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BOUNDEDNESS OF HARDY-LITTLEWOOD MAXIMAL OPERATOR IN THE FRAMEWORK OF LIZORKIN-TRIEBEL SPACES

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Abstract

We describe a class $\mathcal{O}$ of nonlinear operators which are bounded on the Lizorkin–Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, for $0 < s < 1$ and $1 < p, q < \infty$. As a corollary, we prove that the Hardy-Littlewood maximal operator is bounded on $F^s_{p,q}(\mathbb{R}^n)$, for $0 < s < 1$ and $1 < p, q < \infty$; this extends the result of Kinnunen [9], valid for the Sobolev space $H^1_p(\mathbb{R}^n)$.

1 Introduction

The classical Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on the Lebesgue space $L^1_{loc}(\mathbb{R}^n)$ by setting

$$\forall f \in L^1_{loc}(\mathbb{R}^n), \quad \mathcal{M}_n(f)(x) = \sup_{r>0} \frac{1}{|Q_r|} \int_{Q_r} |f(x-y)|\, dy,$$

for every $x \in \mathbb{R}^n$; here $|Q_r|$ denotes the volume of the cube

$$Q_r = \Big\{ y \in \mathbb{R}^n : \max_{i=1,\dots,n} |y_i| \le r \Big\}.$$

The maximal function is a classical tool in harmonic analysis but recently it has been successfully used in studying Sobolev functions and partial differential equations, see Bojarski–Hajlasz [4] and Lewis [10]. The celebrated theorem of Hardy, Littlewood and Wiener asserts that the maximal operator is bounded in $L^p(\mathbb{R}^n)$ for all $1 < p \le \infty$ (cf. Stein [15]; we say that a –possibly nonlinear– operator T is bounded from a

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Banach space E to a Banach space F if there exists a constant C such that for every $f \in E$, we have $\|T(f)\|_F \leq C\ \|f\|_E$). This theorem is one of the cornerstones of harmonic analysis but the applications to Sobolev functions and to partial differential equations indicate that it is also useful to know how the maximal operator preserves the differentiability properties of functions. Recently, Kinnunen [9] proved that $\mathcal{M}_n$ is bounded on the Sobolev space $H^1_p(\mathbb{R}^n)$, for $1 < p < +\infty$. It is therefore a natural question to ask whether $\mathcal{M}_n$ is also bounded for every $s \in (0,1)$ on the Sobolev spaces $H^s_p(\mathbb{R}^n)$ defined by the Bessel potentials ([2], [15]) or on its generalizations Lizorkin–Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$.

To our knowledge, there is no general theorem allowing us to interpolate a nonlinear operator T, bounded on $H^0_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $H^1_p(\mathbb{R}^n)$, to an operator bounded on $H^s_p(\mathbb{R}^n)$ for every $0 < s < 1$, even in the special case where T is the Hardy-Littlewood maximal operator. The known results of Böhm [3], Peetre [12] or Tartar [17], do not seem to apply to our situation. Although non linear, the maximal operator $\mathcal{M}_n$ is strongly related to linear operators : we shall introduce below a notion of *linearizable operator*, of which $\mathcal{M}_n$ will be an example. We introduce an alternative of the characterization of $F^s_{p,q}(\mathbb{R}^n)$ by differences which allows us, by the means of the fundamental result of Benedek-Calderón-Panzone [1], to describe a class $\mathcal{O}$ of operators T that are bounded on $F^s_{p,q}(\mathbb{R}^n)$ for all $0 \leq s < 1$ and $1 < p, q < \infty$; our result yields that the Hardy-Littlewood maximal operator is bounded on $F^s_{p,q}(\mathbb{R}^n)$ for all $0 \leq s < 1$ and $1 < p, q < \infty$.

We recall the fundamental result due to Benedek, Calderón and Panzone [1] : let E and F be Banach spaces ; $\mathcal{L}(E,F)$ denotes the space of all bounded linear operators from E to F. An operator U is called a Benedek-Calderón-Panzone operator (*a* **BCP** *operator* for short), if U is bounded from $L^r(\mathbb{R}^n, E)$ to $L^r(\mathbb{R}^n, F)$ for some fixed $r \in (1,\infty)$, and if there exists a strongly measurable $\mathcal{L}(E,F)$-valued kernel K defined on $\mathbb{R}^n$, locally integrable outside the origin such that

1) if f is any E-valued continuous function with compact support $\mathrm{supp}(f) \subset \mathbb{R}^n$ and if $x \notin \mathrm{supp}(f)$, then

$$U(f)(x) = \int_{\mathbb{R}^n} K(x-y).f(y)\ dy;$$

2) (Hörmander's condition) there exists a constant $M \geq 0$ such that

$$\forall y \in \mathbb{R}^n, \quad \int_{|x|>2|y|} \|K(x-y) - K(x)\|_{\mathcal{L}(E,F)} \, dx \leq M.$$

When $E = \mathbb{R}$, the Banach space $\mathcal{L}(E, F)$ is identified with F and the kernel K is identified with a F-valued function.

The result of Benedek, Calderón and Panzone states that under these assumptions, this operator U can be extended to a bounded linear operator from $L^p(\mathbb{R}^n, E)$ to $L^p(\mathbb{R}^n, F)$, for every $p \in (1, \infty)$.

Let us mention an interesting special case of the preceding situation. Let $E = \mathbb{R}$, $F = L^2((0, +\infty); dt/t)$ and let U be the convolution operator with the kernel K defined as follows : since $E = \mathbb{R}$, K is identified with a function from $\mathbb{R}^n$ to F, namely $K(x)(t) = t^{-n} \, \psi(x/t)$ for $x \in \mathbb{R}^n$, $t > 0$, where ψ is a real function on $\mathbb{R}^n$ satisfying the following conditions

$$|\psi(x)| \leq C \, |x|^{-n-\epsilon}, \int_{\mathbb{R}^n} \psi(x) \, dx = 0 \text{ and } \int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)| \, dx \leq C \, |y|^{\epsilon},$$

for some fixed real number $\epsilon > 0$. The operator U corresponding to the kernel K is related to the g-function operator $f \to \mathrm{g}(f)$ defined by setting

$$\mathrm{g}(f)(x) = \sqrt{\int_0^\infty |f * \psi_t(x)|^2 \, \frac{dt}{t}} = \|U(f)(x)\|_F$$

where $\psi_t(x) = t^{-n} \, \psi(x/t)$. The above conditions imply a suitable decay at 0 and infinity for the Fourier transform of ψ, and yield that the g-function operator is bounded on $L^2(\mathbb{R}^n)$. Then, the result of **BCP** yields that U is bounded on $L^p(\mathbb{R}^n)$ for every $p \in (1, \infty)$; we refer to [1] or [6] for more details.

2 Results

The class $\mathcal{O}$ of operators is defined as follows : an operator T belongs to $\mathcal{O}$ if T is a –nonlinear– operator from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for some $r \in (1, +\infty)$, which commutes with translations (*i.e.* for every $\alpha \in \mathbb{R}^n$, $\tau_\alpha T = T \tau_\alpha$, where $\tau_\alpha f(x) = f(x - \alpha)$), and such that

$$\forall f \in L^r(\mathbb{R}^n), \quad T(f)(x) = \|U(f)(x)\|_F$$

where U is a **BCP** operator from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n, F)$ for some Banach space F.

Notice that every operator $T \in \mathcal{O}$ takes values in the positive cone of $L^r(\mathbb{R}^n)$. The operator T satisfies $|T(f) - T(g)| \leq T(f-g)$; so T is continuous on $L^p(\mathbb{R}^n)$ if it is bounded, and T is indeed bounded on every $L^p(\mathbb{R}^n)$, $1 < p < +\infty$ by the Benedek-Calderón-Panzone result. This class $\mathcal{O}$ contains the Littlewood-Paley g-function operator mentioned above.

Theorem 1. *Every operator $T \in \mathcal{O}$ satisfies the following properties:*
(**i**) *for all $1 < p < \infty$, T is bounded on $H^1_p(\mathbb{R}^n)$, and for all $f \in H^1_p(\mathbb{R}^n)$*

$$|\partial_k T(f)| \leq T(\partial_k f), \quad k = 1, \dots, n; \tag{1}$$

(**ii**) *for all $0 < s < 1$ and $1 < p, q < \infty$, T is bounded on $F^s_{p,q}(\mathbb{R}^n)$.*

Corollary 1. *The maximal operator $\mathcal{M}$ satisfies properties (i) and (ii) of Theorem 1.*

Theorem 2. *There exists a positive function $f \in C^\infty_0(\mathbb{R}^n)$ such that $\mathcal{M}_n(f)$ does not belong to $F^s_{p,q}(\mathbb{R}^n)$ for every $s > 1 + 1/p$ and $1 < p, q < +\infty$.*

3 Key Lemmata

Let us recall the characterization by differences of the Lizorkin–Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$. We fix s such that $0 < s < 1$ and let

$$S_1(f)(x) = \Big(\int_0^1 \Big[\int_{B_n} \Big| \frac{f(x+th) - f(x)}{t^s} \Big| \, dh \Big]^q \frac{dt}{t} \Big)^{1/q}$$

for $f \in F^s_{p,q}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where B_n denotes the unit ball of $\mathbb{R}^n$. Next, consider the norm

$$N_1(f) = \|f\|_p + \|S_1(f)\|_p.$$

This is an equivalent norm on $F^s_{p,q}(\mathbb{R}^n)$. In the case $q = 2$, $F^s_{p,2} = H^s_p$; this characterization is due to Strichartz [16]. The expression $\|S_1(f)\|_p$ appears as the norm in the space

$$L^p(L^q(L^1)) = L^p(\mathbb{R}^n, \, dx, \, L^q((0,1), dt/t, L^1(B_n)))$$

of the function $F(x, t, h) = t^{-s}(f(x + th) - f(x))$ that depends linearly upon f. In other words, $F^s_{p,q}(\mathbb{R}^n)$ is a Banach space isomorphic to a subspace of $L^p(L^q(L^1))$.

Every operator $T \in \mathcal{O}$ satisfies the following inequality (because any **BCP** operator satisfies it, see for example Fefferman [5] or García Cuerva and Rubio de Francia [6]) : for all $1 < p < \infty$ and $1 < r \leq \infty$, there exists a constant $C \geq 0$ such that for all sequences $(f_j)_{j\in\mathbb{Z}}$ in $L^p(\mathbb{R}^n)$, we have

$$\left\| \left(\sum_j |T(f_j)|^r \right)^{1/r} \right\|_p \leq C \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_p . \tag{2}$$

This inequality in $L^p(\ell^r)$ can be easily extended to the continuous case $L^p(L^r)$, or even to $L^p(L^q(L^r))$, $(1 < p, q, r < \infty)$; this is proved in our Lemma 2 below. Since T satisfies

$$|T(f) - T(g)| \leq T(f - g)$$

and commutes with translations, we obtain the following pointwise inequality

$$\Big| T(f)(x + th) - T(f)(x) \Big| \leq T\Big(\tau_{th} f - f \Big)(x).$$

Computing $N_1(Tf)$, the last estimate yields

$$\Big\| S_1\big(T(f)\big) \Big\|_p \leq \|T(\tilde{f})\|_{L^{\vec{p}}} \tag{3}$$

where $\tilde{f}(x, t, h) = t^{-1-2s}\,[f(x+th) - f(x)]$, $\vec{p} = (p, q, 1)$ and $L^{\vec{p}}$ is defined below. In order to conclude the proof of Property (**ii**) in Theorem 1, it is enough to have the following inequality

$$\|T(g)\|_{L^{\vec{p}}} \leq C\, \|g\|_{L^{\vec{p}}}\,;$$

but this last estimate is not true in general for the space $L^{\vec{p}}$, because (2) is not valid when $r = 1$ (otherwise, it would be valid for the maximal operator, and this is known to be false, see [14] page 75) ; since $r = 1$ is precisely what we need, the characterization by differences of $F^s_{p,q}(\mathbb{R}^n)$ is not adequate ; we shall rather use the following characterization which is a special case of a Triebel's result (cf. [19], page 194) :

Lemma 1. *Let* $0 < s < 1$, $1 < p, q < \infty$ *and* $1 \leq r < \min(p, q)$ *; then*

$$N_r(f) = \|f\|_p + \|S_r(f)\|_p$$

defines an equivalent norm on $F^s_{p,q}(\mathbb{R}^n)$, *where*

$$S_r(f)(x) = \Big(\int_0^1 \Big[\int_{B_n} \Big|\frac{f(x+th) - f(x)}{t^s}\Big|^r dh\Big]^{q/r} \frac{dt}{t}\Big)^{1/q}.$$

Let $1 < p_1, p_2, p_3 < \infty$ and set $\vec{p} = (p_1, p_2, p_3)$; given a Banach space F, we denote by $L^{\vec{p}}(F)$ the space of all measurable F-valued functions f defined on $\mathbb{R}^n \times (0,1) \times B_n$, such that

$$\|f\|_{L^{\vec{p}}(F)} = \left\{ \int_{\mathbb{R}^n} \left[\int_0^1 \left(\int_{B_n} \|f(x_1, x_2, x_3)\|_F^{p_3} dx_3 \right)^{p_2/p_3} dx_2 \right]^{p_1/p_2} dx_1 \right\}^{1/p_1} < \infty.$$

When $F = \mathbb{R}$, we denote simply by $L^{\vec{p}}$ the corresponding space. If U is a **BCP** operator from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n, F)$, we associate to it the operator $\widetilde{U}$ defined by :

$$\widetilde{U}(f)(x_1, x_2, x_3) = U\big(f_{x_2,x_3}\big)(x_1)$$

for every F-valued continuous function f defined on $\mathbb{R}^n \times (0,1) \times B_n$ with compact support, where f_{x_2,x_3} denotes the function $x_1 \to f(x_1, x_2, x_3)$. Under these hypothesis, we have the following result:

Lemma 2. *The operator* $\widetilde{U}$ *can be extended to a bounded operator from* $L^{\vec{p}}$ *to* $L^{\vec{p}}(F)$.

4 Proofs

4.1 Proof of Lemma 2

The proof is just an iteration of the fundamental result of Benedek, Calderón and Panzone which we recalled in the introduction. Let U be a **BCP** operator from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n, F)$, for some $r \in (1, +\infty)$; its kernel K is a function from $\mathbb{R}^n$ to F. A first application of **BCP** yields that U is bounded from $L^{p_3}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n, F)$. For every continuous function g with compact support in $\mathbb{R}^n \times B_n$ and for every $x_3 \in B_n$, let

$g_{x_3}(x_1) = g(x_1, x_3)$, and let $(U_1 g)(x_1, x_3) = (U g_{x_3})(x_1)$; for every x_3, we have

$$\int_{\mathbb{R}^n} \|U g_{x_3}(x_1)\|_F^{p_3}\, dx_1 \le C^{p_3} \int_{\mathbb{R}^n} |g_{x_3}(x_1)|^{p_3}\, dx_1.$$

By integrating with respect to the variable $x_3 \in B_n$ and applying Fubini's theorem, we deduce from this an operator U_1 defined from $L^{p_3}(\mathbb{R}^n, E_1)$ to $L^{p_3}(\mathbb{R}^n, F_1)$, where $E_1 = L^{p_3}(B_n)$ and $F_1 = L^{p_3}(B_n, F)$; this operator U_1 is a new **BCP** operator: its kernel K_1, where $K_1(x_1) \in \mathcal{L}(E_1, F_1)$ is defined by setting

$$(K_1(x_1).f)(x_3) = f(x_3)K(x_1).$$

The operator norm of $K_1(x_1)$ and $K(x_1)$ coincide, and similarly

$$\|K_1(x_1 - y_1) - K_1(x_1)\|_{\mathcal{L}(E_1,F_1)} = \|K(x_1 - y_1) - K(x_1)\|_F.$$

So, the Hörmander condition for K_1 results immediately from that of K. Second, we apply the result of **BCP**, with $r = p_3$, to deduce that U_1 defines a bounded operator from $L^{p_2}(\mathbb{R}^n, E_1)$ to $L^{p_2}(\mathbb{R}^n, F_1)$; again Fubini's theorem gives an operator U_2 defined on $L^{p_2}(\mathbb{R}^n, E_2)$ with values in $L^{p_2}(\mathbb{R}^n, F_2)$, where $E_2 = L^{p_2}((0, 1), E_1)$ and $F_2 = L^{p_2}((0, 1), F_1)$, its kernel K_2, where $K_2(x_1) \in \mathcal{L}(E_2, F_2)$ is defined by setting

$$\Big(K_2(x_1)f\Big)(x_2, x_3) = \Big(K_1(x_1)f(x_3)\Big)(x_2);$$

again the operator norm of $K_2(x_1)$ and $K_1(x_1)$ coincide, and similarly

$$\|K_2(x_1 - y_1) - K_2(x_1)\|_{\mathcal{L}(E_2,F_2)} = \|K_1(x_1 - y_1) - K_1(x_1)\|_{\mathcal{L}(E_1,F_1)}.$$

So, the Hörmander condition for K_2 results immediately from that of K_1. We finish the proof by a third application of the **BCP** result, which shows that U_2 defines a bounded operator from $L^{p_1}(\mathbb{R}^n, E_2)$ to $L^{p_1}(\mathbb{R}^n, F_2)$. ■

4.2 Proof of Theorem 1

(i) Let $(e_i)_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$; the characterization of $H_p^1(\mathbb{R}^n)$ using the modulus of continuity $\omega_p(h) = \|\tau_{he_i} f - f\|_p$ (cf. Stein [15], page 139), the inequality

$$|\tau_\alpha T(f) - T(f)| \le T(\tau_\alpha f - f)$$

and the boundedness of T in $L^p(\mathbb{R}^n)$ for every $p \in (1, \infty)$ yield that T is bounded on $H^1_p(\mathbb{R}^n)$. Now, we prove the pointwise inequality (1). We have the following pointwise inequality

$$\varepsilon_m^{-1} \left| T(f)(x + \varepsilon_m e_i) - T(f)(x) \right| \leq T\big(\varepsilon_m^{-1}\{\tau_{\varepsilon_m e_i} f - f\}\big)(x), \qquad (4)$$

where $(\varepsilon_m)_{m\in\mathbb{N}}$ is a sequence of real numbers such that $\varepsilon_m > 0$. The condition $f \in H^1_p(\mathbb{R}^n)$ and the boundedness of T on $H^1_p(\mathbb{R}^n)$ yield

$$T\left(\frac{\tau_{\varepsilon_m e_i} f - f}{\varepsilon_m}\right) \xrightarrow[m\to\infty]{} T(\partial_i f) \quad \text{in } L^p(\mathbb{R}^n). \qquad (5)$$

and

$$\frac{\tau_{\varepsilon_m e_i} T(f) - T(f)}{\varepsilon_m} \xrightarrow[m\to\infty]{} \partial_i T(f) \quad \text{in } L^p(\mathbb{R}^n). \qquad (6)$$

Without loss of generality, we may assume that (5) and (6) hold a.e. So by passing to the limit in (4), the Property (**i**) is completely proved. ■

(ii) The case $s = 0$ is the case $L^p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$, and it is given by the **BCP** result; the proof for $0 < s < 1$ consists in using the fact that

$$\left| \frac{T(f)(x + th) - T(f)(x)}{t^{1+2s}} \right| \leq T\left(\frac{\tau_{th} f - f}{t^{1+2s}}\right)(x) \quad \text{a.e.}$$

Consequently

$$\|S_r\big(T(f)\big)\|_p \leq C\ \|\widetilde{U}(\widetilde{f})\|_{L^{\vec{p}}(F)}, \qquad (7)$$

where $\vec{p} = (p, q, r)$, $1 < p, q < \infty$, $1 < r < \min(p, q)$ and

$$\widetilde{f}(x, t, h) = \frac{f(x + th) - f(x)}{t^{1+2s}}.$$

Lemma 2 and inequality (7) conclude the proof of Property (**ii**). ■

4.3 Proof of Corollary 1

Step 1. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be radial such that $\mathrm{supp}(\phi) \subset \{x : |x| \leq 1\}$; we define the operator $\mathcal{M}_\phi$ by setting

$$\forall f \in L^1_{loc}(\mathbb{R}^n), \quad \mathcal{M}_\phi(f)(x) = \sup_{\delta>0} |f * \phi_\delta(x)|,$$

where $\phi_\delta(x) = \frac{1}{\delta^n}\phi\big(\frac{x}{\delta}\big)$. The operator $\mathcal{M}_\phi$ is a **BCP** operator: it is linearizable, and its linearization $U_\phi : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, F)$, where $F = L^\infty((0,\infty), dt)$, is defined by saying that $U_\phi(f)(x)$ is the bounded function $\delta > 0 \to f * \phi_\delta(x)$. The operator U_ϕ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n, F)$, because $\mathcal{M}_\phi(f)$ is majorized by $C\ \mathcal{M}(f)$ for some constant C. The kernel corresponding to U_ϕ is the F-valued function K_ϕ defined by

$$K_\phi(x) : \delta > 0 \to \delta^{-n}\phi(x/\delta);$$

this function K_ϕ is differentiable (from $\mathbb{R}^n$ to F) away from the origin, and

$$\|K'_\phi(x)\|_F \leq C\ |x|^{-n-1};$$

it results from it that the Hörmander condition is satisfied: we use the mean value theorem and the polar coordinates, we obtain

$$\begin{aligned}\int_{|x|>2|y|} \|K_\phi(x-y) - K_\phi(x)\|_F\ dx &\leq C\ |y| \int_{|x|>2|y|} |x|^{-n-1}\ dx \\ &\leq C\ |y| \int_{\rho>2|y|} \frac{d\rho}{\rho^2} \leq C\end{aligned}$$

(C denotes a universal constant which may change from line to line); this shows that $\mathcal{M}_\phi$ satisfies the assumptions of Theorem 1.

Step 2. If moreover the function ϕ of step 1 satisfies the conditions $\phi \geq 0$ and $\phi(0) > 0$, there exists a constant C such that $\mathcal{M}_n(f) \leq C\ \mathcal{M}_\phi(|f|)$. So

$$\begin{aligned}\left|\frac{\mathcal{M}_n(f)(x+th) - \mathcal{M}_n(f)(x)}{t^{1+2s}}\right| &\leq \mathcal{M}_n\left(\frac{\tau_{th}f - f}{t^{1+2s}}\right)(x), \\ &\leq C\ \mathcal{M}_\phi\left(\left|\frac{\tau_{th}f - f}{t^{1+2s}}\right|\right)(x).\end{aligned}$$

Therefore the Lemma 1 gives

$$\|S_r(\mathcal{M}_n(f))\|_p \leq C\ \|\widetilde{U}_\phi(\widetilde{f})\|_{L^{\vec{p}}(F)}, \tag{8}$$

($1 < p, q < \infty$, $1 < r < \min(p,q)$ and $\vec{p} = (p,q,r)$) where

$$\widetilde{f}(x,t,h) = \left|\frac{f(x+th) - f(x)}{t^{1+2s}}\right|.$$

Since the step 1 yields that U_ϕ is a **BCP** operator, so the Lemma 2 and the inequality (8) give that the maximal operator satisfies the Property (**ii**) of Theorem 1. ■

4.4 Proof of Theorem 2

We split the proof of Theorem 2 into two steps.

Step 1. Here, we deal with the one-dimensional situation. We shall prove that there exists a positive function $f \in C_0^\infty(\mathbb{R})$ such that, for every $\varepsilon > 0$, $\mathcal{M}_1(f)$ is not $(1+\varepsilon)$-Hölder function in a neighbourhood of 0. This yields, by the following embedding (cf. Triebel [18])

$$F^v_{p,q}(\mathbb{R}) \hookrightarrow C^{v-\frac{1}{p}}(\mathbb{R}) \quad \text{for every} \quad v > 1/p,$$

that $\mathcal{M}_1(f)$ does not belong to $H^s_p(\mathbb{R})$ whenever $s > 1 + 1/p$.

Now, consider a positive function $f \in C_0^\infty(\mathbb{R})$ satisfying the following conditions

- $\operatorname{supp}(f) \subset [1/3, 2]$;
- f is increasing on $[1/3, 2/3]$ and decreasing on $[2/3, 2]$;
- $\int_0^1 f(x)\,dx = 1$, $f(1) = 1$ and $f^{(m)}(1) = 0$ for every $m \geq 1$.

First, let us explain how to compute $\mathcal{M}_1(f)(x)$ for every $x < 1/3$. Obviously, for $x < 1/3$, we have

$$\mathcal{M}_1(f)(x) = \sup_{y>0} \frac{1}{2y} \int_{x-y}^{x+y} f(t)\ dt = \frac{1}{2} \sup_{u>x} \frac{F(u) - F(x)}{u - x},$$

where $F(u) = \int_{-\infty}^u f(t)\,dt$. Using the convexity of F and the fact that $F \in L^\infty(\mathbb{R})$, we observe that there exists one and only one real number $u > x$ such that

$$\mathcal{M}_1(f)(x) = \frac{1}{2} \frac{F(u) - F(x)}{u - x}.$$

So the relation between u and x is

$$x = u - \frac{F(u)}{f(u)} = u - \frac{F(u)}{F'(u)};$$

Finally, for every $x < 1/3$, we compute $\mathcal{M}_1(f)(x)$ by the following algorithm

$$\begin{cases} \mathcal{M}_1(f)(x) &= \dfrac{F'(u)}{2}, \\ x &= u - \dfrac{F(u)}{F'(u)}. \end{cases}$$

On a neigbourhood of $x = 0$ and $u = 1$, we have, for every natural number $N \geq 1$, $F(u) = u + O((u-1)^N)$ and $F'(u) = 1 + O((u-1)^N)$. Therefore, for every $N \geq 1$, $x = x(u) = O((u-1)^N)$; so the mapping $x \to u$ is not regular. However, we have $F'(u) = 1 + O((u-1)^N)$, and the situation is not clear. We have to refine our analysis, which we shall do in the following particular case : we assume that

$$F(u) = u - \exp(-\frac{1}{(u-1)^2}) \text{ in a neighbourhood of } 1;$$

this is compatible with the conditions given above. Therefore, we have

$$F'(u) = 1 - \frac{2}{(u-1)^3} \exp\left(-\frac{1}{(u-1)^2}\right),$$

and

$$\begin{aligned} x &= u - \frac{F(u)}{F'(u)} \\ &= \exp\left(\frac{-1}{(u-1)^2}\right)\left\{-\frac{2}{(u-1)^3} - \frac{2}{(u-1)^2} + 1 + \cdots\right\}. \end{aligned}$$

So, it follows that

$$\begin{cases} 2\,\mathcal{M}_1(f)(x) &= 1 - \dfrac{2}{(u-1)^3} \exp\left(\dfrac{-1}{(u-1)^2}\right) \\ x &= -\dfrac{2}{(u-1)^3} \exp\left(\dfrac{-1}{(u-1)^2}\right) - \dfrac{2}{(u-1)^2} \exp\left(\dfrac{-1}{(u-1)^2}\right) \\ &\quad + \exp\left(\dfrac{-1}{(u-1)^2}\right) + \cdots . \end{cases}$$

Hence, we obtain $2\,\mathcal{M}_1 f(x) = 1 + x + O(x)$. Nevertheless, $2\,\mathcal{M}_1 f(x)$ does not equal to $1 + x + O(x^{1+\varepsilon})$. We proceed by contradiction by assuming that

$$2\,\mathcal{M}_1 f(x) = 1 + x + O(x^{1+\varepsilon}).$$

Since

$$\begin{aligned}2\mathcal{M}_1 f(x) - (1+x) &= \left\{1 - \frac{2}{(u-1)^3}\exp\left(\frac{-1}{(u-1)^2}\right) - 1\right\} \\ &\quad + \frac{2}{(u-1)^3}\exp\left(\frac{-1}{(u-1)^2}\right) + \frac{2}{(u-1)^2}\exp\left(\frac{-1}{(u-1)^2}\right) \\ &\quad - \exp\left(\frac{-1}{(u-1)^2}\right) + \cdots \\ &= -\frac{2}{(u-1)^2}\ \exp\left(\frac{-1}{(u-1)^2}\right) + \cdots \\ &= (u-1)x + \cdots\end{aligned}$$

This yields that $u-1 = O(x^{\varepsilon})$. But this is impossible. Hence, for every $\varepsilon > 0$, $\mathcal{M}_1(f)$ is not $(1+\varepsilon)$-Hölder function on a neighbourhood of 0.

Step 2. Let us recall that the function f, given in the step 1, satisfies the following proposition : for every $0 < \delta < 1/3$, there exists $\eta > 0$ such that the equality

$$\mathcal{M}_1(f)(x_1) = \sup_{0<r<\eta} \frac{1}{2r}\int_{-r}^{r} f(x_1 - u_1)\ du_1.$$

holds for every $x_1 \in (-\delta, \delta)$. Define a function $\tilde{f}$ by setting

$$\tilde{f}(x) = f(x_1)\ \theta(x'), \quad x' \in \mathbb{R}^{n-1},$$

where $\theta \in C_0^{\infty}(\mathbb{R}^{n-1})$ is a smooth function satisfying $0 \le \theta \le 1$ and $\theta(0) > 0$. Since $f \in L^{\infty}(\mathbb{R})$, then there exists $\gamma > \eta$ (independent of θ) such that, for every $x_1 \in (-\delta, \delta)$ and every $x' \in \mathbb{R}^{n-1}$,

$$\begin{aligned}&\mathcal{M}_n(\widetilde{f})(x_1, x') = \\ &= \sup_{0<r<\gamma} \frac{1}{|Q_r|}\int_{Q_r} f(u_1 - x_1)\ \theta(u_2 - x_2, \ldots, u_n - x_n)\ du_1 \ldots du_n.\end{aligned}$$

Now, we choose θ such that $\theta(u_2, \ldots, u_n) = 1$ on

$$\widetilde{Q}_{\gamma+1} = \left\{(v_2, \ldots, v_n) \in \mathbb{R}^{n-1} : \max_{2\le i\le n} |v_i| \le 1 + \gamma\right\}.$$

Therefore, for every $x_1 \in (-\delta, \delta)$ and every $x' \in \widetilde{Q}_1$, we have

$$\mathcal{M}_n(\widetilde{f})(x_1, x') = \mathcal{M}_1(f)(x_1).$$

Obviously, $\widetilde{f}$ is a positive function belonging to $C_0^\infty(\mathbb{R}^n)$. However, for every real number $s > 1 + \frac{1}{p}$, the function $\mathcal{M}_n(\widetilde{f})$ does not belong to $F^s_{p,q}(\mathbb{R}^n)$. If this would be false, via the so-called Fubini property (cf. Triebel [20], Theorem 4.4, page 36 ; but in the cases considered here it can also be found in a somewhat hidden way in a paper by Kaljabin [7] and [8]) :

$$\|\mathcal{M}_n(\widetilde{f})\|_{F^s_{p,q}} \sim \sum_{j=1}^{n} \Big\| \, \|\mathcal{M}_n(\widetilde{f})(x_1, \dots, x_{j-1}, \bullet, x_j, \dots, x_n)\|_{F^s_{p,q}(\mathbb{R})} \Big\|_{L^p(\mathbb{R}^{n-1})},$$

we obtain that, for almost every $x' \in \widetilde{Q}_1$, the function $x_1 \to \mathcal{M}_n(\widetilde{f})(x_1, x')$ belongs to $F^s_{p,q}(\mathbb{R})$. But, according to step 1, this is impossible. This concludes the proof of Theorem 2. ■

Before we conclude, we would like to make some remarks.

Remark 1. Consider the following variant of the Hardy-Littlewood maximal operator, defined by the means of Gauss semigroups $(\varphi_t)_{t>0}$ by setting :

$$T(f) = \sup_{t>0} |f * \varphi_t(x)|,$$

where $\varphi_t(x) = t^{-n/2} \ \varphi(x/\sqrt{t})$ and $\varphi(x) = (2\pi)^{-n/2} \ \exp(-|x|^2/2)$. It is simple to chek that $T \in \mathcal{O}$. However, if we choose $f = \partial/\partial_{x_1}\varphi$ which belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$; the function $T(f)$ does not belong to $F^s_{p,q}(\mathbb{R}^n)$ for every $s \geq 1 + 1/p$ and $q < +\infty$. Indeed, by using the identity $\varphi_t * f = \partial/\partial_{x_1}\varphi_{t+1}$, we obtain $T(f)(x) = |x_1| \ \varphi(x)$ for every $|x| \leq 1$. According to the fact that $|x_1|$ does not belong, locally on a neighbourhood of the origin, to $F^s_{p,q}(\mathbb{R}^n)$ as soon as $s \geq 1 + 1/p$ and $q < +\infty$ (due to the fact that, in the one-dimensional situation, the characteristic function $\chi_{[-1,1]}$ does not belongs to $F^{s-1}_{p,q}(\mathbb{R})$ whenever $s \geq 1 + 1/p$ and $q < +\infty$), this completes our claim. ■

Remark 2. Denote by $x = (x_1, \dots, x_n)$ points in $\mathbb{R}^n$. For a locally integrable function f on $\mathbb{R}^n$, define

$$(\mathcal{N}_n f)(x) = \sup_{a_1<x_1<b_1} \cdots \sup_{a_n<x_n<b_n} \frac{1}{(b_1-a_1)\cdots(b_n-a_n)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(y_1, \dots, y_n) \, dy_n \cdots dy_1.$$

The operator $\mathcal{N}_n$ is called the “strong” maximal function on $\mathbb{R}^n$.

Corollary 2. *Let* $0 < s < 1$ *and* $p, q \in (1, +\infty)$*. Then, the operator* $\mathcal{N}$ *is bounded on* $F^s_{p,q}(\mathbb{R}^n)$*.*

Proof. Observe that there exists a constant c_n such that

$$\mathcal{N}_n \leq c_n \, \mathcal{M}_1^{(1)} \circ \cdots \circ \mathcal{M}_1^{(n)},$$

where $\mathcal{M}_1^{(j)}$ denotes the maximal centered operator $\mathcal{M}_1$ applied to x_j coordinate. Therefore, we obtain

$$\begin{aligned} \left| \frac{\mathcal{N}_n(f)(x+th) - \mathcal{N}_n(f)(x)}{t^{1+2s}} \right| &\leq \mathcal{N}_n \left(\frac{f(\cdot + th) - f(\cdot)}{t^{1+2s}} \right)(x), \\ &\leq C_n \, \mathcal{M}_1^{(1)} \circ \cdots \circ \mathcal{M}_1^{(n)} \big(\widetilde{f}(\cdot, t, h) \big)(x) \end{aligned}$$

where

$$\widetilde{f}(x,t,h) = \left| \frac{f(x+th) - f(x)}{t^{1+2s}} \right|$$

An iteration of Lemma 2 concludes the proof of Corollary 2. ■

Remark 3. Note that by real interpolation of nonlinear operators the boundedness of the maximal operator on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $0 < s < 1$ becomes obvious : One has by real interpolation

$$B^{s\theta}_{p,q}(\mathbb{R}^n) = \Big(L_p(\mathbb{R}^n), H^s_p(\mathbb{R}^n) \Big)_{\theta,q}$$

where

$$1 < p < \infty, \quad 0 < q \leq \infty, \quad ,0 < \theta < 1, \quad s \in \mathbb{R},$$

one can remplace H^s_p by $F^s_{p,r}$.
Furthermore $\mathcal{M}$ is sublinear with the consequence

$$\|\mathcal{M}(f) - \mathcal{M}(g)\|_p \leq C \, \|\mathcal{M}(f-g)\|_p \leq C' \, \|f-g\|_p, \quad 1 < p < \infty.$$

Together with the proved boundedness of $\mathcal{M}$ in H^s_p or $F^s_{p,r}$ one can use the nonlinear real interpolation by Peetre [12] and Tartar [17]. An appropriate formulation can be found in Runst–Sickel [13], page 88. It comes out that $\mathcal{M}$ is also bounded at least is $B^s_{p,q}$ with $0 < s < 1$, $1 < p < \infty$, $0 < q \leq \infty$. ■

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