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UNRAMIFIED NONSPECIAL REAL SPACE CURVES HAVING MANY REAL BRANCHES AND FEW OVALS

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Abstract

Let $C \subseteq \mathbb{P}^n$ be an unramified nonspecial real space curve having many real branches and few ovals. We show that C is a rational normal curve if n is even, and that C is an M-curve having no ovals if n is odd.

1 Introduction

Let n be a natural integer satisfying $n \geq 2$. Let C be a smooth geometrically integral real algebraic curve in real projective space $\mathbb{P}^n$ [2]. The curve C is *nondegenerate* if C is not contained in a real hyperplane of $\mathbb{P}^n$. Suppose that C is nondegenerate. We say that C is *unramified* [4] if, for all real hyperplanes H of $\mathbb{P}^n$, one has

$$\deg(H \cdot C) - \deg(H \cdot C)_{\mathrm{red}} \leq n - 1,$$

where $_{\mathrm{red}}$ means the associated reduced divisor. In particular, an unramified real curve does not have real inflection points. The converse, however, does not hold. Indeed, a suitable small deformation of the real plane curve defined by the affine equation $(x^2+y^2-1)(x^2+y^2-2) = 0$, is ramified but does not have real inflection points.

The corresponding notion of an unramified complex algebraic curve in complex projective space is well understood. Indeed, any unramified complex algebraic curve is a rational normal curve and conversely [1, p. 270]. For real algebraic curves, the situation seems to be much more interesting.

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In [4], it is shown that there are unramified real curves of any genus in any odd dimensional projective space. More precisely, let g and n be natural integers, with $n \geq 3$ and n odd. Let C be an M-curve of genus g, i.e., C is a smooth geometrically integral proper real curve such that the number of connected components of $C(\mathbb{R})$ is equal to $g+1$. Choose a divisor D on C of degree $g+n$ such that D has odd degree on any connected component of $C(\mathbb{R})$. This is possible since $C(\mathbb{R})$ has $g+1$ connected components and $n-1$ is even. Then the linear system $|D|$ induces an unramified embedding of C in $\mathbb{P}^n$. It is conjectured that any unramified real space curve in $\mathbb{P}^n$, for n odd, is obtained in this way [4]. For n even, it is conjectured that any unramified real space curve in $\mathbb{P}^n$ is a rational normal curve, i.e., its genus is equal to 0 and its degree is equal to n [4]. The latter conjecture has been proved for $n = 2$ [6].

The object of this paper is to prove both conjectures for nonspecial space curves having many real branches and having few ovals.

2 Real space curves having many real branches and few ovals

Let C be a smooth geometrically integral proper real curve. A *real branch* of C is a connected component of the set $C(\mathbb{R})$ of real points of C. Let g be the genus of C. By Harnack's Inequality [3], the number of real branches of C is less than or equal to $g+1$. The curve C is called an *M-curve* (resp. *$(M-1)$-curve*) if the number of real branches of C is equal to $g+1$ (resp. g). We say that *C has many real branches* if C has at least g real branches, i.e., if C is either an M-curve or an $(M-1)$-curve.

We need to recall the following result [5, Theorem 2.1].

Theorem 1. *Let C be a smooth geometrically integral proper real algebraic curve having many real branches. Let g be the genus of C. Let $B_1, \ldots, B_g$ be mutually distinct real branches of C and put*

$$B = \prod_{i=1}^{g} B_i.$$

Let $e_1, \ldots, e_g$ be nonzero natural integers and let

$$\varphi\colon B \longrightarrow \operatorname{Pic}(C)$$

be the map defined by $\varphi(P) = \mathrm{cl}(\sum_{i=1}^{g} e_i P_i)$*, where* cl *denotes the divisor class. Then,* φ *is a topological covering of its image of degree* $\prod_{i=1}^{g} e_i$*. In particular,* φ *is surjective on a connected component of* $\mathrm{Pic}(C)$. ■

Let $C \subseteq \mathbb{P}^n$ be a smooth geometrically integral real curve. A real branch B of C is a compact connected smooth real analytic curve in the real projective space $\mathbb{P}^n(\mathbb{R})$. The branch B is an *oval* if B is contractable in $\mathbb{P}^n(\mathbb{R})$. Otherwise, B is a *pseudo-line*. To put it otherwise, B is an oval if and only if each real hyperplane in $\mathbb{P}^n(\mathbb{R})$ intersects B in an even number of points, when counted with multiplicities. The branch B is a pseudo-line if and only if each real hyperplane in $\mathbb{P}^n(\mathbb{R})$ intersects B in an odd number of points, when counted with multiplicities. Now, suppose that C has many real branches. Let ε be the number of ovals of C. We say that C *has few ovals* if

$$\varepsilon \leq \begin{cases} n & \text{if } C \text{ is an } (M-1)\text{-curve, and} \\ n+1 & \text{if } C \text{ is an } M\text{-curve.} \end{cases} \tag{1}$$

In order to put it otherwise, let δ be the number of pseudo-lines of C. Since $\varepsilon + \delta$ is equal to the number of real branches of C, the curve C has few ovals if and only if $g - \delta \leq n$.

Let $C \subseteq \mathbb{P}^n$ be a smooth geometrically integral real curve. Recall that C is *normal* if the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \longrightarrow H^0(C, \mathcal{O}(1))$$

is an isomorphism. In particular, C is nondegenerate if it is normal. Let g be the genus of C and let d be its degree. Recall that C is *nonspecial* if C is normal and $n = d - g$.

The following statement is an affirmative answer to the conjecture mentioned in the Introduction, for nonspecial space curves having many real branches and few ovals.

Theorem 2. *Let* $C \subseteq \mathbb{P}^n$ *be an unramified nonspecial real space curve having many real branches and few ovals. If* n *is even then* C *is a rational normal curve. If* n *is odd then* C *is an* M*-curve and each real branch of* C *is a pseudo-line.*

Proof. Let $C \subseteq \mathbb{P}^n$ be an unramified nonspecial real space curve having many real branches and having few ovals. Let g be the genus of C and let d be the degree of C. Let D be a hyperplane section of C.

Suppose that n is even. We have to show that $g = 0$. Suppose, therefore, that $g \neq 0$. Since C is nonspecial, $d = g + n$. Since n is even, $d \equiv g \pmod 2$. Let δ be the number of pseudo-lines of C. Since $d \equiv \delta \pmod 2$, one has $\delta \equiv g \pmod 2$. It follows that C has at most g pseudo-lines. Choose distinct real branches $B_1, \ldots, B_g$ of C such that any pseudo-line of C is equal to one of the B_i's. Since $d \geq g$, $g > 0$, $g - \delta \leq n$ and $\delta \equiv d \pmod 2$, there are nonzero natural integers $e_1, \ldots, e_g$ such that $\sum e_i = d$, and e_i is odd if and only if B_i is a pseudo-line. Put $B = \prod B_i$ and let $\varphi\colon B \to \operatorname{Pic}(C)$ be the map as in Theorem 1. Then, by Theorem 1, the image of φ is surjective on the connected component of $\operatorname{Pic}(C)$ containing $\operatorname{cl}(D)$. In particular, there are points $P_i \in B_i$ such that the divisor $\sum e_i P_i$ belongs to the linear system $|D|$. This means that there is a real hyperplane H in $\mathbb{P}^n$ such that $H \cdot C = \sum e_i P_i$. Since $\sum (e_i - 1) = d - g = n$, the real space curve C is ramified. Hence, $g = 0$ and C is a rational normal curve.

Suppose that n is odd. Let, again, δ be the number of pseudo-lines of C. We have to show that $\delta = g + 1$. Suppose, therefore, that $\delta \neq g + 1$. Then $\delta \leq g$, i.e., C has at most g pseudo-lines. Since $\delta \equiv d \pmod 2$ and $d = g + n$, one has $\delta \equiv g + n \pmod 2$. Since n is odd, $\delta \not\equiv g \pmod 2$. Therefore, $\delta < g$. In particular, $g \neq 0$. As before, one derives a contradiction using Theorem 1. Therefore, $\delta = g + 1$. ∎

Let us mention, in conclusion, that there are many nonspecial real space curves having many real branches and few ovals. Indeed, let C be any smooth proper geometrically integral real curve having many real branches. Let n be any natural integer satisfying $n \geq 3$. Let ε be any natural integer satisfying the inequality (1). Let g be the genus of C. Choose a general effective divisor D on C of degree $g + n$ such that D is of even degree on exactly ε real branches of C. This is possible if $\varepsilon \equiv n \pmod 2$ in case C is an $(M-1)$-curve, and if $\varepsilon \not\equiv n \pmod 2$ in case C is an M-curve. Then, the complete linear system $|D|$ induces an embedding of C in $\mathbb{P}^n$, and the image curve is a nonspecial real space curve having many real branches and few ovals. Theorem 2 states that, among all these real space curves, the only ones that are unramified are rational normal curves if n is even, and M-curves having no ovals, if n is odd.

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