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DELTA LINK-HOMOTOPY ON SPATIAL GRAPHS

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Abstract

We study new equivalence relations in spatial graph theory. We consider natural generalizations of delta link-homotopy on links, which is an equivalence relation generated by delta moves on the same component and ambient isotopies. They are stronger than edge-homotopy and vertex-homotopy on spatial graphs which are natural generalizations of link-homotopy on links. Relationship to existing familiar equivalence relations on spatial graphs are stated, and several invariants are defined by using the second coefficient of the Conway polynomial and the third derivative at 1 of the Jones polynomial of a knot.

1 Introduction and results

Throughout this paper we work in the piecewise linear category. Let G be a finite simple graph, namely it has no loops and multiedges. We denote the set of all edges of G by $E(G)$. We consider G as a topological space in the usual way and study an embedding of G into the 3-sphere S^3, called a *spatial embedding* of G or simply a *spatial graph*.

In [28], eight equivalence relations (1) *ambient isotopy*, (2) *cobordism*, (3) *isotopy*, (4) *I-equivalence*, (5) *edge-homotopy*, (6) *vertex-homotopy*, (7) *homology*, (8) $\mathbf{Z}_2$*-homology* on spatial graphs are introduced and the following implication between them are stated [28, FUNDAMENTAL THEOREM]:

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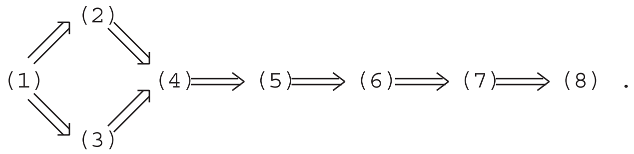


Moreover these eight equivalence relations are different equivalence relations. We refer the reader to [28] for precise definitions. Specially, (5) and (6) (In [28], (5) and (6) was called *homotopy* and *weak homotopy*, respectively) were introduced as natural generalizations of *link-homotopy* [12] on links. Two spatial embeddings $f, g : G \to S^3$ are said to be *edge-homotopic* if f and g can be transformed into one another by *self-crossing changes* and ambient isotopies, where a self-crossing change is a crossing change on an edge, and *vertex-homotopic* if f and g can be transformed into one another by crossing changes on adjacent edges and ambient isotopies.

In this paper we define new equivalence relations and discuss them. A *delta move* is a local move as illustrated in Fig. 1.1. It is known that this move is an *unknotting operation*, namely any knot can be transformed into a trivial one by delta moves and ambient isotopies [11] [16].

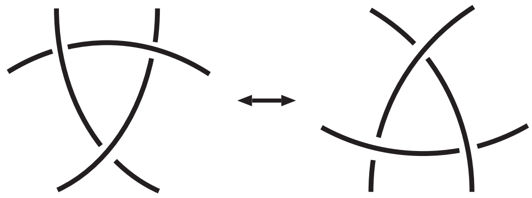
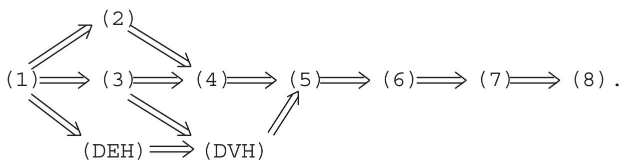


Fig. 1.1.

We say that two spatial embeddings $f, g : G \to S^3$ are (DEH) *delta edge-homotopic* if f and g can be transformed into one another by *self-delta moves* and ambient isotopies, where a self-delta move is a delta move on an edge, and (DVH) *delta vertex-homotopic* if f and g can be transformed into one another by *quasi-adjacent delta moves* and ambient isotopies, where a quasi-adjacent delta move is a delta move on exactly two adjacent edges. We note that these equivalence relations coincides with *delta link-homotopy* (or *self delta-equivalence*) on links [25] if G is homeomorphic to a mutually disjoint union of 1-spheres. Namely these are natural generalizations of delta link-homotopy on links. We refer the reader to [26], [27], [20], [17], [18] and [19] for works related to delta link-homotopy on links.

It is natural to ask how strong are the above-mentioned equivalence relations. We show the following relations between (1),(2),. . .,(8) and (DEH), (DVH).

Theorem 1.1.



Moreover these ten equivalence relations are different equivalence relations.

A graph H is called a *minor* of G if H is obtained from G by a finite sequence of the following two operations: (1) edge contraction, (2) taking a subgraph. We note that any subgraph of G is a minor of G. For delta vertex-homotopy, we have the following as a corollary of Theorem 1.1 and [28, Theorem B].

Corollary 1.2. *For a graph G, the following conditions are mutually equivalent:*
(i) *Any two spatial embeddings $f, g : G \to S^3$ are delta vertex-homotopic.*
(ii) *None of G_1, G_2 and G_3 as illustrated in Fig. 1.2 is a minor of G.* ■

A graph satisfying with the condition of Corollary 1.2 is called a *generalized bouquet* [28]. Therefore Corollary 1.2 means that a quasi-adjacent delta move is an "unknotting operation" for spatial embeddings of a generalized bouquet.

To detect the equivalence class, we construct some delta edge (resp. vertex)-homotopy invariants. A *cycle* is a subgraph of G which is homeomorphic to the 1-sphere S^1, and a *k-cycle* is a cycle which contains exactly k vertices. We regard $f(\gamma)$ as a knot for a cycle γ and a spatial

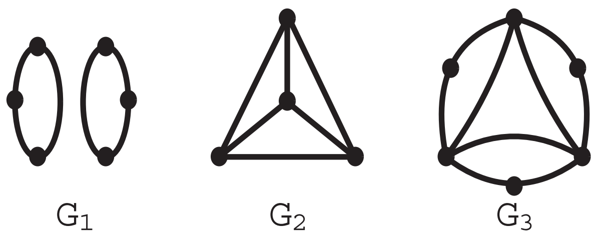


Fig. 1.2.

embedding $f : G \to S^3$. We denote the set of all cycles of G, the set of all cycles containing an edge e of G and the set of all cycles containing edges e_1, e_2 of G by $\Gamma(G)$, $\Gamma_e(G)$ and $\Gamma_{e_1,e_2}(G)$, respectively. Let $\mathbf{Z}_r = \{0, 1, \ldots, r-1\}$ for a positive integer r and $\mathbf{Z}_0 = \mathbf{Z}$, where $\mathbf{Z}$ is the integers (We regard $\mathbf{Z}_r$ $(r \geq 1)$ as the cyclic group of order r whenever we consider $\mathbf{Z}_r$ a group). Then a map $\omega : \Gamma(G) \to \mathbf{Z}_r$ is called a *weight* on $\Gamma(G)$. For a weight $\omega : \Gamma(G) \to \mathbf{Z}_r$ and a spatial embedding $f : G \to S^3$, we set

$$\tilde{\alpha}_\omega(f) \equiv \sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) \pmod{r},$$

where $a_2(J)$ denotes the second coefficient of the *Conway polynomial* [2] of a knot J. Then we have the following.

Theorem 1.3. (1) *If a weight ω is weakly balanced on each of edges of G, then $\tilde{\alpha}_\omega$ is a delta edge-homotopy invariant.*
(2) *If a weight ω is weakly balanced on each pair of adjacent edges of G, then $\tilde{\alpha}_\omega$ is a delta vertex-homotopy invariant.*

We give the definition of "weakly balanced" in section 3. Let d_s be the greatest common divisor of $\{\sharp\Gamma_e(G) \mid e \in E(G)\}$, where $\sharp\{\cdot\}$ means the number of elements of the set. If $d_s \geq 2$, we define a weight $\omega_s : \Gamma(G) \to \mathbf{Z}_{d_s}$ by $\omega_s(\gamma) = 1$ for any $\gamma \in \Gamma(G)$. Let d_{ad} be the greatest common divisor of $\{\sharp\Gamma_{e_1,e_2}(G) \mid \text{adjacent edges } e_1, e_2 \in E(G)\}$. If $d_{ad} \geq 2$, we define a weight $\omega_{ad} : \Gamma(G) \to \mathbf{Z}_{d_{ad}}$ by $\omega_s(\gamma) = 1$ for any $\gamma \in \Gamma(G)$. Then by Theorem 1.3 we have the following.

Corollary 1.4. (1) *$\tilde{\alpha}_{\omega_s}$ is a delta edge-homotopy invariant.*
(2) *$\tilde{\alpha}_{\omega_{ad}}$ is a delta vertex-homotopy invariant.*

By using $\tilde{\alpha}_{\omega_s}$, we show that there exists a non-trivial θ-curve up to delta edge-homotopy (Example 4.1).

For a weight $\omega : \Gamma(G) \to \mathbf{Z}_r$ and a spatial embedding $f : G \to S^3$, we set

$$n_\omega(f) \equiv \frac{1}{18} \sum_{\gamma \in \Gamma(G)} \omega(\gamma) V^{(3)}_{f(\gamma)}(1) \pmod r,$$

where $V_J^{(k)}(1)$ denotes the k-th derivative at 1 of the *Jones polynomial* $V_J(t)$ [6] of a knot J. We note that $(1/18)\sum_{\gamma\in\Gamma(G)} \omega(\gamma)V^{(3)}_{f(\gamma)}(1)$ is integer-valued (cf. Remark 3.4 (1)). Then we have the following.

Theorem 1.5. (1) *If a weight ω is balanced* [29] *on each of edges of G, then n_ω is a delta edge-homotopy invariant.*
(2) *If a weight ω is balanced on each pair of adjacent edges of G, then n_ω is a delta vertex-homotopy invariant.* We give the definition of

"balanced" in section 3. By Theorem 1.1, we have the following immediately.

Corollary 1.6. (1) *If a weight ω is weakly balanced on each pair of adjacent edges of G, then $\tilde{\alpha}_\omega$ is an isotopy invariant.*
(2) *If a weight ω is balanced on each pair of adjacent edges of G, then n_ω is an isotopy invariant.* ■

Remark 1.7. If a weight $\omega : \Gamma(G) \to \mathbf{Z}_r$ is balanced on each of edges (resp. each pair of adjacent edges) of G, then our $\tilde{\alpha}_\omega$ coincides with the Taniyama's edge (resp. vertex)-homotopy invariants α_ω [29].

In the next section we prove Theorem 1.1. The proofs of Theorems 1.3 and 1.5 are given in section 3. We give specific examples in section 4. In this paper we calculate the Jones polynomial of a knot by the skein relation $tV_{J_+}(t) - t^{-1}V_{J_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{J_0}(t)$.

2 Proof of Theorem 1.1

Lemma 2.1. (1) $\Rightarrow$ (DEH) $\Rightarrow$ (DVH) $\Rightarrow$ (5). *Moreover these four equivalence relations are different equivalence relations.*

Proof. By the definition, (1) ⇒ (DEH) is clear. Since a self-delta move is realized by quasi-adjacent delta moves (see Fig. 2.1) and a quasi-adjacent delta move is realized by two self-crossing changes (see Fig. 2.2), we have that (DEH) ⇒ (DVH) ⇒ (5).

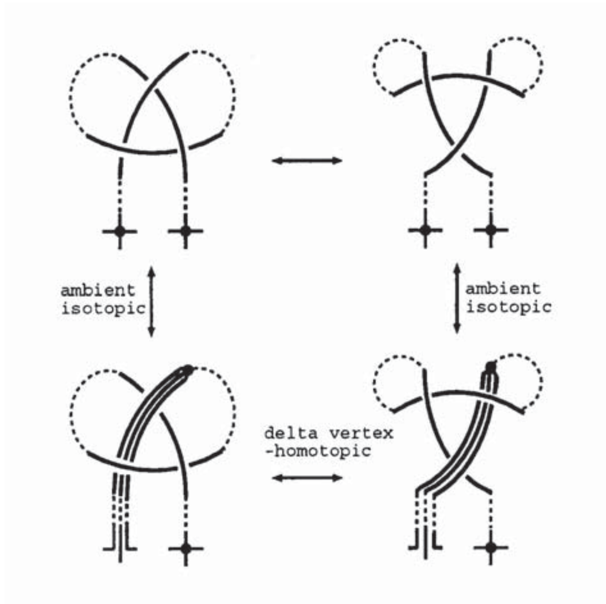


Fig. 2.1.

(1) and (DEH) are different because a trefoil knot is delta-edge homotopic to a trivial knot but they are not ambient isotopic. In section 4, we show that there are delta vertex-homotopic spatial graphs which are not delta edge-homotopic (Example 4.1 and Example 4.2), and edge-homotopic spatial graphs which are not delta vertex-homotopic (Example 4.3). Therefore (1), (DEH), (DVH) and (5) are different equivalence relations. ■

In the following we investigate more about isotopy and delta vertex-homotopy. For a spatial embedding $f : G \to S^3$ and a 3-ball B in S^3, we say that the pair $(B, B \cap f(G))$ is a *ball-star pair* (cf. [5]) if $B \cap f(G)$ is either a proper arc or a star of degree n, where n is a natural number, that is, int$B$ contains only one vertex $f(v)$, and $B \cap f(G)$ consists of n edges $f(e_i)$ that are incident to $f(v)$ and $f(\partial e_i) - \{f(v)\} \subset \partial B$ (when $B \cap f(G)$ is a proper arc, it is regarded as a star of degree 2 even if

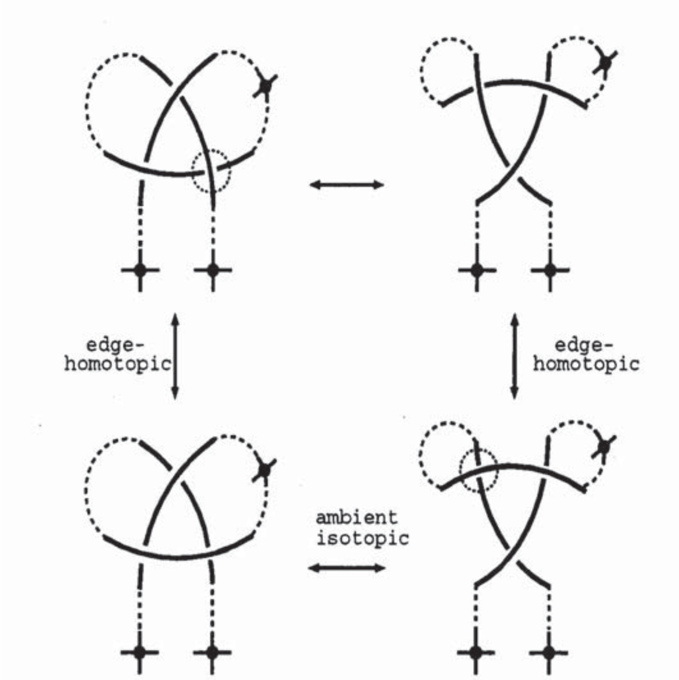


Fig. 2.2.

it does not contain vertices of $f(G)$). A ball-star pair $(B, B \cap f(G))$ is said to be *standard* if there exists a properly embedded 2-disk D in B with $D \supset B \cap f(G)$. We set $J = G - f^{-1}(\mathrm{int}B)$. Then a spatial embedding $g : G \to S^3$ is said to be obtained from f by a *blowing-down* in B if $g|_J = f|_J$ and $(B, B \cap g(G))$ is standard. Conversely f is said to be obtained from g by a *blowing-up* occurring in B. It is known that spatial embeddings $f, g : G \to S^3$ are isotopic if and only if they can be transformed into one another by blowing-downs, blowing-ups and ambient isotopies [28] (see also [24] for links). The following lemma shows that (3) implies (5).

Lemma 2.2. ([28, Lemma 2.1]) *A blowing down is realized by self-crossing changes.* ∎

Moreover we have the following.

Lemma 2.3. *Isotopy implies delta vertex-homotopy.*

Proof. It is sufficient to show that a blowing-down is realized by quasi-adjacent delta moves. Let $f, g : G \to S^3$ be spatial embeddings such that g is obtained from f by a blowing-down in B. By Lemma 2.2, f is obtained from g by self-crossing changes on $B \cap g(G)$, where $(B, B \cap g(G))$ is a standard ball-star pair. For each of crossing points, we deform f up to ambient isotopy as illustrated in Fig. 2.3. Therefore we can deform f up to ambient isotopy so that a "band sum of Hopf links and g" (cf. [31] and [30]). We note that the deformations as illustrated in Fig. 2.4

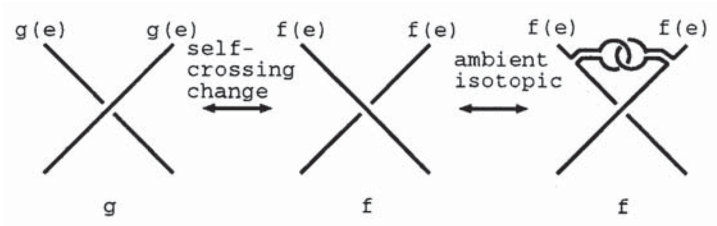


Fig. 2.3.

are realized by delta moves [30]. For (1) and (2), see Fig. 2.5 (1) and (2), respectively. (3) is realized by ambient isotopies only. (4) is clear by (2). For (5), see Fig. 2.5 (5). By using the deformation of Fig. 2.4 (1), we can undo the linking between Hopf bands whose attaching edges are different (see Fig. 2.6). Moreover Hopf bands attached to the same edge are gone by self-delta moves by the deformations of Fig. 2.4 (see Fig. 2.7), thus by quasi-adjacent delta moves. Therefore we can remove all Hopf bands by quasi-adjacent delta moves, namely we can obtain g from f up to delta vertex-homotopy. This completes the proof. ■

Lemma 2.4. (i) *Delta edge-homotopy and delta vertex-homotopy do not imply cobordism, isotopy and I-equivalence.*
(ii) *Cobordism and I-equivalence do not imply delta edge-homotopy and delta vertex-homotopy.*
(iii) *Isotopy does not imply delta edge-homotopy.*

Proof. (i) Let M be a 2-component link as illustrated in Fig. 2.8. We have that M is delta edge-homotopic to a trivial 2-component link (by using the deformation of Fig. 2.4 (1)) but not I-equivalent to a trivial 2-component link because they have different *Milnor $\bar{\mu}$-invariant* [13] (see also [1]). Thus we also have that they are not cobordant and isotopic.

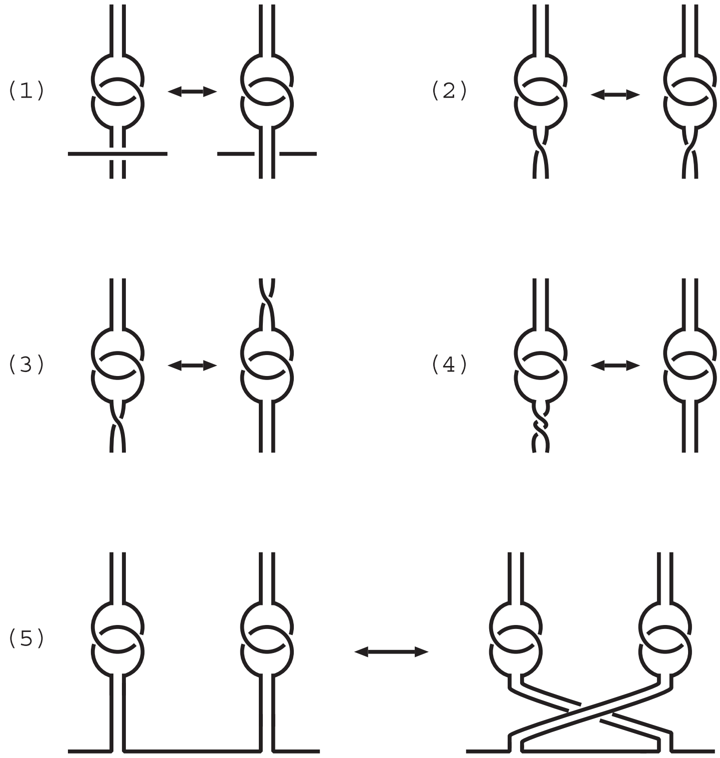


Fig. 2.4.

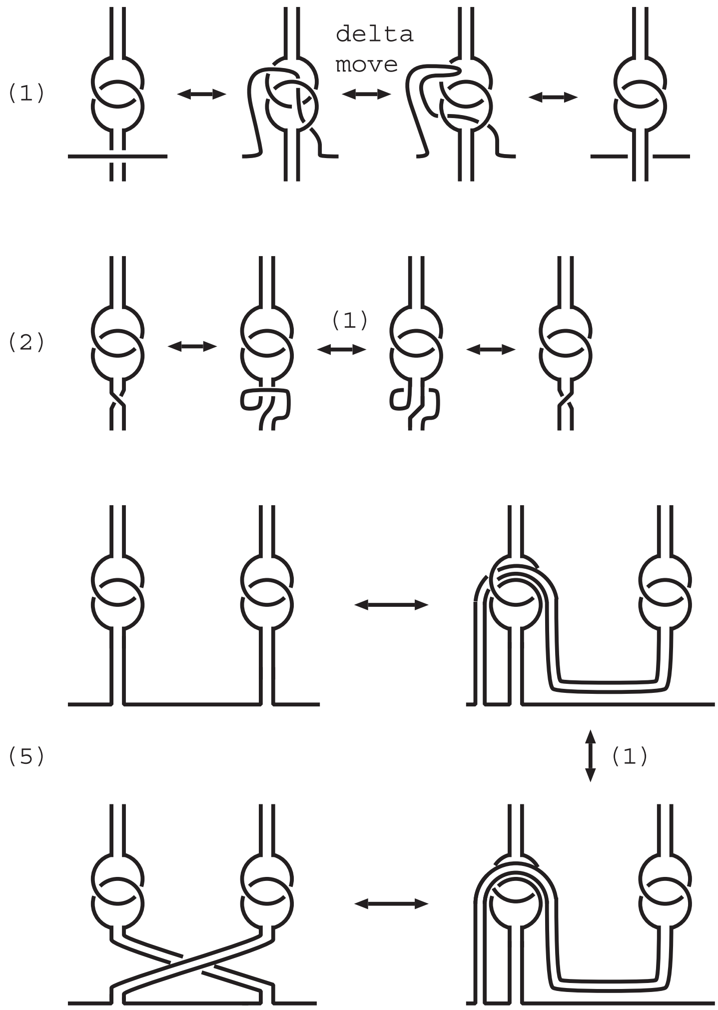


Fig. 2.5.

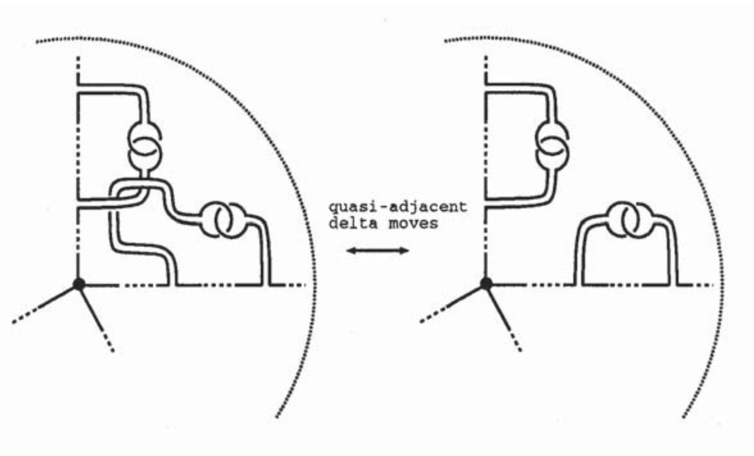

quasi-adjacent
delta moves

Fig. 2.6.

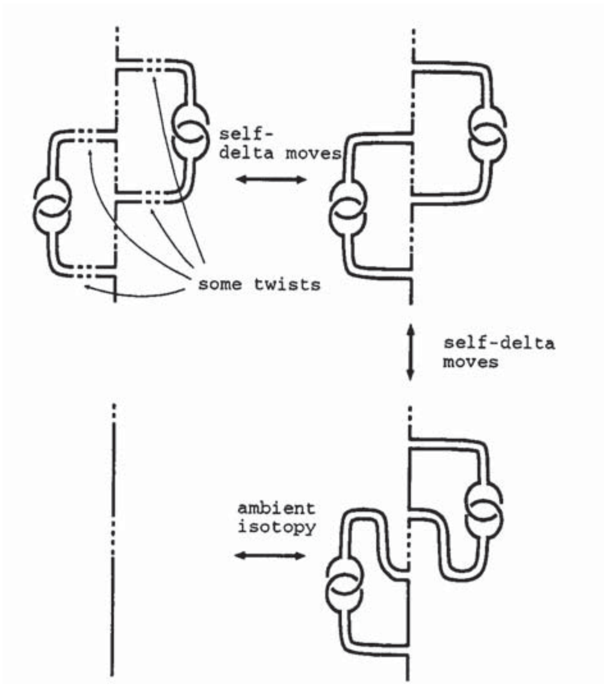

self-
delta moves
some twists
self-delta
moves
ambient
isotopy

Fig. 2.7.

Therefore delta edge-homotopy and delta vertex-homotopy do not imply cobordism, isotopy and I-equivalence.

(ii) Let L be a 2-component link as illustrated in Fig. 2.8. It is known that L and a Hopf link are cobordant but not delta vertex-homotopic [20, Claim 4.5]. Thus a cobordism does not imply delta edge-homotopy and delta vertex-homotopy for links. Thus we also have that I-equivalence does not imply delta edge-homotopy and delta vertex-homotopy.

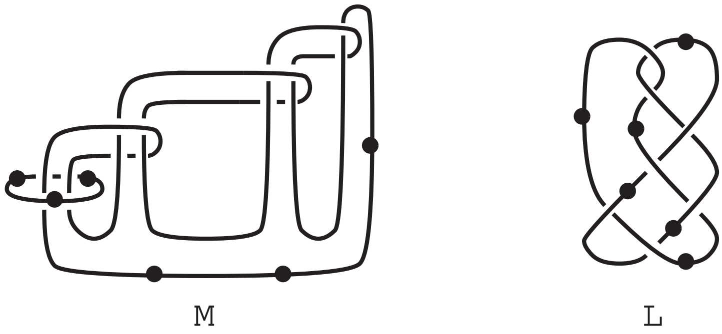


Fig. 2.8.

(iii) In Example 4.1, we show that there exist θ-curves which are not delta edge-homotopic. Since any θ-curves are isotopic (see [28, Theorem B]), we have the desired conclusion. ■

Proof of Theorem 1.1. By Lemmas 2.1, 2.3 and 2.4, we have the result. ■

We have the following immediately as well.

Corollary 2.5. *Isotopy implies delta link-homotopy on links.* ■

3 Delta edge and vertex-homotopy invariants

In this section we prove Theorems 1.3, 1.5 and their corollaries. Let $\omega : \Gamma(G) \to \mathbf{Z}_r$ be a weight. Let e be an edge of G. Then we say that ω is *weakly balanced on e* if

$$\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \equiv 0 \pmod{r}.$$

Let e_1 and e_2 be adjacent edges of G. Then we say that ω is *weakly balanced on* e_1, e_2 if

$$\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma) \equiv 0 \pmod r.$$

Lemma 3.1. (1) *Let e be an edge of G and $\omega : \Gamma(G) \to \mathbf{Z}_r$ a weight which is weakly balanced on e. Let $f, g : G \to S^3$ be spatial embeddings such that g is obtained from f by a self-delta move on $f(e)$. Then $\tilde{\alpha}_\omega(f) \equiv \tilde{\alpha}_\omega(g) \pmod r$.*
(2) *Let e_1 and e_2 be adjacent edges of G and $\omega : \Gamma(G) \to \mathbf{Z}_r$ a weight which is weakly balanced on e_1, e_2. Let $f, g : G \to S^3$ be spatial embeddings such that g is obtained from f by a quasi-adjacent delta move on $f(e_1)$ and $f(e_2)$. Then $\tilde{\alpha}_\omega(f) \equiv \tilde{\alpha}_\omega(g) \pmod r$.*

To prove Lemma 3.1, we recall the M. Okada's work. She showed that the variation of a_2 of knots which differed by a single delta move is ± 1 [22, Theorem 1.1]. We can rewrite the fact above as the following:

$$a_2(K_+) - a_2(K_-) = 1, \tag{3.1}$$

where K_+, K_- are knots as illustrated in Fig. 3.1. We remark here

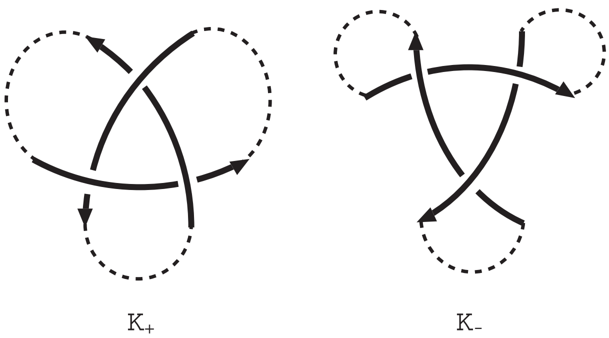


Fig. 3.1.

that each of the following two local moves (i) and (ii) got from the one as illustrated in Fig. 3.1: (i) the orientation of one string is changed, (ii) the upper-lower relations at all crossings are reversed, is obtained by using the move as illustrated in Fig. 3.1 (cf. [16]). Therefore any application of a delta move can be regarded just as the one in Fig. 3.1.

Proof of Lemma 3.1. (1) Let $f, g : G \to S^3$ be spatial embeddings such that g is obtained from f by a single self-delta move on $f(e)$. It is sufficient to show that $\tilde{\alpha}_\omega(f) \equiv \tilde{\alpha}_\omega(g) \pmod{r}$. For $\gamma \in \Gamma_e(G)$, we may assume that $f(\gamma)$ and $g(\gamma)$ are as illustrated in Fig. 3.2 without loss of generality. Then by (3.1) we have that

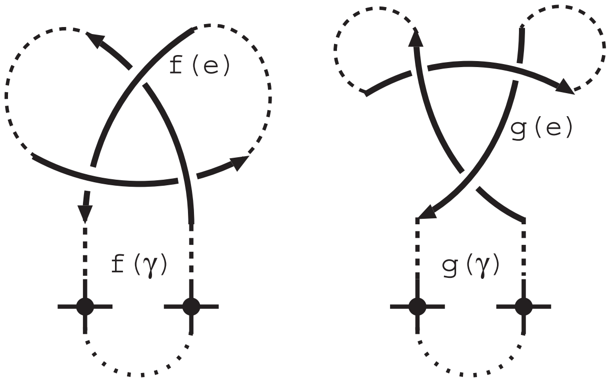


Fig. 3.2.

$$\begin{aligned}
\tilde{\alpha}_\omega(f) - \tilde{\alpha}_\omega(g) &\equiv \sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) - \sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(g(\gamma)) \\
&= \sum_{\gamma \in \Gamma(G)} \omega(\gamma) \left\{ a_2(f(\gamma)) - a_2(g(\gamma)) \right\} \\
&= \sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \left\{ a_2(f(\gamma)) - a_2(g(\gamma)) \right\} \\
&= \sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \\
&\equiv 0 \pmod{r}.
\end{aligned}$$

Therefore we have that $\tilde{\alpha}_\omega(f) \equiv \tilde{\alpha}_\omega(g) \pmod{r}$.

(2) We can prove in the same way as (1) (see Fig. 3.3). ■

Proof of Theorem 1.3. It is clear by Lemma 3.1. ■

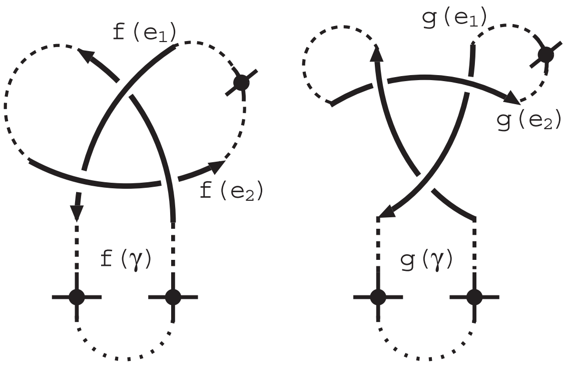


Fig. 3.3.

Proof of Corollary 1.4. It is clear that the weight $\omega_s : \Gamma(G) \to \mathbf{Z}_{d_s}$ is weakly balanced on each of edges of G and the weight $\omega_{ad} : \Gamma(G) \to \mathbf{Z}_{d_{ad}}$ is weakly balanced on each pair of adjacent edges of G. Therefore by Theorem 1.3 we have the desired conclusion. ∎

In [29], Taniyama defined edge (resp. vertex)-homotopy invariants α_ω of spatial graphs by taking advantage of the following fact which is well-known in knot theory (cf. [8]):

$$a_2(J_+) - a_2(J_-) = lk(J_0), \tag{3.2}$$

where J_+, J_- and J_0 are knots and a 2-component link which are identical except inside the depicted regions as illustrated in Fig. 3.4, and lk denotes the *linking number*. We note that α_ω is a delta edge (vertex)-

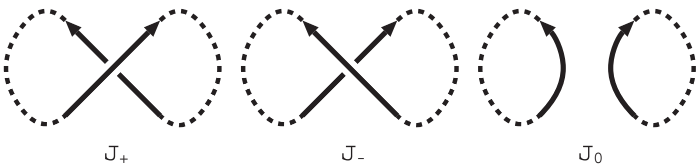


Fig. 3.4.

homotopy invariant by Theorem 1.1. The essential advantage of (3.2) was that the variation of a_2 of knots which differed by a single crossing change is determined by the linking number which is a homological invariant (cf. [23]). On the other hand, we have the following.

Theorem 3.2.

$$\frac{1}{18}V^{(3)}_{K_+}(1) - \frac{1}{18}V^{(3)}_{K_-}(1) = 2Lk(K_0) - 1,$$

where K_+, K_- and K_0 are knots and a 3-component link which are identical except inside the depicted regions as illustrated in Fig. 3.5 and Lk denotes the total linking number (namely the summation of all linking numbers of 2-component sublinks). ■

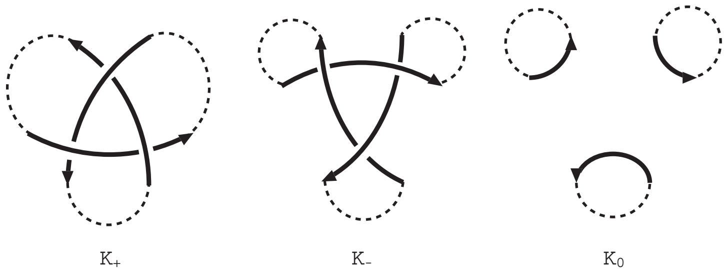


Fig. 3.5.

Therefore the variation of $V^{(3)}(1)$ of knots which differed by a single delta move is also determined by the linking number. We define some delta edge (resp. vertex)-homotopy invariants by taking advantage of Theorem 3.2 in a similar way as Taniyama's construction of edge (resp. vertex)-homotopy invariants.

To prove Theorem 3.2, we show the following relation between a single delta move on a knot and the Jones polynomial.

Lemma 3.3.

$$V_{K_+}(t) - V_{K_-}(t) = (t - 2t^2 + t^3)\left\{V_{K_\infty}(t) - V_{K_0}(t)\right\}, \qquad (3.3)$$

where K_+, K_-, K_∞ and K_0 are knots and a 3-component link which are identical except inside the depicted regions as illustrated in Fig. 3.5 and Fig. 3.6.

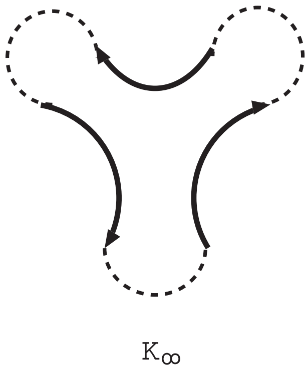


Fig. 3.6.

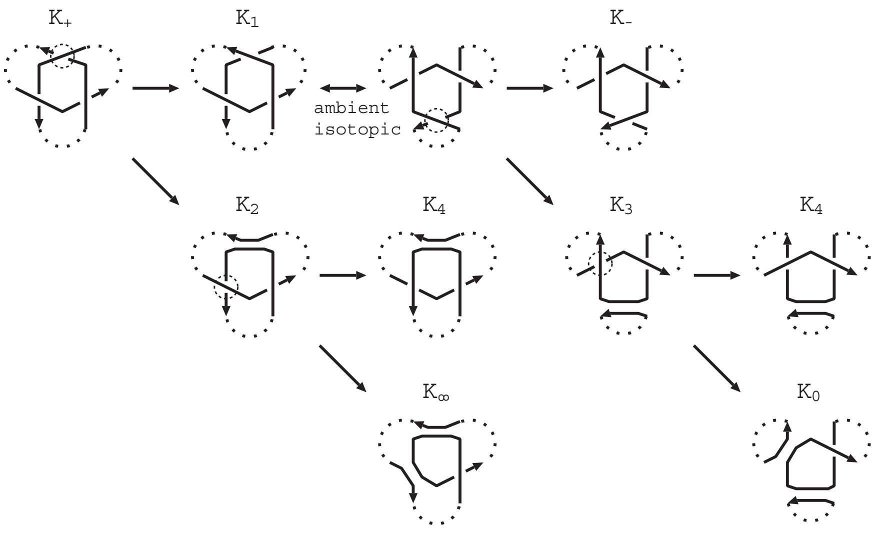


Fig. 3.7.

Proof. Consider a skein tree as illustrated in Fig. 3.7. For K_2, K_4 and K_∞ we have that

$$tV_{K_4}(t) - t^{-1}V_{K_2}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{K_\infty}(t). \tag{3.4}$$

For K_+, K_1 and K_2 we have that

$$tV_{K_1}(t) - t^{-1}V_{K_+}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{K_2}(t). \tag{3.5}$$

By (3.4) and (3.5) we have that

$$\begin{aligned} &V_{K_1}(t) - t^{-2}V_{K_+}(t) \\ &= t(t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{K_4}(t) - (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 V_{K_\infty}(t). \end{aligned} \tag{3.6}$$

On the other hand, for K_3, K_4 and K_0 we have that

$$tV_{K_4}(t) - t^{-1}V_{K_3}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{K_0}(t). \tag{3.7}$$

For K_1, K_- and K_3 we have that

$$tV_{K_1}(t) - t^{-1}V_{K_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{K_3}(t). \tag{3.8}$$

By (3.7) and (3.8) we have that

$$\begin{aligned} &V_{K_1}(t) - t^{-2}V_{K_-}(t) \\ &= t(t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{K_4}(t) - (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 V_{K_0}(t). \end{aligned} \tag{3.9}$$

By (3.6) and (3.9), we have the desired conclusion. ■

Proof of Theorem 3.2. We set $f(t) = t - 2t^2 + t^3$ and denote the k-th derivative of $f(t)$ by $f^{(k)}(t)$. By differentiating both sides of (3.3), we have that

$$\begin{aligned} V^{(3)}_{K_+}(t) - V^{(3)}_{K_-}(t) = f^{(3)}(t)\left\{V_{K_\infty}(t) - V_{K_0}(t)\right\} \\ +3f^{(2)}(t)\left\{V^{(1)}_{K_\infty}(t) - V^{(1)}_{K_0}(t)\right\} \\ +3f^{(1)}(t)\left\{V^{(2)}_{K_\infty}(t) - V^{(2)}_{K_0}(t)\right\} \\ +f(t)\left\{V^{(3)}_{K_\infty}(t) - V^{(3)}_{K_0}(t)\right\}. \end{aligned}$$

Thus it is clear that

$$V_{K_+}^{(3)}(1) - V_{K_-}^{(3)}(1) = 6\{V_{K_\infty}(1) - V_{K_0}(1)\} \\ +6\left\{V_{K_\infty}^{(1)}(1) - V_{K_0}^{(1)}(1)\right\}. \qquad (3.10)$$

Since it is known that

$$V_L(1) = (-2)^{n-1} \quad ([6, \text{Theorem } 15]) \qquad (3.11)$$

and

$$V_L^{(1)}(1) = \begin{cases} 0 & \text{if } n = 1, \\ 3(-2)^{n-2}Lk(L) & \text{if } n \geq 2, \end{cases} \qquad (3.12)$$
$$([6, \text{Theorem } 16], [15, \text{Theorem } 1])$$

for an n-component link L, therefore by (3.11), (3.12) and (3.10) we have that

$$V_{K_+}^{(3)}(1) - V_{K_-}^{(3)}(1) = 6\left\{1 - (-2)^2\right\} + 6\{-3(-2)Lk(K_0)\} \\ = 36Lk(K_0) - 18.$$

This completes the proof. ■.

Remark 3.4. (1) Since the delta move is an unknotting operation for knots, Theorem 3.2 implies that $(1/18)V_J^{(3)}(1)$ is an integer for any knot J.
(2) The essential idea of the proof of Lemma 3.3 has already appeared in [22] (namely in the proof of (3.1)) and extended to a self-delta move on links in terms of the Conway polynomial [17]. Moreover it can be applied to the *HOMFLY polynomial* [4]. We are due to mention it in the future paper [7].

Let e be an edge of G. We give an orientation to e. We give an orientation to each $\gamma \in \Gamma_e(G)$ by the orientation of e. Since $H_1(G; \mathbf{Z}_r) = \mathrm{Ker}\partial_1$, we denote $[\gamma] \in H_1(G; \mathbf{Z}_r)$ by γ for a cycle γ. Then a weight $\omega : \Gamma(G) \to \mathbf{Z}_r$ is said to be *balanced on e* if

$$\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma)\gamma = 0 \ \in H_1(G; \mathbf{Z}_r).$$

Note that this definition does not depend on the choice of the orientation of e.

Let e_1, e_2 be adjacent edges of G. We give an orientation to e_1. We give an orientation to each $\gamma \in \Gamma_{e_1,e_2}(G)$ by the orientation of e_1. Then a weight $\omega : \Gamma(G) \to \mathbf{Z}_r$ is said to be *balanced on* e_1, e_2 if

$$\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma)\gamma = 0 \ \in H_1(G;\mathbf{Z}_r).$$

We also note that this definition does not depend on the choice of the orientation of e_1. We remark here that a weight $\omega : \Gamma(G) \to \mathbf{Z}_r$ which is balanced on an edge e (resp. adjacent edges e_1, e_2) is weakly balanced on an edge e (resp. adjacent edges e_1, e_2), namely it is easy to see that if $\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma)\gamma = 0$ (resp. $\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma)\gamma = 0$) in $H_1(G;\mathbf{Z}_r)$ then $\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \equiv 0 \pmod r$ (resp. $\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma) \equiv 0 \pmod r$).

Lemma 3.5. (1) *Let e be an edge of G and $\omega : \Gamma(G) \to \mathbf{Z}_r$ a weight which is balanced on e. Let $f, g : G \to S^3$ be spatial embeddings such that g is obtained from f by a self-delta move on $f(e)$. Then $n_\omega(f) \equiv n_\omega(g) \pmod r$.*
(2) *Let e_1 and e_2 be adjacent edges of G and $\omega : \Gamma(G) \to \mathbf{Z}_r$ a weight which is balanced on e_1, e_2. Let $f, g : G \to S^3$ be spatial embeddings such that g is obtained from f by a quasi-adjacent delta move on $f(e_1)$ and $f(e_2)$. Then $n_\omega(f) \equiv n_\omega(g) \pmod r$.*

Proof. (1) Note that the orientation of $\gamma \in \Gamma_e(G)$ is given by the one of e. We may assume that $f(\gamma)$ and $g(\gamma)$ are as illustrated in Fig. 3.8 without loss of generality. Let $L_{f,g}(\gamma) = l^{(1)}_{f,g}(\gamma) \cup l^{(2)}_{f,g} \cup l^{(3)}_{f,g}$ be a 3-component link as the right-hand figure in Fig. 3.8. Note that $l^{(2)}_{f,g}$ and $l^{(3)}_{f,g}$ are common to all $\gamma \in \Gamma_e(G)$. Then we have

$$\begin{aligned}
&n_\omega(f) - n_\omega(g) \\
\equiv &\frac{1}{18}\sum_{\gamma \in \Gamma(G)} \omega(\gamma) V^{(3)}_{f(\gamma)}(1) - \frac{1}{18}\sum_{\gamma \in \Gamma(G)} \omega(\gamma) V^{(3)}_{g(\gamma)}(1) \\
= &\sum_{\gamma \in \Gamma(G)} \omega(\gamma) \left\{ \frac{1}{18} V^{(3)}_{f(\gamma)}(1) - \frac{1}{18} V^{(3)}_{g(\gamma)}(1) \right\}
\end{aligned}$$

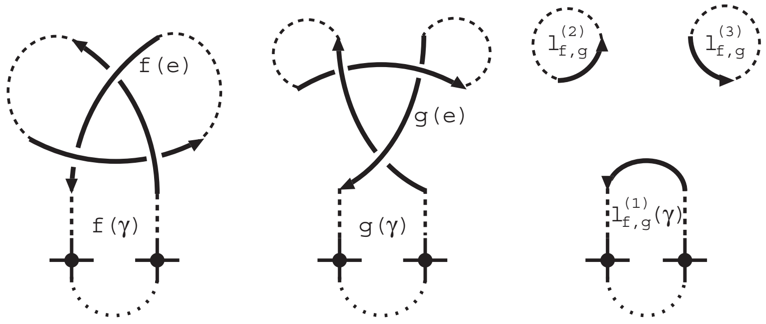


Fig. 3.8.

$$
\begin{aligned}
&= \sum_{\gamma\in\Gamma_e(G)} \omega(\gamma)\left\{\frac{1}{18}V^{(3)}_{f(\gamma)}(1) - \frac{1}{18}V^{(3)}_{g(\gamma)}(1)\right\} \\
&= \sum_{\gamma\in\Gamma_e(G)} \omega(\gamma)\left\{2lk(l^{(1)}_{f,g}(\gamma), l^{(2)}_{f,g}) + 2lk(l^{(1)}_{f,g}(\gamma), l^{(3)}_{f,g}) + 2lk(l^{(2)}_{f,g}, l^{(3)}_{f,g}) - 1\right\} \\
&= 2\sum_{k=2,3}\sum_{\gamma\in\Gamma_e(G)} \omega(\gamma) lk(l^{(1)}_{f,g}(\gamma), l^{(k)}_{f,g}) + \left\{2lk(l^{(2)}_{f,g}, l^{(3)}_{f,g}) - 1\right\}\sum_{\gamma\in\Gamma_e(G)}\omega(\gamma) \\
&\equiv 2\sum_{k=2,3} lk(\sum_{\gamma\in\Gamma_e(G)} \omega(\gamma) l^{(1)}_{f,g}(\gamma), l^{(k)}_{f,g}) \\
&\equiv 2\sum_{k=2,3} lk(0, l^{(k)}_{f,g}) \\
&\equiv 0 \pmod{r}.
\end{aligned}
$$

Thus we have the desired conclusion.
(2) We can complete the proof in a similar way as (1) (see Fig. 3.9). ■

Proof of Theorem 1.5. It is clear by Lemma 3.5. ■

4 Examples

Example 4.1. Let f_m be a θ-curve as illustrated in Fig. 4.1, where m is a non-negative integer. Note that f_i and f_j are delta vertex-homotopic for any i, j. It is easy to see that f_m contains a unique non-trivial knot J which is a connected sum of m trefoil knots for $m \neq 0$. Since $\sharp\Gamma_e(\theta) = 2$

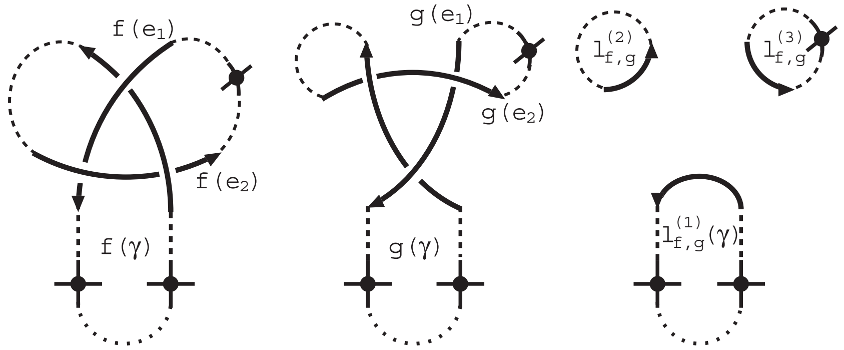


Fig. 3.9.

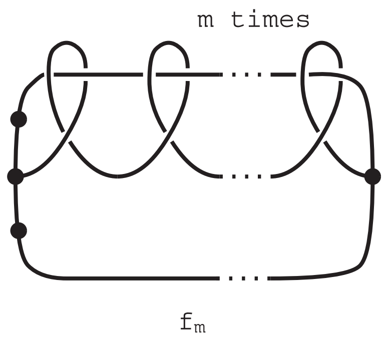


Fig. 4.1.

for any edge e, we have that $d_s = 2$. By a calculation we have that $a_2(J) = m$. Therefore we have that $\tilde{\alpha}_{\omega_s}(f_m) \equiv 0 \pmod 2$ if m is even and $1 \pmod 2$ if m is odd. Thus by Corollary 1.4 (1) we have that f_i and f_j are not delta edge-homotopic if the one of i, j is odd and the other is even. Especially f_m is not delta edge-homotopic to the trivial embedding f_0 if m is odd.

Example 4.2. (cf. [29, Example 3.1]) Let K_4 be the complete graph on 4 vertices (namely the graph G_2 as illustrated in Fig. 1.2). Let f_m and g be spatial embeddings of K_4 as illustrated in Fig. 4.2, where m is an integer. Note that f_m and g are delta vertex-homotopic (Undo the local Borromean ring in $f(K_4)$ by a quasi-adjacent delta move). It

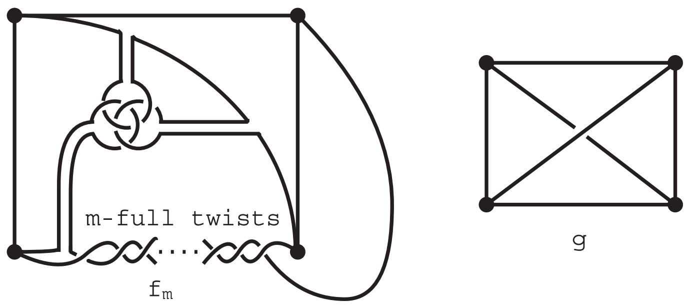


Fig. 4.2.

is easy to see that $f_m(K_4)$ contains two non-trivial knots J_1 and J_2 as illustrated in Fig. 4.3. Since $\sharp\Gamma_e(K_4) = 4$ for any edge $e \in E(K_4)$, we have that $d_s = 4$. By a calculation we have that $a_2(J_1) = a_2(J_2) = 1$. Therefore we have that $\tilde{\alpha}_{\omega_s}(f_m) \equiv 2 \pmod 4$ for any integer m. Since $\tilde{\alpha}_{\omega_s}(g) \equiv 0 \pmod 4$, by Theorem 1.4 (1) we have that f_m and g are not delta edge-homotopic for any integer m.

Let $\omega : \Gamma(K_4) \to \mathbf{Z}$ be a weight defined by

$$\omega(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a 4-cycle,} \\ -1 & \text{if } \gamma \text{ is a 3-cycle.} \end{cases}$$

Then we can see that ω is balanced on each of edges of K_4. Thus by Theorem 1.5 (1), n_ω is a delta edge-homotopy invariant. By Theorem 3.2 we have that $V^{(3)}_{J_1}(1) = 36m - 18$ and $V^{(3)}_{J_2}(1) = -18$. Therefore we have that $n_\omega(f_m) = 2m$. This shows that f_i and f_j are not delta edge-homotopic for $i \neq j$.

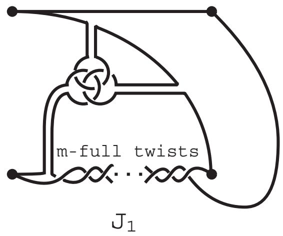


Fig. 4.3.

Example 4.3. (cf. [29, Example 3.2]) Let K_5 be the complete graph on 5 vertices. Let f_m be a spatial embedding of K_5 as illustrated in Fig. 4.4, where m is a non-negative integer. Note that f_i and f_j are edge-homotopic for any i, j (Undo the local Borromean ring in $f_m(K_5)$ by two self-crossing changes). It is easy to see that $f_m(K_5)$ contains six

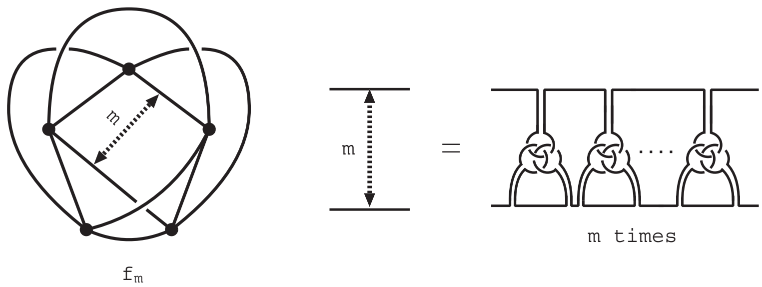


Fig. 4.4.

non-trivial knots J_k $(k = 1, 2, \ldots, 6)$ for $m \neq 0$ as illustrated in Fig. 4.5. Note that J_k is the image of a 4-cycle if $k = 1, 2$ and the image of a 5-cycle if $k = 3, 4, 5, 6$. Since $\sharp\Gamma_{e_1,e_2}(K_5) = 5$ for any adjacent edges $e_1, e_2 \in E(K_5)$, we have that $d_{ad} = 5$. By a calculation we have that $a_2(J_1) = a_2(J_3) = a_2(J_5) = m$ and $a_2(J_2) = a_2(J_4) = a_2(J_6) = -m$. Therefore we have that $\tilde{\alpha}_{\omega_{ad}}(f_m) \equiv 0 \pmod 5$. Since $\tilde{\alpha}_{\omega_{ad}}(g) \equiv 0$

(mod 5), We can not distinguish f_i and f_j for $i \neq j$ up to delta vertex-homotopy by using $\tilde{\alpha}_{\omega_{ad}}$.

Let $\omega : \Gamma(K_5) \to \mathbf{Z}$ be a weight defined by

$$\omega(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a 5-cycle,} \\ -1 & \text{if } \gamma \text{ is a 4-cycle and} \\ 0 & \text{if } \gamma \text{ is a 3-cycle.} \end{cases}$$

Then we can see that ω is balanced on each pair of adjacent edges of K_5. Thus by Theorem 1.5 (2), n_ω is a delta vertex-homotopy invariant. By Theorem 3.2 we have that $V_{J_k}^{(3)}(1) = -18m$ $(k = 1, 2, \ldots, 6)$. Therefore we have that $n_\omega(f_m) = -2m$. This shows that f_i and f_j are not delta vertex-homotopic for $i \neq j$.

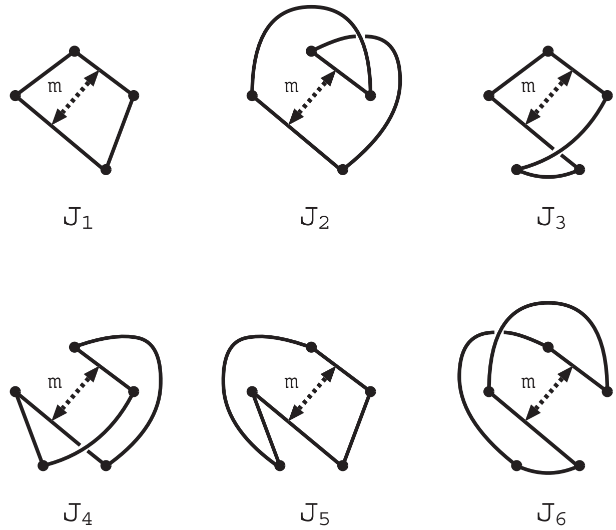


Fig. 4.5.

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