing (3) by

$$\mathcal{Q}(u,u) \geq \delta \|\nabla^m u\|_2^2 - \lambda \|u\|_2^2 \ , \qquad \lambda \geq 0 \ ,$$

(this inequality is the one we frequently meet in practice). In that case, the operator $L+\lambda$, restricted to $\mathcal{D}(L)$, is maximal accretive of type $\omega < \frac{\pi}{2}$ and $-(L+\lambda)$ is the generator of a contraction semigroup $e^{-t(L+\lambda)}$ on $L^2(\mathbb{R}^n)$. Writing $e^{-tL} = e^{-t(L+\lambda)}e^{\lambda t}$, we get

$$\|e^{-tL}\|_{L^2 \to L^2} \ \leq \ e^{\lambda t} \ .$$

2.3 De-Giorgi estimates and the Gaussian property for the heat kernel

Let us start by a few definitions useful for our purpose.

L-harmonic functions. Let Ω be an open bounded set in $\mathbb{R}^n$. A L-harmonic function u in Ω is a solution of $Lu = 0$ in Ω in the weak sense:

$$u \in H^m(\Omega) \text{ and for all } \varphi \in H_0^m(\Omega) \quad , \quad \mathcal{Q}(u,\varphi) = 0 \ .$$

Heat kernel. By $\mathcal{K}_t(x,y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, we denote the distributional kernel of the semigroup e^{-tL}, i.e., $\mathcal{K}_t$ is defined by

$$(e^{-tL} f \ , \ g) \ = \ (\mathcal{K}_t^L \ , \ g \otimes f) \ ,$$

for all $f, g \in C_0^\infty(\mathbb{R}^n)$. We refer to this kernel as to the heat kernel of L.

Remark 2. For our purpose, the heat kernel is , formally, the fundamental solution of the heat equation with the Dirac mass δ_y as initial data.

Now, let us recall the theorem concerning the Gaussian estimates for the heat kernel (see [2]).

Theorem 1. *Are equivalent:*

(i) There exists a constant $c_0 > 0$ such that for all $R > 0$, for all $x_0 \in \mathbb{R}^n$ and for all L-harmonic function v in $B_R(x_0)$, one has

$$\int_{B_r(x_0)} \mid \nabla^m v \mid^2 \leq \ c_0 \left(\frac{r}{R}\right)^{n-2m+2\mu_0} \int_{B_R(x_0)} \mid \nabla^m v \mid^2 \qquad (D)$$

provided $0 < r \leq R$. Note that $\mu_0 \in \big(\max(0, m - n/2), m\big]$, the exponent $n - 2m + 2\mu_0$ is nonnegative and $B_\rho(x_0)$ stands for the Euclidean ball of centre x_0 and radius $\rho > 0$.

(ii) There exist $l \in \{0, 1, \cdots, m-1\}$, $\nu \in (0,1)$ and two constants c and $a > 0$ such that for all $t > 0$, for all $x, y, h \in \mathbb{R}^n$ and for all multi-index $\gamma \in \mathbb{N}^n$, one has

$$|D_x^\gamma \mathcal{K}_t(x,y)| \ + \ |D_y^\gamma \mathcal{K}_t(x,y)| \ \leq \ \frac{c}{t^{\frac{n+|\gamma|}{2m}}} \ \exp\Bigg(-a\Big(\frac{|x-y|}{t^{\frac{1}{2m}}}\Big)^{\frac{2m}{2m-1}}\Bigg) \,,$$

when $|\gamma| \leq l$, and

$$|D_x^\gamma \mathcal{K}_t(x+h,y) - D_x^\gamma \mathcal{K}_t(x,y)| \ \leq \ \frac{c}{t^{\frac{n+|\gamma|}{2m}}} \ \left(\frac{|h|}{t^{1/2m}}\right)^\nu \,,$$

$$|D_y^\gamma \mathcal{K}_t(x,y+h) - D_y^\gamma \mathcal{K}_t(x,y)| \ \leq \ \frac{c}{t^{\frac{n+|\gamma|}{2m}}} \ \left(\frac{|h|}{t^{1/2m}}\right)^\nu \,,$$

when $|\gamma| = l$.

This means that the kernel $\mathcal{K}_t(x,y)$ belongs to the Hölder space $\mathcal{C}^{l,\nu}(\mathbb{R}^n)$ in each variable.

Remark 3. The relationship between μ_0 and $\mu = l + \nu$ is such that, if (i) is verified for μ_0 then (ii) is satisfied for all $\mu \in (0, \mu_0)$. Note that $\mu \notin \mathbb{N}$.

The interest of Theorem 1 (seemingly new) is that the elliptic property (D) applies appropriately to operators with little smoothness such as uniformly continuous or VMO (Vanish Mean Oscillation) coefficients.

The equivalence applies as well to operators such as $\mathcal{L}_0$ defined in the introduction. Indeed, it was shown in ([2], Proposition 51) that $\mathcal{L}_0$ verifies (D) for all $\mu_0 \in [0, 2m)$, thus by Theorem 1, we get the following Gaussian estimates for the heat kernel:

Proposition 2. *Let $\mu \in [0, 2m) \setminus \mathbb{N}$. There exist constants c and $b_1 > 0$ such that for all $t \in (0, \infty)$ and all $x, h \in \mathbb{R}^n$, we have for any multi-index $\gamma \in \mathbb{N}^n$ such that $|\gamma| \le E(\mu)$*

$$|D_x^\gamma K_t(x,y)| \le \frac{c}{t^{\frac{n+|\gamma|}{4m}}}\ g_{m,b_1}\Big(\frac{|x-y|}{t^{\frac{1}{4m}}}\Big)\ , \tag{G_1}$$

and if $|\gamma| = E(\mu)$

$$|D_x^\gamma K_t(x+h,y) - D_x^\gamma K_t(x,y)| \le \frac{c}{t^{\frac{n+|\gamma|}{4m}}}\Big(\frac{|h|}{t^{\frac{1}{4m}}}\Big)^{\mu - E(\mu)}\ , \tag{G_2}$$

where $K_t(x,y)$ denotes the heat kernel of the semigroup generated by the operator $\mathcal{L}_0$ and $g_{m,\delta}(y) = \exp\Big(-\delta y^{\frac{4m}{4m-1}}\Big)$ for $\delta > 0$.

From now on, $K_t(x)$ stands for $K_t(x,0)$.

3 Statement of the main result

We consider the Cauchy problem (1). An adaptation of the argument (Banach fixed point theorem) used in $\big($[4], Proposition 1$\big)$ shows that

Proposition 3. *There exists a unique solution $u \in \mathcal{C}\big([0,\infty); L^1(\mathbb{R}^n)\big)$ of (1) such that*

$$u \in \mathcal{C}\big((0,\infty); W^{4m,p}(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big((0,\infty); L^p(\mathbb{R}^n)\big)$$

for all $p \in (1, \infty)$.

The main result of this paper is the following

Theorem 4. *For all $u_0 \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} u_0(x)dx = M$, the solution u of (1) satisfies for all $p \in [1, \infty]$*

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} ||D_x^\gamma u(x,t) - M D_x^\gamma K_t(x+at)||_p = 0, \tag{4}$$

for all multi-index $\gamma \in \mathbb{N}^n$ such that $|\gamma| \in \{0, 1, \cdots, 2m-1\}$.

Remark 4. The restriction $|\gamma| \in \{0, 1, \cdots, 2m-1\}$ is justified by Theorem 1 and Remark 3. Indeed, as mentioned above, $\mathcal{L}_0$ verifies (D) for all $\mu_0 \in [0, 2m)$.

To prove Theorem 4, note first that if u is the solution of (1) then $v(x,t) := u(x - at, t)$ satisfies

$$\begin{cases} v_t + \mathcal{L}_0 v = 0 \quad \text{on } \mathbb{R}^n \times (0, \infty), \\ v(0) = u_0. \end{cases} \tag{5}$$

This remark then reduces Theorem 4 to proving

Proposition 5. *Let $u_0 \in L^1(\mathbb{R}^n)$ be such that $M = \int_{\mathbb{R}^n} u_0(x)dx$. Then the solution v of (5) verifies*

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} ||D_x^\gamma v - M D_x^\gamma K_t||_p = 0, \tag{6}$$

for all $p \in [1, \infty]$ and all $\gamma \in \mathbb{N}^n$ such that $|\gamma| \in \{0, 1, \cdots, 2m-1\}$.

4 Proof of the main result

This section is devoted to the proof of Proposition 5, hence Theorem 4. We include an argument adapted from [4] and the proof will be divided into three steps.

- *Step 1: Estimation of $||D_x^\gamma u(., t)||_p$.*

Lemma 6. *For all $p \in [1, \infty]$ there exists a constant $C_{p,m}$ such that for all $t > 0$, the solution u of (5) satisfies*

$$t^{|\gamma|/4m} ||D_x^\gamma u(., t)||_p \le C_{p,m} t^{-\frac{n}{4m}(1-\frac{1}{p})} ||u_0||_1 \tag{7}$$

for all $|\gamma| \le E(\mu)$.

Proof. We have $u(x,t) = K_t \star u_0(x)$ and $D_x^\gamma u(x,t) = D_x^\gamma K_t \star u_0(x)$. Then by the Young inequality we get

$$||D_x^\gamma u(., t)||_p = ||D_x^\gamma K_t \star u_0||_p \le ||D_x^\gamma K_t||_p ||u_0||_1.$$

On the other hand, using (G_1) yields

$$||D_x^\gamma K_t||_p^p \leq \frac{C}{t^{p\frac{n+|\gamma|}{4m}}} \int_{\mathbb{R}^n} g_{m,pb_1}\Big(\frac{|x|}{t^{\frac{1}{4m}}}\Big)dx = \frac{C}{t^{p\frac{n+|\gamma|}{4m}-\frac{n}{4m}}} \int_{\mathbb{R}^n} g_{m,pb_1}(|x|)dx.$$

Hence

$$||D_x^\gamma K_t||_p \leq C_{p,m} t^{-\frac{n+|\gamma|}{4m}+\frac{n}{4mp}}$$

and (7) follows.

Because of the density of $L^1(\mathbb{R}^n; 1+|x|)$ into $L^1(\mathbb{R}^n)$, the proof of Proposition 5 is reduced to proving the following result.

• *Step 2: The heart of the matter.*

The key to the proof is the following result from which Proposition 5 (and hence Theorem 4) follows straightforwardly.

Theorem 7. *1. For all $p \in [1,\infty]$ there exists a contant $C_p > 0$ such that for all $t > 0$ and all $\varphi \in L^1(\mathbb{R}^n; 1+|x|)$ such that $\int_{\mathbb{R}^n} \varphi(x)dx = 0$,*

$$t^{|\gamma|/4m} ||D^\gamma K_t \star \varphi||_p \leq \begin{cases} C_p t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{1}{4m}} ||\varphi||_{L^1(\mathbb{R}^n;|x|)} & \text{if } |\gamma| < E(\mu) \\ C_p t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{\nu}{8m}} ||\varphi||_{L^1(\mathbb{R}^n;|x|^{\nu/2})} & \text{if } |\gamma| = E(\mu), \end{cases} \tag{8}$$

where $\nu = \mu - E(\mu)$ and μ is as in Proposition 2.

2. For all $p \in [1,\infty]$ there exists a contant $C'_p > 0$ such that if $\varphi \in L^1(\mathbb{R}^n; 1+|x|)$ with $\int_{\mathbb{R}^n} \varphi(x)dx = M$, then for all $t > 0$ $t^{|\gamma|/4m}||D^\gamma K_t \star \varphi - MD^\gamma K_t||_p \leq$

$$\begin{cases} C'_p t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{1}{4m}} ||\varphi||_{L^1(\mathbb{R}^n;|x|)} & \text{if } |\gamma| < E(\mu) \\ C'_p t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{\nu}{8m}} ||\varphi||_{L^1(\mathbb{R}^n;|x|^{\nu/2})} & \text{if } |\gamma| = E(\mu). \end{cases} \tag{9}$$

Proof. We only prove (8). The same argument applies to inequalities (9).

1. Let $\varphi \in L^1(\mathbb{R}^n; 1+|x|)$ be such that $\int_{\mathbb{R}^n} \varphi(x)dx = 0$. We get

$$D^\gamma K_t \star \varphi(x) = \int_{\mathbb{R}^n} (D^\gamma K_t(x-y) - D^\gamma K_t(x))\varphi(y)dy.$$

$*$ *Case* 1 : $|\gamma| < E(\mu)$.

Using successively Taylor's formula and (G_1) we obtain

$$D^\gamma K_t(x-y) - D^\gamma K_t(x) = y \int_0^1 D^\tau K_t(x-\theta y) d\theta,$$

where $\tau \in \mathbb{N}^n$ is such that $|\tau| = |\gamma| + 1$. It then follows that

$$|D^\gamma K_t(x-y) - D^\gamma K_t(x)| \leq |y| \int_0^1 \frac{c}{t^{\frac{n+|\tau|}{4m}}} g_{m,b_1}\Big(\frac{|x-\theta y|}{t^{1/4m}}\Big) d\theta.$$

Hence,

$$|D^\gamma K_t \star \varphi(x)| \leq \frac{c}{t^{\frac{n+|\tau|}{4m}}} \int_0^1 \int_{\mathbb{R}^n} g_{m,b_1}\Big(\frac{|x-\theta y|}{t^{1/4m}}\Big) |y||\varphi(y)| dy d\theta.$$

It follows that

$$\begin{aligned} ||D^\gamma K_t \star \varphi||_\infty &\leq \frac{c}{t^{\frac{n+|\tau|}{4m}}} ||\varphi||_{L^1(\mathbb{R}^n;|x|)} \sup_{x,y\in\mathbb{R}^n} \int_0^1 g_{m,b_1}\Big(\frac{|x-\theta y|}{t^{1/4m}}\Big) d\theta \\ &\leq Ct^{-\frac{n+|\tau|}{4m}} ||\varphi||_{L^1(\mathbb{R}^n;|x|)}, \end{aligned}$$

which is (8) for $p = \infty$ in this case. Similarly, we have for $p = 1$,

$$||D^\gamma K_t \star \varphi||_1 \leq \frac{c}{t^{\frac{n+|\tau|}{4m}}} \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g_{m,b_1}\Big(\frac{|x-\theta y|}{t^{1/4m}}\Big) |y||\varphi(y)| dx dy d\theta,$$

and since

$$\int g_{m,b_1}\Big(\frac{|x-\theta y|}{t^{1/4m}}\Big) dx = t^{n/4m} \int g_{m,b_1}(|u|) du,$$

$$||D^\gamma K_t \star \varphi||_1 \leq Ct^{-\frac{|\tau|}{4m}} ||\varphi||_{L^1(\mathbb{R}^n;|x|)},$$

this is (8) when $p = 1$.

Finally, the case $p \in (1, \infty)$ is easily obtained by interpolation. Indeed,

$$\begin{aligned} ||D^\gamma K_t \star \varphi||_p &\leq ||D^\gamma K_t \star \varphi||_1^{1/p} ||D^\gamma K_t \star \varphi||_\infty^{1-1/p} \\ &\leq C' t^{-\frac{n}{4m}(1-\frac{1}{p}) - \frac{\tau}{4m}} ||\varphi||_{L^1(\mathbb{R}^n;|x|)}. \end{aligned}$$

Inequality (8) is proved for all $p \in [1, \infty]$ when $|\gamma| < E(\mu)$.

$*$ *Case* 2 : $|\gamma| = E(\mu)$.

We have by inequalities (G_1) and (G_2)

$$\begin{aligned} &|D^\gamma K_t(x-y) - D^\gamma K_t(x)| = \\ &= |D^\gamma K_t(x-y) - D^\gamma K_t(x)|^{1/2} |D^\gamma K_t(x-y) - D^\gamma K_t(x)|^{1/2} \\ &\leq \frac{c}{t^{\frac{n+|\gamma|}{4m}}} \Big(\frac{|y|}{t^{\frac{1}{4m}}} \Big)^{\nu/2} \Big[g_{m,b_1}\Big(\frac{|x|}{t^{\frac{1}{4m}}} \Big) + g_{m,b_1}\Big(\frac{|x-y|}{t^{\frac{1}{4m}}} \Big) \Big]^{1/2} \\ &\leq \frac{c'}{t^{\frac{n+|\gamma|}{4m} + \frac{\nu}{8m}}} |y|^{\nu/2} \Big[g_{m,b_1/2}\Big(\frac{|x|}{t^{\frac{1}{4m}}} \Big) + g_{m,b_1/2}\Big(\frac{|x-y|}{t^{\frac{1}{4m}}} \Big) \Big] \ . \end{aligned}$$

Thus,

$$|D^\gamma K_t \star \varphi(x)| \leq$$
$$\frac{c}{t^{\frac{n+|\gamma|}{4m} + \frac{\nu}{8m}}} \int_{\mathbb{R}^n} \Big[g_{m,b_1/2}\Big(\frac{|x|}{t^{\frac{1}{4m}}} \Big) + g_{m,b_1/2}\Big(\frac{|x-y|}{t^{\frac{1}{4m}}} \Big) \Big] |y|^{\nu/2} |\varphi(y)| dy \ .$$

It then follows that

$$||D^\gamma K_t \star \varphi||_\infty \leq \frac{C}{t^{\frac{n+|\gamma|}{4m} + \frac{\nu}{8m}}} ||\varphi||_{L^1(\mathbb{R}^n; |x|^{\nu/2})} \ .$$

This is (8) for $p = \infty$. For the case $p = 1$, we have

$$\begin{aligned} &||D^\gamma K_t \star \varphi||_1 \ \leq \\ &\leq \frac{c}{t^{\frac{n+|\gamma|}{4m} + \frac{\nu}{8m}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[g_{m,b_1/2}\Big(\frac{|x|}{t^{\frac{1}{4m}}} \Big) + g_{m,b_1/2}\Big(\frac{|x-y|}{t^{\frac{1}{4m}}} \Big) \Big] |y|^{\nu/2} |\varphi(y)| dx dy \\ &\leq \frac{C}{t^{\frac{n+|\gamma|}{4m} + \frac{\nu}{8m}}} t^{\frac{n}{4m}} ||\varphi||_{L^1(\mathbb{R}^n; |x|^{\nu/2})} \end{aligned}$$

since, as before,

$$\int g_{m,b_1/2}\Big(\frac{|x-y|}{t^{1/4m}} \Big) dx = \int g_{m,b_1/2}\Big(\frac{|x|}{t^{1/4m}} \Big) dx = t^{n/4m} \int g_{m,b_1/2}(|u|) du \ .$$

Hence

$$||D^\gamma K_t \star \varphi||_1 \leq \frac{C}{t^{\frac{|\gamma|}{4m} + \frac{\nu}{8m}}} ||\varphi||_{L^1(\mathbb{R}^n; |x|^{\nu/2})}$$

which corresponds to (8) when $p = 1$ and $|\gamma| = E(\mu)$.

As previously, the case $p \in (1,\infty)$ follows by interpolation.

2. We show (9) in the same way by writing

$$\begin{aligned}(D^\gamma K_t \star \varphi - MD^\gamma K_t)(x) &= \int_{\mathbb{R}^n} D^\gamma K_t(x-y)\varphi(y)dy - D^\gamma K_t(x)\int_{\mathbb{R}^n}\varphi(y)dy \\ &= \int_{\mathbb{R}^n}(D^\gamma K_t(x-y) - D^\gamma K_t(x))\varphi(y)dy.\end{aligned}$$

Theorem 7 is completely proved.

Now, we are in a position to prove Proposition 5, hence Theorem 4. We recall that the argument of density used here appears in [4]. Recall that $L^1(\mathbb{R}^n; 1+|x|)$ is dense in $L^1(\mathbb{R}^n)$.

• *Step 3: A density argument.*

Let $u_0 \in L^1(\mathbb{R}^n)$ and $u_N \in L^1(\mathbb{R}^n; 1+|x|)$ be such that $\int_{\mathbb{R}^n} u_N(x)dx = M$ and $lim_{N\to+\infty} u_N = u_0$ in $L^1(\mathbb{R}^n)$. We have

$$\begin{aligned}&t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}||D^\gamma K_t \star u_0 - MD^\gamma K_t||_p \leq \\ &\quad t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}\Big(||D^\gamma K_t \star u_N - MD^\gamma K_t||_p + ||D^\gamma K_t \star (u_N - u_0)||_p\Big)\end{aligned}$$

and by Lemma 6,

$$||D^\gamma K_t \star (u_N - u_0)||_p \leq C_p t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}||u_N - u_0||_1.$$

Let $\varepsilon > 0$, then there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$,

$$\sup_{t>0}\Big(t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}||D^\gamma K_t \star (u_N - u_0)||_p\Big) \leq \frac{\varepsilon}{2}.$$

On the other hand, Theorem 7 yields

$$\lim_{t\to+\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}||D^\gamma K_t \star u_N - MD^\gamma K_t||_p = 0,$$

i.e. there exists $t_0 > 0$ such that for all $t \geq t_0$,

$$t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}||D^\gamma K_t \star u_N - MD^\gamma K_t||_p \leq \frac{\varepsilon}{2}.$$

Hence, for all $\varepsilon > 0$, there is $t_0 > 0$ such that for all $t \geq t_0$,

$$t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}||D^\gamma K_t \star u_0 - MD^\gamma K_t||_p \leq \varepsilon.$$

The proof of Proposition 5 is complete.

5 Concluding remarks

1. The result of Theorem 4 can be improved in the case where $\mathcal{L}_0 = \Delta^{2m}$. Indeed, in that case, K_t is "explicitly" given by

$$K_t(x) = \frac{1}{t^{n/4m}} K\Big(\frac{x}{t^{1/4m}}\Big)$$

where

$$K(x) = \frac{1}{(2\pi)^n} \int e^{ix.\xi} e^{-|\xi|^{4m}} d\xi.$$

Then, by the Fourier transform we get

$$|D^\gamma K(x)| \leq C_\gamma e^{-|x|^{\frac{4m}{4m-1}}} \quad \text{for all } \gamma.$$

It then follows that Theorem 4 is valid for **all** $\gamma \in \mathbb{N}^n$ instead of $|\gamma| \leq E(\mu)$. We thank P. Auscher for bringing this remark to our attention.

2. It is possible to prove a similar result to Theorem 4 for the solution of the equation (2) when the coefficients $a_{\alpha\beta}$, associated to $\tilde{L}_0$, are as follows:

(i) constants or

(ii) $a_{\alpha\beta} \in BUC$ (Bounded Uniformly Continuous) or

(iii) $a_{\alpha\beta} \in L^\infty \cap VMO$ (Vanish Mean Oscillation) or

(iv) $a_{\alpha\beta} \in BMO$ (Bounded Mean Oscillation) with small BMO norm.

Indeed, in these cases, the kernel associated to $\tilde{L}_0$, elliptic in the sense of the Gårding inequality, satisfies the gaussian estimates (ii) of Theorem 1 (see [2], section: examples).

References

[1] S. Agmon, *Lectures on elliptic boundary problems*, Van Nostrand, 1966.

[2] P. Auscher and M. Qafsaoui, *Equivalence between regularity theorems and heat kernel estimates for higher order elliptic operators and systems under divergence form*, J. Funt. Anal. **177** (2000).

[3] S. Campanato, *Sistemi ellittici in forma divergenza, regolarità all'interno*, Scu. Nor. Sup. Pisa, 1980.

[4] M. Escobedo and E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^n$, J. Funt. Anal. **100 (1) : 119-161** (1991).

[5] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, New-York, 1966.

[6] M. Qafsaoui, *Opérateurs elliptiques d'ordre supérieur sous forme divergence à coefficients peu réguliers, estimations gaussiennes de leurs noyaux de la chaleur et applications*, PhD thesis, LAMFA-CNRS, Université de Picardie, 1999.

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