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# ON THE NONSQUARE CONSTANTS OF $L^{(\Phi)}[0,+\infty)$

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**Abstract**

Let $L^{(\Phi)}[0,+\infty)$ be the Orlicz function space generated by $N-$function $\Phi(u)$ with Luxemburg norm. We show the exact nonsquare constant of it when the right derivative $\phi(t)$ of $\Phi(u)$ is convex or concave.

## 1 Introduction

Let $X$ be a Banach space and $S(X) = \{x : \|x\| = 1, x \in X\}$ denotes the unit sphere of $X$. The nonsquare constants in the sense of James $J(X)$ and in the sense of Schaffer $g(X)$ are defined as:

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S(X)\}, \tag{1}$$

$$g(X) = \inf\{\max(\|x+y\|, \|x-y\|) : x, y \in S(X)\}. \tag{2}$$

Clearly, if $\dim X \geq 2$, then $1 \leq g(X) \leq \sqrt{2} \leq J(X) \leq 2$. Ji and Wang [5] asserted

$$g(X) \cdot J(X) = 2 \tag{3}$$

for $\dim X \geq 2$.

It is proved[1] that $J(X) = 2$ if $X$ fails to be reflexive. Nonsquareness is an important geometric property of Banach spaces which expose the intrinsic construction of a space according to the "shape" of the unit ball of the spaces. Therefore, it is interesting to investigate it in classical Banach spaces, for example, Orlicz spaces. It is showed[5] that $J(L^p) = \max(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}})(1 < p < \infty)$. However, examples for values of $J(X)$ for $X$ to be reflexive except $L^p$ remains unknown. In this paper, we deal with $J(X)$ when $X$ is an Orlicz function space with Luxemburg norm.

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Let $\Phi(u) = \int_0^{|u|} \phi(t)dt$ be an $N-$function ,i.e., $\phi(0) = 0, \phi$ is right continuous and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$. The Orlicz function space $L^{(\Phi)}[0,\infty)$ is defined to be the set

$$L^{(\Phi)}[0,\infty) = \left\{ x(t) : \rho_\Phi(\lambda x) = \int_{[0,\infty)} \Phi(\lambda |x(t)|)dt < \infty \text{ for some } \lambda > 0 \right\}.$$

The Luxemburg norm is expressed as

$$\|x\|_{(\Phi)} = \inf \left\{ c > 0 : \rho_\Phi(\frac{x}{c}) \leq 1 \right\}.$$

$\Phi(u)$ is said to satisfy the $\Delta_2-$condition for for all $u \geq 0$, in symbol $\Phi \in \Delta_2$, if there exists $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for $u \geq 0$. In what follows, we will frequently use Semenove indices of $\Phi(u)$:

$$\overline{\alpha}_\Phi = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \overline{\beta}_\Phi = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}. \tag{4}$$

## 2 Main Results

We first consider the lower bounds of $L^{(\Phi)}[0,\infty)$.The following result is refined from Ren[8].

**Theorem 1.** *Let $\Phi(u)$ be an $N-$function. Then the nonsquare constant of $L^{(\Phi)}[0,\infty)$, in the sense of James, satisfies*

$$\max\left(\frac{1}{\overline{\alpha}_\Phi}, 2\overline{\beta}_\Phi\right) \leq J(L^{(\Phi)}[0,\infty)). \tag{5}$$

**Proof.** To prove (5), we first show

$$\frac{1}{\overline{\alpha}_\Phi} \leq J(L^{(\Phi)}[0,\infty)). \tag{6}$$

Take a real number $u \in (0,\infty)$, choose measurable subsets $G_1$ and $G_2$ in $[0,\infty)$ such that $G_1 \cap G_2 = \emptyset$. and $\mu(G_1) = \mu(G_2) = \frac{1}{2u}$. Put

$$x(t) = \Phi^{-1}(2u)\chi_{G_1}(t) \text{ and } y(t) = \Phi^{-1}(2u)\chi_{G_2}(t),$$

where $\chi_{G_1}$ is the characteristic function of $G_1$. Note that

$$\|\chi_{G_1}\|_{(\Phi)} = \|\chi_{G_2}\|_{(\Phi)} = \frac{1}{\Phi^{-1}(\frac{1}{\mu(G_1)})} = \frac{1}{\Phi^{-1}(2u)}.$$

We have $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x-y\|_{(\Phi)} = \|x+y\|_{(\Phi)} = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}.$$

Taking the supremum over $u \in (0,\infty)$, since the function $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is right continuous at 0 and takes value on $[\frac{1}{2},1]$, we deduce that

$$J(L^{(\Phi)}[0,\infty)) \geq \sup_{u\in(0,\infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \sup_{u\in[0,\infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \frac{1}{\overline{\alpha}_\Phi}.$$

Finally we show

$$2\overline{\beta}_\Phi \leq J(L^{(\Phi)}[0,\infty)). \tag{7}$$

For every real number $v > 0$, choose measurable subsets $E_1, E_2$ in $[0,\infty)$ such that $E_1 \cap E_2 = \emptyset$ and $\mu(E_1) = \mu(E_2) = \frac{1}{2v}$. Put

$$x(t) = \Phi^{-1}(v)[\chi_{E_1}(t) + \chi_{E_2}(t)] \text{ and } y(t) = \Phi^{-1}(v)[\chi_{E_1}(t) - \chi_{E_2}(t)],$$

Then $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x-y\|_{(\Phi)} = \|x+y\|_{(\Phi)} = \frac{2\Phi^{-1}(v)}{\Phi^{-1}(2v)}.$$

Taking the supremum over $v \in (0,\infty)$ we also have

$$J(L^{(\Phi)}[0,\infty)) \geq 2\overline{\beta}_\Phi.$$

Hence (5) follows from (6) and (7). ∎

Assume $\Phi$ satisfies $\Delta_2-$condition for all $u$. Ji and Wang([5],Theorem 3) offered a couple of formulas:

(i) If $\phi(t)$ is a concave function, then

$$g(L^{(\Phi)}[0,\infty)) = \inf\left\{k_x > 0 : \rho_\Phi(\frac{2x}{k_x}) = 2, \rho_\Phi(x) = 1\right\}; \tag{8}$$

(ii) If $\phi(t)$ is convex, then

$$J(L^{(\Phi)}[0,\infty)) = \sup\left\{k_x > 0 : \rho_\Phi(\frac{2x}{k_x}) = 2, \rho_\Phi(x) = 1\right\}. \tag{9}$$

We now extend the above representatives and deduce the upper bounds.

**Theorem 2.** *Suppose $\phi(t)$ be the right derivative of $\Phi(u)$. We have*

(i) *If $\phi(u)$ is concave , then*

$$J(L^{(\Phi)}[0,\infty)) \leq \frac{1}{\overline{\alpha}_\Phi}; \tag{10}$$

(ii)*If $\phi(u)$ is convex , then*

$$J(L^{(\Phi)}[0,\infty)) \leq 2\overline{\beta}_\Phi. \tag{11}$$

**Proof.** If $\Phi \notin \Delta_2$, which is equivalent to $\overline{\beta}_\Phi = 1$, then $L^{(\Phi)}[0,\infty)$ is nonreflexive and hence $J(L^{(\Phi)}[0,\infty)) = 2$ according to the results in Chen[1] or Hudzik[4]. Since $\phi(t)$ is concave implies $\Phi \in \Delta_2$(see Krasnoselskiĭ and Rutickiĭ[6],p.26), we only need to check (11) when $\phi(t)$ is convex, but this is trivial since $J(l^{(\Phi)}) = 2 = 2\beta^0_\Phi = 2\tilde{\beta}_\Phi$. Therefore it suffices for us to prove (10) and (11) for $\Phi \in \Delta_2$.

We first prove (10) for $\Phi(u) \in \Delta_2$, which is equal to

$$g(L^{(\Phi)}[0,\infty)) \geq 2\overline{\alpha}_\Phi \tag{12}$$

when $\phi(t)$ is concave in view of (3) and (8).

Let $H_\Phi(u) = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}$, then $\Phi^{-1}(2u) = H_\Phi(u) \cdot \Phi^{-1}(u)$. Put $x = \Phi^{-1}(u)$, then $u = \Phi(x)$ and

$$2\Phi(x) = \Phi[H_\Phi(\Phi(x)) \cdot x]. \tag{13}$$

Therefore, when $u = \Phi(x(t)) \geq 0$ we have

$$\begin{aligned}\rho_\Phi\left(\tfrac{2x(t)}{2\overline{\alpha}_\Phi}\right) &= \rho_\Phi\left(\tfrac{x(t)}{\overline{\alpha}_\Phi}\right) \geq \rho_\Phi\left(\tfrac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} \cdot x(t)\right)\\ &= \rho_\Phi[H_\Phi(u) \cdot x(t)] = 2\rho_\Phi(x(t)) = 2\end{aligned}$$

for $\rho_\Phi(x(t)) = 1$. It follows that (12) and hence (10) holds.

One can prove (11) analogously by (9). ■

We obtain the main result from the above theorems:

**Theorem 3.** *Let $\Phi(u)$ be an $N-$function, $\phi(t)$ be the right derivative of $\Phi(u)$. Then*

(i)*If $\phi(t)$ is concave , then*

$$J(L^{(\Phi)}[0,\infty)) = \frac{1}{\overline{\alpha}_\Phi}; \tag{14}$$

(ii)*If $\phi(t)$ is convex , then*

$$J(L^{(\Phi)}[0,\infty)) = 2\overline{\beta}_\Phi. \tag{15}$$

**Remark 4.** If the index function $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is decreasing or increasing on interval $[0,\infty)$, then the indices $\overline{\alpha}_\Phi$ and $\overline{\beta}_\Phi$ take the values at either end of the interval. The author[10] found that if $F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}$ is increasing(decreasing) on $(0, \Phi^{-1}(u_0)]$ then $G_\Phi(u)$ is also increasing(decreasing) on $(0, \frac{u_0}{2}]$, respectively. Rao and Ren[7] gave interrelations between Semenove and Simonenko indices:

$$2^{-\frac{1}{A_\Phi}} \le \alpha_\Phi \le \beta_\Phi \le 2^{-\frac{1}{B_\Phi}}, \qquad 2^{-\frac{1}{A^0_\Phi}} \le \alpha^0_\Phi \le \beta^0_\Phi \le 2^{-\frac{1}{B^0_\Phi}},$$

where

$$A_\Phi = \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}, \qquad B_\Phi = \limsup_{t\to\infty} \frac{t\phi(t)}{\Phi(t)};$$

$$A^0_\Phi = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)}, \qquad B^0_\Phi = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)};$$

and

$$\alpha_\Phi = \liminf_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)};$$

$$\alpha^0_\Phi = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta^0_\Phi = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

When the index function $F_\Phi(t)$ is monotonic, the limits $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$ and $C_\Phi^0 = \lim_{t\to 0} F_\Phi(t)$ must exist and we have

$$\alpha_\Phi = \beta_\Phi = \lim_{u\to\infty} G_\Phi(u) = 2^{-\frac{1}{C_\Phi}}, \qquad \alpha_\Phi^0 = \beta_\Phi^0 = \lim_{u\to 0} G_\Phi(u) = 2^{-\frac{1}{C_\Phi^0}}. \tag{16}$$

This makes it easier to calculate the indices in Theorem 3.

**Example 5.** Observe the $N-$function(see Gallardo[2])

$$\Phi_{p,r}(u) = |u|^p \ln^r(1+|u|), \qquad 1 \le p < \infty, 0 < r < \infty.$$

It is easy to check the right derivative of $\Phi_{p,r}(u)$, $\phi(t)$ is convex when $1 \le p < \infty, 2 \le r < \infty$. The index function

$$F_{\Phi_{p,r}}(t) = \frac{t\Phi'_{p,r}(t)}{\Phi_{p,r}(t)} = p + \frac{rt}{(1+t)\ln(1+t)}$$

is decreasing from $p+r$ to $p$ on $[0,\infty)$ since

$$\frac{d}{dt}\Phi_{p,r}(t) = \frac{r[\ln(1+t)-t]}{(1+t)^2\ln^2(1+t)} < 0.$$

So $C^0_{\Phi_{p,r}}(t) = \lim_{t\to 0} F_{\Phi_{p,r}}(t) = p+r$. According to (16) in the above remark and Theorem 3 we have

$$J(L^{(\Phi_{p,r})}[0,\infty)) = 2\overline{\beta}_{\Phi_{p,r}} = 2\beta^0_{\Phi_{p,r}} = 2\cdot 2^{-\frac{1}{p+r}} = 2^{1-\frac{1}{p+r}}. \tag{17}$$

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