REVISTA MATEMÁTICA COMPLUTENSE ISSN 1139-1138
Vol. 15 Núm. 2 (2002), 533-542
http://dx.doi.org/10.5209/rev_REMA.2002.v15.n2.16919

REPRESENTING OPEN 3-MANIFOLDS AS 3-FOLD BRANCHED COVERINGS

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En recuerdo de mi padre, Lorenzo Montesinos

Abstract

It is proved that the Freudenthal compactification of an open, connected, oriented 3-manifold is a 3-fold branched covering of S^3, and in some cases, a 2-fold branched covering of S^3.The branching set is a locally finite disjoint union of strings.

La compactificación de Freudenthal de una 3-variedad abierta conexa y orientable es una cubierta de 3 hojas ramificada sobre S^3 y, en ciertos casos, de dos hojas. La ramificación es una unión localmente finita y disjunta de cuerdas.

1 Introduction

H. Hilden ([9],[10]) and the author ([16],[17]) proved independently that every closed, oriented 3-manifold is a 3-fold, dihedral covering of S^3, branched over a knot. Bobby Neals Winters asked me, some years ago, if such a result could be generalized to open 3-manifolds with the obvious restrictions on the base space. The purpose of this paper is to answer this question in the affirmative giving a proof of the following theorem.

Theorem 1. *Let M be an open, connected, oriented 3-manifold. Let $\widehat{M}$ denote its Freudenthal compactification.Then, there exist a 3-fold simple branched covering $p : \widehat{M} \to S^3$ such that p maps the end space $E(M)$ of M homeomorphically onto a tame subset T of S^3.The 3-fold branched covering $p \mid M : M \to S^3 - T$ is simple, and the branching set is a locally finite disjoint union of strings (properly embedded arcs).*

The number of sheets in the statement of Theorem 1 cannot be reduced to 2 (take $\widehat{M}$ to be any closed and oriented 3-manifold which

2000 Mathematics Subject Classification: 57M12, 57M30, 57N10.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2002

is not a 2-fold branched covering of S^3, [6] together with [4]; or [1]). However there are cases in which the Freudenthal compactification of an open 3-manifold is a 2-fold branched covering of S^3, and we will give a sufficient condition.

These results were announced in [19].

2 Some mixed preliminaries

Following Fox [5], we say that a space X is locally connected in a space Y if there is a basis of Y such that $V \cap X$ is connected for every basic open set V. Freudenthal [7] (see [5]) has shown that every connected, locally connected, locally compact, with base numerable, regular space X is contained in a connected, locally connected, compact, with base numerable, regular space Y in such a way that X is dense, open and locally connected in Y, and the *end space* $E(X) := Y - X$ is totally disconnected. Moreover, this compactification (Freudenthal compactification) is determined by these properties. If the space X is a locally finite contractible, connected 1-complex Γ(a *tree*) we can define an *end* as an injective simplicial map $\mathbf{e}$: $[0,\infty) \to \Gamma$, such that $\mathbf{e}(0) = v$, where v is a fixed base vertex of Γ, and the tree $[0,\infty)$ has some fixed simplicial structure. A *cofinal* of $\mathbf{e}$ will be $\mathbf{e}$ $([x,\infty))$ for some $x \geq 0$. An open neighbourhood of $\mathbf{e}$ will be the union of a connected component V of $\Gamma-$ (compact set), containing a cofinal of $\mathbf{e}$, together with the set of ends having cofinals in V.

Manifolds of dimension 2 and 3 in this paper will be separable metric spaces. Then, they are triangulated by locally finite simplicial complexes [14] (see also[15]). An *open 3-manifold* will be, in this paper, a non compact, connected, oriented, 3-manifold (with empty boundary). If M is an open 3-manifold we denote by $\widehat{M}$ its Freudenthal compactification and by $E(M)$ the *end space* $\widehat{M} - M$. The starting point for the proof of Theorem 1 is the following representation of open 3-manifolds due to Hoste [11] (compare[3] and Lemma 8 in [12]). If M is an open, connected, oriented 3-manifold there exist a sequence $\{M_1, M_2, ...\}$ of compact, connected submanifolds of M such that each M_i is contained in the interior of M_{i+1}; M is the union of the M_i's; and no two components of BdM_i (the boundary of M_i) can be joined by a path in the closure of $M - M_i$. Hoste [11] associates a locally finite tree Γ (or $\Gamma(M)$) to M by placing one vertex in each connected component of $M_i - IntM_{i-1}$for $i \geqq 1$, and joining two vertices with an edge whenever those two components share a common boundary. Note that the subspace $\Gamma + E(M)$ of $\widehat{M}$ is the Freudenthal compactification of the tree Γ. Therefore, the end space $E(\Gamma)$ of Γ is the same as the end space $E(M)$ of M.

A branched covering will be understood in the sense of Fox [5]. The *Compactification Theorem* of Fox [5],page 249, can be generalized easily giving the following useful condition:

Theorem 2. *Let $f : X \to B$ be a branched covering. Assume X and B are connected, locally connected, locally compact, with base numerable and regular, but no compact.Let $\widehat{B}$ be the Freudenthal compactification of B, and let j be the inclusion $j : B \subset \widehat{B}$. Let $g : Y \to \widehat{B}$ be the branched covering which is the Fox completion of $j \circ f : X \to \widehat{B}$. Then, Y is the Freudenthal compactification of X if $\widehat{B}$ has a basis such that, for each basic open set W, the number of components of $f^{-1}(W)$ is finite.*

3 A Lemma on compact 3-manifolds

The following Lemma is the building block to construct the proof of Theorem 1. It generalizes to compact 3-manifolds with boundary the Theorem of Hilden and the author refered to in the introduction. The proof will be an adaptation of the argument in [18].

Lemma 3. *Let X be a compact, oriented 3-manifold with n boundary components $\Sigma = \Sigma_1 + ... + \Sigma_n$.Then, there exists a 3-fold simple covering $p : X \to S^3_n$ branched over a set of disjoint arcs with their ends in the boundary of the n-punctured 3-sphere S^3_n, such that the restriction $p \mid \Sigma_i$ is a 3-fold simple branched covering onto the i-th boundary component S_i of S^3_n.*

Proof. First cap-off each boundary component Σ_i of X with a handlebody to obtain a closed, oriented 3-manifold Y. The handlebodies can be viewed as regular neighbourhoods of disjoint finite graphs G_i in Y. By a Theorem of Lickorish [13] and Wallace [20] (independently) there exists a link L in Y such that S^3 can be obtained by integral Dehn-surgery on L. Since $G = \cup_{i=1}^n G_i$ is 1-dimensional we may assume that L does not intersect G. Next, we use Hempel's trick [8] liberally to unknot and unlink the components of G and to unknot the components of L so that finally we arrive to the following surgery description of the pair (Y, G). In $S^3 = R^3 + \infty$ we place a graph $F = F_1 + ... + F_n$, where $\chi(F_i) = \chi(G_i)$, as follows. The component F_j of F is a union $C_j + T_j + H_j$ of a bouquet C_j of circles with common point $(a, b_j, 0)$, lying in the $(x, y)-$ plane and symmetric with respect to the plane $x = a$ (for some real number a), together with the"tail" $T_j = \{(x, b_j, 0) : -a \leq x \leq 2a\}$, together with the set of half circles H_j with common point $(-a, b_j, 0)$, lying in the $(x, y)-$ plane and symmetric with part of C_j with respect to the plane $x = 0$ (see Figure 1); the numbers $b_1 < b_2 < ... < b_n$ are suitably selected. (We remark that the number of half circles in H_j is the same as the number of circles in C_j.)

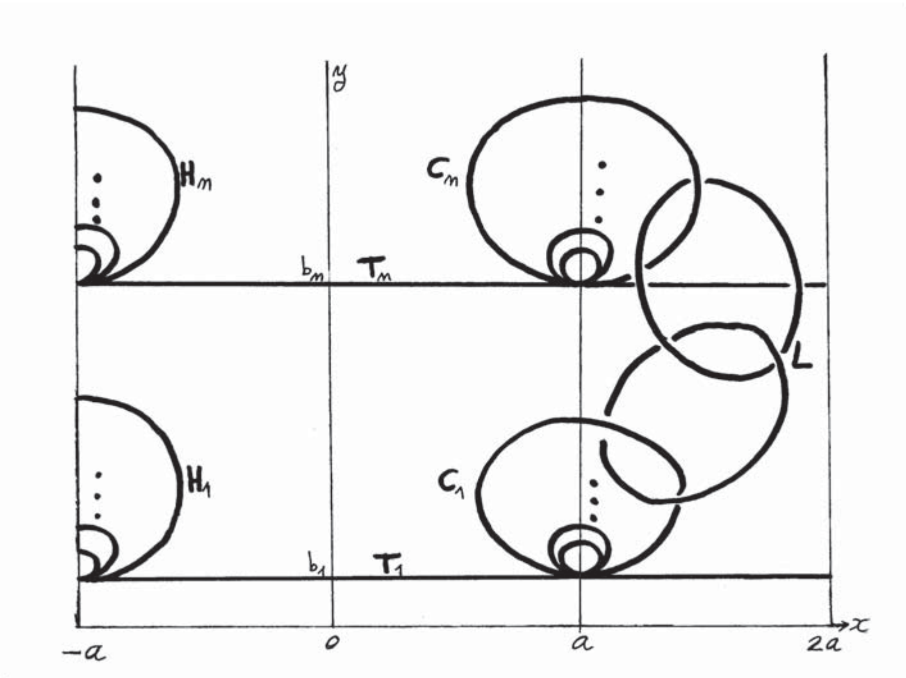

Figure 1

Next, there is a link L made of unknoted components, disjoint from F, whose projection in the $(x, y)-$ plane has only double points and lies entirely on the strip $\{z = 0, a < x < 2a\}$. Then (Y, G) is (S^3, F) in which some integral Dehn-surgery is performed in the link L. (The link

L depicted in Figure 1 exhibits all possible complications.)

Now we continue as in [18] but taking care of the extra-complication produced by the eventual linking between L and F. The details are left to the reader, but here are some guiding principles to the proof. The main point is to *symmetrize* the surgery instructions given by the framed link L,with respect to the standard 3-fold covering $f : S^3 \to S^3$, defined by folding S^3 around the axes $x = 0$, $x = 2a$ of Figure 1.

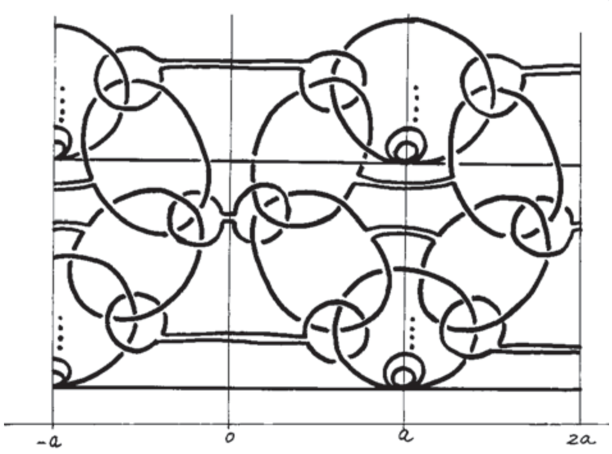

Figure 2a
The space X

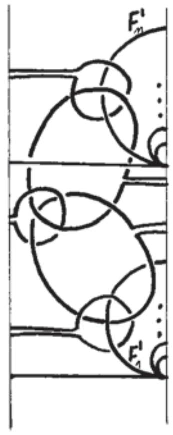

Figure 2b
The space S^3_n

This standard 3-fold covering $f : S^3 \to S^3$ is branched over the trivial link of two components depicted in part as the two vertical lines of Figure 2b. To achieve this symmetrization process we need to consider first the link $L_1 = f^{-1} \circ f(L)$. This link L_1 has three parts: *left part* L_{1l}, *middle part* $L_{1m}$and *right part* $L = L_{1r}$. Fuse each component of the right part L_{1r} with its symmetric partner in the middle part L_{1m} by the boundary of a band, so that the resulting trivial knot is symmetric with respect to the 180 degree rotation around the axis $x = a$. We get $(L_{1m} + L_{1r})$. Consider $L_2 = f^{-1} \circ f(L_{1m} + L_{1r})$. Then L_2 has two parts: the right part L_{2r} and the left part L_{2l}. The left part L_{2l} is a system of arcs (and is negligeable for reasons to be explained latter) and the right part L_{2r} is $(L_{1m} + L_{1r})$. This right part L_{2r} is not isotopic to the original link L due to the introduction of undesired linking between components of L_{1m} themselves and between components of L_{1m} and the circles of

$C_1 \cup ... \cup C_n$. We get rid off these undesired linking by applying again Hempel's trick along the little circles shown in the area $0 < x < a$ of Figure 2a. These circles K are to be made symmetric, but this time with respect to the 180 degree rotation around axis $x = 0$. As we did before with L, we consider $K_1 = f^{-1} \circ f(K)$. This link K_1 has three parts: *left part* K_{1l}, *middle part* $K = K_{1m}$ and *right part* K_{1r}. Fuse each component of the middle part K_{1m} with its symmetric partner in the left part K_{1l} by the boundary of a band, so that the resulting trivial knot is symmetric with respect to the 180 degree rotation around the axis $x = 0$. We get $(K_{1m} + K_{1l})$. Consider $K_2 = f^{-1} \circ f(K_{1m} + K_{1l})$. Then K_2 has two parts: the right part K_{2r} and the left part K_{2l}. The right part K_{2r} is a system of arcs (and is negligeable for reasons to be explained latter) and the left part K_{2l} is $(K_{1m} + K_{1l})$.This time K_{2l} is isotopic to K outside $L_2 \cup K_{2r} \cup F$. Adjust the framings of the Dehn surgeries in $L_{2r} \cup K_{2l}$ so that they provide a surgery description of the pair (Y, G). The projection $f(L_{2r} \cup K_{2l})$ is shown in Figure 2b, as a system of arcs with their end-points lying on the branching set of the standard 3-fold covering f. Suitable modifications inside regular neighbourhoods of these arcs are lifted to desired Dehn-surgeries in $L_{2r} \cup K_{2l}$, and negligeable modifications inside regular neighbourhoods of the arcs $(L_{2l} \cup K_{2r})$.(Modifications inside regular neighbourhoods of arcs are called *negligeable* because they have no effect on the topology of the pair (Y, G).) In this way the branched covering $p : X \to S^3_n$ is constructed. Figure 2 shows the construction of the branched covering $p : X \to S^3_n$ for the example of Figure 1.

Note that the graph F is projected under the standard 3-fold covering in the disjoint union F' of n trees. Therefore this standard 3-fold covering restricted to $S^3 - U(F)$ has base space $S^3 - U(F') \cong S^3_n$. To complete the proof we have to show that the branching set of the covering can be converted into a number of arcs running from component to component of BdS^3_n. There are exactly $g_1 + g_2 + ... + g_n + 2n$ arcs in the branching set, where $g_i = genus(\Sigma_i)$. This is because the branched covering restricted to Σ_i is a 3-fold simple covering of S_i with $2g_i + 4$ points of ramification.If the branching set of $p : X \to S^3_n$ contains some knots, they can be connected to the strings using moves as in [18]. This completes the proof of the Lemma.

4 Proof of Theorem 1

Let Γ be a tree representing the open, connected, oriented 3-manifold M. Thus, to each vertex v of Γ is associated a compact, connected, oriented 3-manifold X_v. The boundary components of X_v are in one to one correspondence with the edges of Γ touching v. To each edge e of Γ we associate an orientation reversing homeomorphism $f_e : \Sigma_{v,e} \to \Sigma_{w,e}$ between the corresponding boundary components $\Sigma_{v,e}$ of X_v and $\Sigma_{w,e}$ of X_w.The manifold M is obtained by pasting together the pieces X_v by means of the homeomorphisms f_e.

The same tree Γ gives rise to another open, connected, oriented 3-manifold S^3_* as follows. To each vertex v of Γ of valence n_v corresponds a copy S^3_v of the n_v-punctured 3-sphere. These pieces are pasted together by cylinders $C^3_e = S^2 \times [0,1]$ corresponding to the edges of Γ. (We introduce these cylinders for technical reasons.) It is not hard to see that the Freudenthal compactification of S^3_* is S^3 and that the end space T is tamely embedded. In fact the Freudenthal compactification of Γ can be embedded in R^2 in such a way that its end space T lies in a straigth line (see [11]). In Figure 3 we see an example.

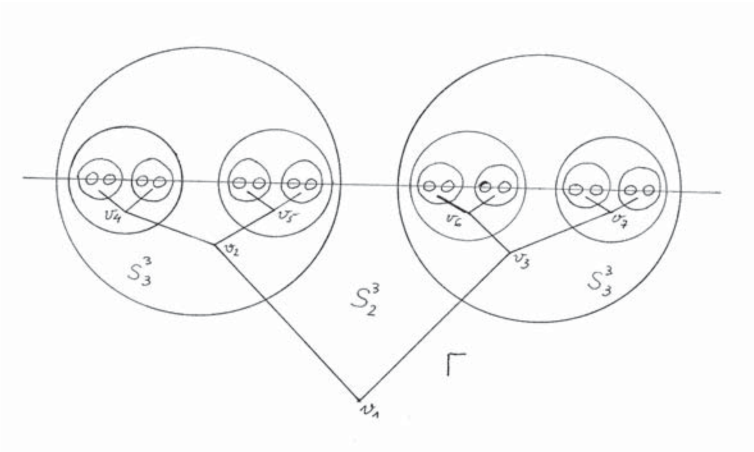

Figure 3

According to Lemma 3, for each vertex v of Γ we have a 3-fold simple covering $p_v : X_v \to S^3_v$ branched over a disjoint union of arcs. Therefore, for each edge e we have two 3-fold simple branched coverings obtained by restriction, namely $p_v \mid \Sigma_{v,e} : \Sigma_{v,e} \to S_{v,e}$ and $p_w \mid \Sigma_{w,e} : \Sigma_{w,e} \to S_{w,e}$, where the spheres $S_{v,e}$ and $S_{w,e}$ are the boundary components of the cylinder C_e. According to a Theorem of Hilden [10] there exists an orientation reversing homeomorphism $b_e : S_{v,e} \to S_{w,e}$ such that $(p_w \mid \Sigma_{w,e}) \circ f_e = b_e \circ (p_v \mid \Sigma_{v,e})$. We therefore can paste together the branched coverings p_v to obtain a 3-fold simple branched covering with total space M and base space S^3_*. Its branching set is obtained by sewing together the branching sets of the different p_v's by means of the braids associated to the homeomorphisms b_e realized in the cylinders C_e. If this branching set contains knots, they can be connected to suitable strings by using moves as in the proof of Lemma 3. In this way we have constructed a 3-fold simple branched covering $p' : M \to S^3_*$ with branching set a locally finite collection of disjoint strings.

Consider the inclusion $j : S^3 \subset S^3_*$. Let $p : M' \to S^3$ be the Fox completion of $j \circ p' : M \to S^3$. The preimage $p^{-1}(t)$, $t \in T$, in this branched covering, is composed precisely of one point because for every open connected neighbourhood V of t in S^3 the preimage $(j \circ p')^{-1}V$ is connected, as it is evident from the construction of the branched covering p'. From this, and Theorem 2, it follows immediately that M' is in fact the Freudenthal compactification of M and that p sends the end space $E(M)$ of M homeomorphically onto the tame subset T of S^3. Of course, the branched index of $p^{-1}(t)$, $t \in T$, is 3. This ends the proof of Theorem 1.

Corollary 4. *Let M be an open, connected, oriented 3-manifold with just one end. Then there exist a 3-fold branched covering $p : M \to R^3$ onto Euclidean 3-space branched upon a locally finite disjoint union of strings.*

This is the case of the uncountably many open, contractible 3-manifolds. In a forthcoming paper we will deal with some concrete examples.

5 2-fold coverings

In some cases it is possible to prove that a particular open 3-manifold is a 2-fold branched covering of $S^3 - T$.

Theorem 5. *Let M be an open, connected, oriented 3-manifold such that each vertex v of the tree $\Gamma(M)$ corresponds to a compact, oriented*

3-manifold X_v such that (i) BdX_v is composed of n_v connected components of genus $\leqq 2$, and (ii) X_v is a 2-fold branched covering of the 3-sphere minus the interior of n_v disjoint 3-balls. Then, there exist a 2-fold branched covering $p : \widehat{M} \to S^3$ such that p maps the end space $E(M)$ of M homeomorphically onto a tame subset T of S^3. The branching set of the 2-fold branched covering $p \mid M : M \to S^3 - T$ is a locally finite disjoint union of knots and strings.

Proof. The 2-fold branched coverings $p_v : X_v \to S^3_v$ can be pasted together. In fact, if $f : \Sigma \to S$ is a 2-fold branched covering of a surface Σ of genus $\leqq 2$ onto the 2-sphere S, every homeomorphism of Σ is the lifting of some homeomorphism of S, up to isotopy [2].

Corollary 6. *The open contractible Whitehead manifold is a 2-fold covering of R^3 branched over a string.*

In a forthcoming paper we will offer a number of examples.

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Recibido: 3 de Diciembre de 2001
Revisado: 8 de Mayo de 2002