ON THE NONSQUARE CONSTANTS OF 
$L(\Phi)[0, +\infty)$

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Abstract

Let $L(\Phi)[0, +\infty)$ be the Orlicz function space generated by $N-$function $\Phi(u)$ with Luxemburg norm. We show the exact nonsquare constant of it when the right derivative $\phi(t)$ of $\Phi(u)$ is convex or concave.

1 Introduction

Let $X$ be a Banach space and $S(X) = \{x : \|x\| = 1, x \in X\}$ denotes the unit sphere of $X$. The nonsquare constants in the sense of James $J(X)$ and in the sense of Schaffer $g(X)$ are defined as:

\[ J(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in S(X)\}, \]

\[ g(X) = \inf\{\max(\|x + y\|, \|x - y\|) : x, y \in S(X)\}. \]

Clearly, if $\dim X \geq 2$, then $1 \leq g(X) \leq \sqrt{2} \leq J(X) \leq 2$. Ji and Wang [5] asserted

\[ g(X) \cdot J(X) = 2 \]

for $\dim X \geq 2$.

It is proved[1] that $J(X) = 2$ if $X$ fails to be reflexive. Nonsquareness is an important geometric property of Banach spaces which expose the intrinsic construction of a space according to the “shape” of the unit ball of the spaces. Therefore, it is interesting to investigate it in classical Banach spaces, for example, Orlicz spaces. It is showed[5] that $J(L^p) = \max(2^{1/2}, 2^{1/2 - 1/p})(1 < p < \infty)$. However, examples for values of $J(X)$ for $X$ to be reflexive except $L^p$ remains unknown. In this paper, we deal with $J(X)$ when $X$ is an Orlicz function space with Luxemburg norm.

2000 Mathematics Subject Classification: 46E30.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2002

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Let $\Phi(u) = \int_0^u \phi(t)dt$ be an $N$–function, i.e., $\phi(0) = 0$, $\phi$ is right continuous and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$. The Orlicz function space $L(\Phi)(0, \infty)$ is defined to be the set

$$L(\Phi)(0, \infty) = \left\{ x(t) : \rho_\Phi(\lambda x) = \int_0^\infty \Phi(\lambda|x(t)|)dt < \infty \text{ for some } \lambda > 0 \right\}.$$  

The Luxemburg norm is expressed as

$$\|x\|_{\Phi} = \inf \left\{ c > 0 : \rho_\Phi\left(\frac{x}{c}\right) \leq 1 \right\}.$$  

$\Phi(u)$ is said to satisfy the $\Delta_2$–condition for any $u \geq 0$, in symbol $\Phi \in \Delta_2$, if there exists $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for $u \geq 0$. In what follows, we will frequently use Semenov indices of $\Phi(u)$:

$$\alpha_{\Phi} = \inf_{u > 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_{\Phi} = \sup_{u > 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$  

(4)

2 Main Results

We first consider the lower bounds of $L(\Phi)(0, \infty)$. The following result is refined from Ren[8].

**Theorem 1.** Let $\Phi(u)$ be an $N$–function. Then the nonsquare constant of $L(\Phi)(0, \infty)$, in the sense of James, satisfies

$$\max\left(\frac{1}{\alpha_{\Phi}}, 2\beta_{\Phi}\right) \leq J(L(\Phi)(0, \infty)).$$  

(5)

**Proof.** To prove (5), we first show

$$\frac{1}{\alpha_{\Phi}} \leq J(L(\Phi)(0, \infty)).$$  

(6)

Take a real number $u \in (0, \infty)$, choose measurable subsets $G_1$ and $G_2$ in $[0, \infty)$ such that $G_1 \cap G_2 = \emptyset$ and $\mu(G_1) = \mu(G_2) = \frac{1}{2u}$. Put

$$x(t) = \Phi^{-1}(2u)\chi_{G_1}(t) \text{ and } y(t) = \Phi^{-1}(2u)\chi_{G_2}(t),$$

where $\chi_A(t)$ is the characteristic function of the set $A$. Then

$$\rho_\Phi(\lambda x) = \int_0^\infty \Phi(\lambda\chi_{G_1}(t))dt < \infty \quad \text{for some } \lambda > 0,$$

and

$$\rho_\Phi(\lambda y) = \int_0^\infty \Phi(\lambda\chi_{G_2}(t))dt < \infty \quad \text{for some } \lambda > 0,$$

so that $x(t)$ and $y(t)$ are in $L(\Phi)(0, \infty)$. It is easy to verify that

$$\frac{1}{\alpha_{\Phi}} = \frac{1}{\inf_{u > 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}} = \frac{1}{\alpha_{\Phi}}$$

and

$$2\beta_{\Phi} = \sup_{u > 0} \frac{2\Phi^{-1}(u)}{\Phi^{-1}(2u)} = \beta_{\Phi}.$$
where $\chi_{G_1}$ is the characteristic function of $G_1$. Note that
\[ \| \chi_{G_1} \|_{(\Phi)} = \| \chi_{G_2} \|_{(\Phi)} = \frac{1}{\Phi^{-1}(\frac{1}{\mu(G_1)})} = \frac{1}{\Phi^{-1}(2u)}. \]
We have $\| x \|_{(\Phi)} = \| y \|_{(\Phi)} = 1$ and
\[ \| x - y \|_{(\Phi)} = \| x + y \|_{(\Phi)} = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}. \]
Taking the supremum over $u \in (0, \infty)$, since the function $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is right continuous at 0 and takes value on $[\frac{1}{2}, 1]$, we deduce that
\[ J(L^{(\Phi)}[0, \infty)) \geq \sup_{u \in (0, \infty)} \Phi^{-1}(2u) = \sup_{u \in (0, \infty)} \Phi^{-1}(2u) = \frac{1}{\delta_\Phi}. \]
Finally we show
\[ 2\beta_\Phi \leq J(L^{(\Phi)}[0, \infty)). \quad (7) \]
For every real number $v > 0$, choose measurable subsets $E_1, E_2$ in $(0, \infty)$ such that $E_1 \cap E_2 = \emptyset$ and $\mu(E_1) = \mu(E_2) = \frac{1}{2v}$. Put
\[ x(t) = \Phi^{-1}(v)[\chi_{E_1}(t) + \chi_{E_2}(t)] \text{ and } y(t) = \Phi^{-1}(v)[\chi_{E_1}(t) - \chi_{E_2}(t)], \]
Then $\| x \|_{(\Phi)} = \| y \|_{(\Phi)} = 1$ and
\[ \| x - y \|_{(\Phi)} = \| x + y \|_{(\Phi)} = \frac{2\Phi^{-1}(v)}{\Phi^{-1}(2v)}. \]
Taking the supremum over $v \in (0, \infty)$ we also have
\[ J(L^{(\Phi)}[0, \infty)) \geq 2\beta_\Phi. \]
Hence (5) follows from (6) and (7). 

Assume $\Phi$ satisfies $\Delta_2$–condition for all $u$. Ji and Wang([5],Theorem 3) offered a couple of formulas:
(i) If $\phi(t)$ is a concave function, then
\[ g(L^{(\Phi)}[0, \infty)) = \inf \left\{ k_x > 0 : \rho_\phi\left(\frac{2x}{k_x}\right) = 2, \rho_\phi(x) = 1 \right\}; \quad (8) \]
(ii) If $\phi(t)$ is convex, then

$$J(L^\Phi([0, \infty))) = \sup \left\{ k_\phi > 0 : \rho_\phi \left( \frac{2x}{k_\phi} \right) = 2, \rho_\phi(x) = 1 \right\}. \quad (9)$$

We now extend the above representatives and deduce the upper bounds.

**Theorem 2.** Suppose $\phi(t)$ be the right derivative of $\Phi(u)$. We have

(i) If $\phi(u)$ is concave, then

$$J(L^\Phi([0, \infty))) \leq \frac{1}{\alpha_\Phi}; \quad (10)$$

(ii) If $\phi(u)$ is convex, then

$$J(L^\Phi([0, \infty))) \leq 2\beta_\Phi. \quad (11)$$

**Proof.** If $\Phi \notin \Delta_2$, which is equivalent to $\beta_\Phi = 1$, then $L^\Phi([0, \infty))$ is nonreflexive and hence $J(L^\Phi([0, \infty))) = 2$ according to the results in Chen[1], or Hudzik[4]. Since $\phi(t)$ is concave implies $\Phi \in \Delta_2$ (see Krasnoselskii and Rutickii[6], p.26), we only need to check (11) when $\phi(t)$ is convex, but this is trivial since $J(t^\Phi) = 2 = 2\beta_\Phi = 2\beta_\Phi$. Therefore it suffices for us to prove (10) and (11) for $\Phi \in \Delta_2$.

We first prove (10) for $\Phi(u) \in \Delta_2$, which is equal to

$$g(L^\Phi([0, \infty))) \geq 2\pi_\Phi$$

when $\phi(t)$ is concave in view of (3) and (8).

Let $H_\Phi(u) = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}$, then $\Phi^{-1}(2u) = H_\Phi(u) \cdot \Phi^{-1}(u)$. Put $x = \Phi^{-1}(u)$, then $u = \Phi(x)$ and

$$2\Phi(x) = \Phi[H_\Phi(\Phi(x)) \cdot x]. \quad (13)$$

Therefore, when $u = \Phi(x(t)) \geq 0$ we have

$$\rho_\Phi \left( \frac{2x(t)}{2\pi_\Phi} \right) = \rho_\Phi \left( \frac{x(t)}{\pi_\Phi} \right) \geq \rho_\Phi \left( \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} \cdot x(t) \right) = \rho_\Phi[H_\Phi(u) \cdot x(t)] = 2\rho_\Phi(x(t)) = 2$$
for \( \rho_{\Phi}(x(t)) = 1 \). It follows that (12) and hence (10) holds.

One can prove (11) analogously by (9).

We obtain the main result from the above theorems:

**Theorem 3.** Let \( \Phi(u) \) be an \( N \)-function, \( \phi(t) \) be the right derivative of \( \Phi(u) \). Then

(i) If \( \phi(t) \) is concave, then

\[
J(L^{(\Phi)}[0, \infty)) = \frac{1}{\alpha_{\Phi}};
\]

(ii) If \( \phi(t) \) is convex, then

\[
J(L^{(\Phi)}[0, \infty)) = 2\beta_{\Phi}.
\]

**Remark 4.** If the index function \( G_{\Phi}(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} \) is decreasing or increasing on interval \([0, \infty)\), then the indices \( \alpha_{\Phi} \) and \( \beta_{\Phi} \) take the values at either end of the interval. The author[10] found that if \( F_{\Phi}(t) = \frac{t\phi(t)}{\Phi(t)} \) is increasing(decreasing) on \((0, \Phi^{-1}(u_0)]\), then \( G_{\Phi}(u) \) is also increasing(decreasing) on \((0, \frac{u_0}{2})\), respectively. Rao and Ren[7] gave interrelations between Semenove and Simonenko indices:

\[
2^{-\frac{1}{\pi_{\Phi}}} \leq \alpha_{\Phi} \leq \beta_{\Phi} \leq 2^{-\frac{1}{\pi_{\Phi}}}, \quad 2^{-\frac{1}{\pi_{\Phi}^0}} \leq \alpha_{\Phi}^0 \leq \beta_{\Phi}^0 \leq 2^{-\frac{1}{\pi_{\Phi}^0}},
\]

where

\[
A_{\Phi} = \liminf_{t \to \infty} \frac{t\phi(t)}{\Phi(t)}, \quad B_{\Phi} = \limsup_{t \to \infty} \frac{t\phi(t)}{\Phi(t)};
\]

\[
A_{\Phi}^0 = \liminf_{t \to 0} \frac{t\phi(t)}{\Phi(t)}, \quad B_{\Phi}^0 = \limsup_{t \to 0} \frac{t\phi(t)}{\Phi(t)};
\]

and

\[
\alpha_{\Phi} = \liminf_{u \to -\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}; \quad \beta_{\Phi} = \limsup_{u \to -\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)};
\]

\[
\alpha_{\Phi}^0 = \liminf_{u \to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}; \quad \beta_{\Phi}^0 = \limsup_{u \to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.
\]
When the index function \( F_\Phi(t) \) is monotonic, the limits \( C_\Phi = \lim_{t \to \infty} F_\Phi(t) \) and \( C_\Phi^0 = \lim_{t \to 0} F_\Phi(t) \) must exist and we have

\[
\alpha_\Phi = \beta_\Phi = \lim_{u \to -\infty} G_\Phi(u) = 2^{-\frac{1}{\alpha_\Phi}}, \quad \alpha_\Phi^0 = \beta_\Phi^0 = \lim_{u \to 0} G_\Phi(u) = 2^{-\frac{1}{\alpha_\Phi^0}}. \quad (16)
\]

This makes it easier to calculate the indices in Theorem 3.

**Example 5.** Observe the \( N \)-function (see Gallardo[2])

\[
\Phi_{p,r}(u) = |u|^p \ln^r (1 + |u|), \quad 1 \leq p < \infty, 0 < r < \infty.
\]

It is easy to check the right derivative of \( \Phi_{p,r}(u) \), \( \phi(t) \) is convex when \( 1 \leq p < \infty, 2 \leq r < \infty \). The index function

\[
F_{\Phi_{p,r}}(t) = \frac{t \Phi'_{p,r}(t)}{\Phi_{p,r}(t)} = p + \frac{rt}{(1+t) \ln(1+t)}
\]

is decreasing from \( p + r \) to \( p \) on \([0, \infty)\) since

\[
\frac{d}{dt} \Phi_{p,r}(t) = \frac{r[\ln(1+t) - t]}{(1+t)^2 \ln^2(1+t)} < 0.
\]

So \( C_{\Phi_{p,r}}^0(t) = \lim_{t \to 0} F_{\Phi_{p,r}}(t) = p + r \). According to (16) in the above remark and Theorem 3 we have

\[
J(L(\Phi_{p,r})[0, \infty)) = 2\beta_{\Phi_{p,r}} = 2\beta_{\Phi_{p,r}}^0 = 2 \cdot 2^{-\frac{1}{\frac{1}{p} + \frac{1}{r}}} = 2^{1 - \frac{1}{p + r}}. \quad (17)
\]

**References**


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Recibido: 28 de Junio de 2001
Revisado: 11 de Marzo de 2002