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ELASTIC WAVE PROPAGATION IN PARALLEL: THE HUYGENS' APPROACH

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Abstract

The use of parallel computers makes it feasible to simulate elastic waves throughout large heterogeneous structures, and new domain decomposition methods can be used to increase their efficiency and decrease the computing time spent in the simulation. In this paper we introduce a simple parallel algorithm for the propagation of elastic waves in complex heterogeneous media after a finite element discretization. This method performs more efficiently than classic domain decomposition techniques based on substructuring (Schur complement methods) and iterative domain decomposition. Some numerical examples are shown, including an analysis of the efficiency and performance of the algorithm.

1 Introduction

Wave propagation problems arise in a large number of applications in fields such as geophysics and structural dynamics. In this paper we focus our attention on elastic waves in a regime of small strain and small displacement, considered as a hyperbolic initial boundary condition problem to be solved using Galerkin finite element procedures that are useful in situations requiring non regular grids or general geometries. Within this framework, implicit time–stepping is preferred over explicit schemes, because the latter imposes a severe constraint on the duration of the time step to ensure stability. Thus, a large system of equations must be solved at each time step. Domain decomposition procedures can be used to break these large computations into several smaller ones. Partitioning the spatial domain into subdomains entails two main benefits: on one hand, numerical complexity is reduced as the corresponding subproblems are of smaller size and can be addressed in parallel; on the

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other hand, it is well suited for dealing with heterogeneous media. The main drawback, common for this type of numerical tools, is that the smaller subdomain problems remain coupled.

General domain decomposition methods include substructuring techniques based on the solution of the Schur complement for a given PDE ([14], [28]). This Schur matrix can also be viewed as the algebraic counterpart of the Steklov–Poincare operator that appears in some iterative decomposition methods ([19], [21], [18]). For the elastic wave propagation problem, several solutions have been proposed based on different approaches: implicit continuity of boundary conditions in the common interfaces [30], spectral resolution multidomain by collocation [31] or partition at the interface of the wavefront into several parts ([15], [6], [7],[8]).

The motivation for choosing a new scheme for parallelism lies on the necessity of increasing the efficiency of the algorithms when simulating real or *non academic* cases. Our numerical approach to elastic wave propagation is very simple, since it introduces a unique coupling procedure between subdomains: explicit calculations on the boundaries between subdomains to predict the transmitted energy into each one. This energy can be calculated by means of the displacement and velocity fields at every time step. Therefore, this procedure is noniterative, and involves dividing the domain into non–overlapping subdomains. The explicit nature of the energy prediction induces a time step limitation that is necessary to preserve stability, this constraint being the same as for the fully explicit method.

The algorithm allows considerable flexibility in the geometry and notably increases the performance of current decomposition techniques based on the substructuring formulation. The paper is organized as follows; in section 2 it is described briefly the elastic wave problem, deriving the discrete solution from the variational formulation of the elastodynamic problem. In section 3 we introduce a parallel technique, "the multiblock method", with both physical and mathematical descriptions, and section 4 studies its performance, including several tests carried out in shared and distributed memory environments.

2 Elastic waves theory

The motion of the elastic waves through 2D complex media is governed by a hyperbolic PDE which can be formulated as follows [2] [22]:

$$\rho\, u_{i,tt} = \sigma_{ij,i} + f_i, \tag{2.1}$$

where subindex seperated by a comma means derivation, $\mathbf{u}$ is the displacement vector, and σ_{ij} and ϵ_{ij} are the stress and strain tensors defined by

$$\sigma_{ij}(\mathbf{u}) = \lambda \nabla \mathbf{u}\, \delta_{ij} + 2\mu \epsilon_{ij}(\mathbf{u}), \qquad \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad i,j = 1,2,$$

λ and μ being the Lamé constants and ρ the mass density. The differential equations (2.1) should be fulfilled by initial conditions on the velocities and displacements:

$$u_i(0) = u_{0i} \qquad \text{and} \qquad u_{i,t}(0) = u_{0i,t}, \qquad i = 1,2 \tag{2.2}$$

In addition, we could prescribe a damping zone surrounding the domain to suppress reflections at the limits by imposing the action of an attenuation factor proportional to the velocity [29]. In general, we will assume that the displacement vanishes in the rigid surface (homogeneous Dirichlet condition), and tractions will vanish on the free surface (homogeneous Neumann condition).

The weak formulation of this hyperbolic PDE can be obtained by using classical methods [26]. The use of the Green formula turns the motion equation to

$$\frac{\partial^2}{\partial t^2} \int_\Omega \rho\, \mathbf{u} \cdot \mathbf{v}\, d\Omega + \sum_{i,j} \int_\Omega \sigma_{ij}(\mathbf{u}) \epsilon_{ij}(\mathbf{v})\, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega, \tag{2.3}$$

for a given function $\mathbf{v} \in \mathcal{H}_0^1(\Omega)$. The global domain Ω under study is a bounded Lipschitz region [20] in $\mathcal{R}$. The spaces of definition are the Sobolev space of first order

$$\mathcal{H}^1(\Omega) \equiv \left\{ v \mid v \in \mathcal{L}^2(\Omega), \quad \frac{\partial v}{\partial x_i} \in \mathcal{L}^2(\Omega), \quad i = 1,2,3 \right\},$$

and its extension to the functions vanishing on the boundary

$$\mathcal{H}_0^1(\Omega) \equiv \left\{ v \mid v \in \mathcal{H}^1(\Omega), \quad v|_{\partial\Omega} = 0, \right\}.$$

It is possible to define the inner product on $\mathcal{L}^2(\Omega)$ as

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\Omega, \tag{2.4}$$

and the elastic bilinear form on $\mathcal{H}^1(\Omega)$

$$a(\mathbf{u}, \mathbf{v}) = \sum_{i,j} \int_\Omega \sigma_{ij}(\mathbf{u})\, \epsilon_{ij}(\mathbf{v})\, d\Omega, \tag{2.5}$$

in order to formulate the variational problem of the elastodynamics as to find a function $\mathbf{u} : (0,T) \to \mathcal{H}_0^1(\Omega)$ such that

$$\left(\rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \mathbf{v}\right) + a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{H}_0^1(\Omega) \tag{2.6}$$

The discrete problem is to find $\mathbf{u}_h : (0,T) \to V_h$ such that

$$\left(\rho \frac{\partial^2 \mathbf{u}_h}{\partial t^2}, \mathbf{v}\right) + a(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in V_h \subset \mathcal{H}_0^1(\Omega). \tag{2.7}$$

The Galerkin approximation to this hyperbolic problem (see [24], [26] or [27]) reduces the PDE to a matrix form such as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \hat{\mathbf{f}}, \qquad \forall t \in (0,T). \tag{2.8}$$

This is a second order ordinary differential equation (ODE), where the mass ($\mathbf{M}$) and stiffness ($\mathbf{K}$) matrices have been built taking into account the boundary conditions for the Dirichlet and Neumann frontiers. Including the damping zone is straightforward in the weak formulation, leading to a system similar to (2.8):

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \hat{\mathbf{f}}, \qquad \forall t \in (0,T), \tag{2.9}$$

where $\mathbf{C}$ is the damping matrix. This description can be coupled to a time stepping scheme to obtain a fully discrete solution. A classical central differences scheme gives

$$\mathbf{M}\,\frac{\mathbf{u}_{n+1} - 2\,\mathbf{u}_n + \mathbf{u}_{n-1}}{\Delta t^2} + \mathbf{C}\,\frac{\mathbf{u}_{n+1} - \mathbf{u}_{n-1}}{2\,\Delta t} + \mathbf{K}\mathbf{u}_n = \mathbf{f}_n, \tag{2.10}$$

where $\mathbf{u}_{n+1} = \mathbf{u}((n+1)\cdot\Delta t)$. The use of diagonal mass and damping matrices (which can always be obtained using grid nodes for the quadrature formulae) avoids the resolution of a linear system of equations, the unknowns being obtained by means of an explicit division. This scheme is centered, explicit, second–order accurate in time, and conditionally stable [4].

Up to now, it is understood that elastic waves propagate because of a seismic source included in the initial problem, $\mathbf{f}$. This is called a Green problem and the initial conditions for displacements and velocities are usually chosen to be homogeneous. In the absence of a source term, $\mathbf{f} = 0$, appropriate non–homogeneous initial conditions also generate a wave (Cauchy problem). It can be proved [23] that both problems are equivalent.

3 The multiblock method

In this section we introduce a parallel algorithm for the elastic wave propagation problem. The idea of decomposing the solution of the wave equation into different domains comes from the classical theorem of existence and uniqueness of the solution for Cauchy's initial value problem of a hyperbolic partial differential equation [11]. Such a solution at a certain point $p \equiv p(\mathbf{r}, t)$ is uniquely determined by Cauchy's initial data only in a bounded domain that contains p. The dependence of the solution on the initial data exhibits the finiteness of the speed of propagation of phenomena governed by hyperbolic equations. A geometrical description based on the wave velocity can be reviewed in [1].

Our solution consists in the partition of a large medium Ω into several non–overlapping subdomains Ω_i (Figure 1). As it was stated in the previous section, on the boundary of the whole domain of definition we prescribe essential and natural boundary conditions. We denote these limits as rigid ($\partial\Omega_D$) and free ($\partial\Omega_N$) boundaries. The *artificial* interface is defined as the set of all the interfaces between two connected subdomains, $\Gamma = \cup\Gamma_{ij}$ with $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. We impose this interface as a free surface. In fact, the division is merely a mathematical artifact: we could apply artificial interfaces anywhere just considering they do not constrain the motion of the particles (that is, being free surfaces).

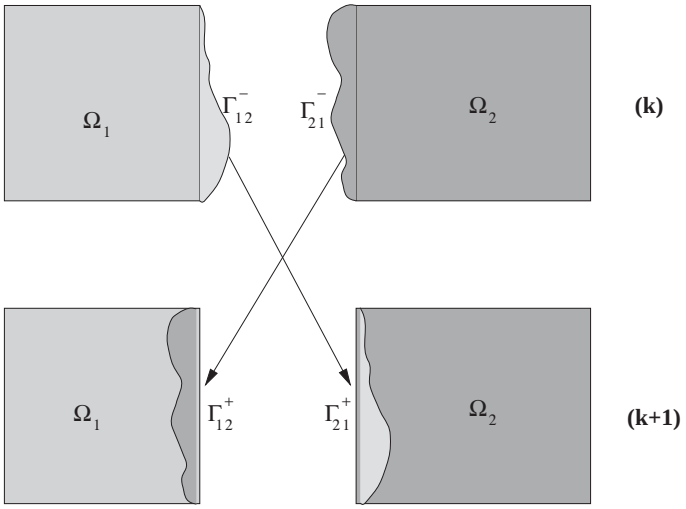

Figure 1: Exchange of information at the common boundary Γ_{12} between two partitions of a given domain Ω. At the end of time $t = k \cdot \Delta t$, the motion of the nodes at the interface is transmitted reciprocally to the neighbour ($\Gamma_{12}^- \rightarrow \Gamma_{12}^+$) and vice versa. In this manner the resulting wavefield at time $t = (k+1) \cdot \Delta t$ is computed as result of the addition of the perturbation generated by the source plus the perturbation at the frontier.

With these considerations, we pose the following Green problem:

$$\begin{cases} \rho\, u_{i,tt} = \sigma_{ij,i} + f_i, & \mathbf{u} \in \Omega, \\ \mathbf{u} = 0, & \mathbf{u} \in \partial\Omega_D, \\ \sigma(\mathbf{u}) \cdot \hat{n} = 0, & \mathbf{u} \in \partial\Omega_N \cup \Gamma, \\ \mathbf{u}(0) = \dot{\mathbf{u}}(0) = 0, & \mathbf{u}, \dot{\mathbf{u}} \in \Omega. \end{cases} \tag{3.1}$$

At each time step $k \cdot \Delta t$, the displacements and velocities lying on the common interfaces Γ are used as Cauchy's initial conditions for the problem having the solution at time $(k-1)\cdot\Delta t$ as initial data transmitted to neighbours:

$$\begin{cases} \mathbf{u}(k)|_{(\Gamma_{ij}^+)} = \mathbf{u}(k-1)|_{(\Gamma_{ji}^-)} \\ \dot{\mathbf{u}}(k)|_{(\Gamma_{ij}^+)} = \dot{\mathbf{u}}(k-1)|_{(\Gamma_{ji}^-)} \end{cases} \tag{3.2}$$

where $\mathbf{u}(k)$ refers to nodal values and means the displacements at $t = k \cdot \Delta t$, and similar for the velocities $\dot{\mathbf{u}}(\mathbf{k})$.

Let us note that interfaces Γ_{ij} and Γ_{ji} are, physically, an unique interface Γ_{ij} divided into two parts at two different moments: the end of a time step (Γ_*^-) and the beginnig of the following time step (Γ_*^+). That does not mean overlapping of parameters between the two subdomains but a common frontier for the exchange of information within the problem. Obviously, the step for the discretization should be larger than $c \cdot \Delta t$, c being the highest wave velocity in each subdomain. This common frontier can be viewed as an external strip of a subdomain or an internal zone of an adjacent subdomain.

After discretizing, using the explicit finite element scheme shown in (2.10), the solution at the interface (nodal values) can be expressed in the following equivalent posing:

$$\begin{aligned} \mathbf{u}(k+1)|_{(\Gamma_{ij}^+)} = \quad & (2\,\mathbf{M} + \mathbf{C}\,\Delta t)^{-1} \cdot [2\,\Delta t^2 \cdot \mathbf{f}(k) \\ & -(2\,\Delta t^2\,\mathbf{K} - 4\,\mathbf{M}) \cdot \mathbf{u}(k)|_{(\Gamma_{ij}^-)} \\ & -(2\,\mathbf{M} - \mathbf{C}\,\Delta t)\mathbf{u}(k-1)|_{(\Gamma_{ij}^-)}], \end{aligned} \tag{3.3}$$

while the expression for the interior points of each subdomain is derived with the same explicit scheme, once the unknowns belonging to the interface problem have been removed.

Because the solution at the interface is an explicit scheme, there remains a time step constraint of the form

$$\Delta t \leq \Delta t_{cr} = \frac{\Delta x}{\sqrt{3(V_p^2 + V_s^2)}} \tag{3.4}$$

which is the Courant condition for the wave propagation problem [4]. Note that in advancing the solution from time level $t = k \cdot \Delta t$ to $t = (k+1) \cdot \Delta t$ one first computes the value of the unknowns at the interface. This step requires a small amount of information from each subdomain. After computing the interface value, there are two separate explicit problems to solve, which can be done in parallel.

3.1 Physical fundamentals

This procedure exhibits the basic features found in the Huygens' principle as an energy conservative law. This principle establishes that each point of an advancing wavefront may be considered as the source of secondary waves. Classically, for an interface with small holes in a barrier, the effect of the barrier is to suppress all propagations of the original disturbance except through the aperture where the displacement perturbations are free to propagate. The Huygens construction offers a useful but essentially qualitative contribution to the analysis of wave propagation. In particular, if waves travelling outward from a source encounter a barrier with only a tiny aperture on it (its width is much smaller than the wavelength), then this aperture appears to act just like a new point source, from which circular waves spread out.

Equation (3.3) for the coupling between two connected subdomains can be viewed as a numerical implementation of the Huygen's principle. Unlike other domain decomposition methods, in this case the PDE to solve is not a boundary condition problem on the exchanged values at the interface, but an initial condition problem with these values. In other words, each point on the interface Γ_{ij} is a source for the transmitted wavefront. With this formulation the problem can be solved by any type of finite dimensional approximation method (such as finite elements, finite differences, spectral or collocation methods). Henceforth, to solve numerically (3.1) one simply needs single–domain solvers, that can deal with boundary conditions describing either Dirichlet or Neumann conditions on $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$.

3.2 Accuracy of the method

The convergence study of this domain decomposition procedure is not necessary since it is a noniterative domain decomposition method. Therefore, the convergence results for the FEM still apply.

This scheme has the property that $\mathbf{u}_(k)$ can be computed on Ω_i and Ω_j, $i \neq j$, completely independent once $\mathbf{u}_(k-1)$ has been computed in the interface Γ. The flux of energy (calculated through the displacement and velocity fields) is derived explicitly at each point of Γ, and the remaining independent interior points are then computed using an explicit or implicit scheme.

Obviously, the energy flux is continuous. As the inner product is positive and the bilinear form (2.5) is $\mathcal{H}^1$–elliptical,

$$a(\mathbf{u},\mathbf{u}) \geq 2\mu|\mathbf{u}|^2,$$

since $\mu > 0$, the energy calculated through the functional

$$J(v_\Gamma) = \frac{1}{2}a(u_\Gamma, v_\Gamma) - f(v_\Gamma)$$

can be established for the variational problem, and hence our method fulfils the requirements of the Lax–Milgram lemma [9]. Indeed, it is true that

$$\forall \mathbf{u},\mathbf{v} \in V_\Gamma, \qquad |a(\mathbf{u},\mathbf{v})| \leq M\|\mathbf{u}\|\|\mathbf{v}\|,$$

and thus, there exists a unique element $Au \in V_\Gamma^{'}$, the dual space of $V_\Gamma \subset \mathcal{H}_0^1(\Gamma)$ (see equation 2.7), such that

$$\forall v \in V, \qquad a(u,v) = Au(v).$$

This result enables us to bound the norm in the space $V^{'}$, $\|\cdot\|^*$,

$$\|Au\|^* = \sup_{v\in V} \frac{|Au(v)|}{\|v\|} \leq M\|u\|.$$

This result allows us to bound the approximate solution obtained after splitting the domain under study:

$$\|u-u_h\|_{\Omega_i} \leq \|u-u_h\|_\Omega \leq \sqrt{\frac{M}{\alpha}} \inf_{v_h\subset V_h} \|u-v_h\|, \tag{3.5}$$

where M and $\alpha < M$ are constants that depend on the order of approximation [9]. A similar bound is allowable for the interface points. The error obtained by this scheme can be obtained from the result

$$\|\mathbf{e}\|_{m,\Omega_i} \leq \|\mathbf{e}\|_{m,\Omega} \leq c\, h^{k+1-m}\, \|\mathbf{u}\|_{k+1} \tag{3.6}$$

where c is a positive constant, h is the stepping of the approach solution, m is the order of the highest derivative involved in the variational problem (for the elasticity, $m = 1$) and k is the degree of approximation of the functions $\mathbf{v}$.

Finally, let us mention that the procedure is correct to the first–order in Δt, as expected, since this would be the case even if no domain decomposition were used.

The proof of this behaviour is similar to the one presented in [12] for general second order hyperbolic problems.

4 Numerical results

Several tests have been done on a IBM–SP2 and a SGI PowerChallenge, using Message Passing Interface (MPI) for implementing parallelism. The election of MPI is based on its functionality, portability and support for heterogeneous parallel architectures. The following examples correspond to a finite element simulation (up to 1.0 seconds) of a wave propagation through a homogeneous medium. The value of the time step is $\Delta t = 0.002s$.

Henceforth, the metrics ([25]) to be used are the computation time for the sequential algorithm (say T_1), the time for the multiprocessor execution (T_p), the speedup ($S_p = T_1/T_p$) and the efficiency ($E_p = S_p/p$), p being the number of processors.

In Table 1 we show some results about timing for the parallel algorithm. We compare it with the Schur complement [14]. In this first case, we vary the number of processors (2 up to 8) for a fixed number of unknowns (63.800). The simulation has been carried out on the SP2, which is a distributed memory architecture. It is interesting to point out that the multiblock method maintains its efficiency more or less constant while varying the processors, which is not the case for the Schur complement algorithm, that shows a good efficiency for a low number of

Table 1: Comparison of parallel metrics for the Schur complement (upper values) and the multiblock (lower values) techniques. In these tests, the number of unknowns has been fixed (63.800) while varying the number of processors (in bold types).

p:	**1**	**2**	**4**	**6**	**8**
T_1	631.63	$--$	$--$	$--$	$--$
T_p	$--$	321.20	178.54	237.97	274.80
		326.08	184.19	128.08	97.47
S_p	$--$	1.96	3.53	2.65	2.30
		1.93	3.43	4.93	6.47
E_p	$--$	0.98	0.88	0.44	0.28
		0.96	0.86	0.82	0.81

subdomains, in agreement with previous work in elastic wave modelling ([16], [17]).

In Table 2 the timing for a similar test is shown. Unlike the previous example, in this case we vary the number of unknowns for a fixed number of processors (4). As in the previous case, the efficiency of the method keeps constant even when the number of unknowns increases (a feature also exhibited by the Schur complement method). The simulation has been performed on the SGI computer, which is a shared–memory machine. A set of different snapshots are shown in Figure 2. It is clear that continuity of the wavefront is preserved during the whole simulation.

We have calculated the errors produced by the parallel algorithm in relation to the sequential one for the same explicit finite element simulation. Table 3 shows the relative difference, measured with the l_2 norm, obtained with the Schur complement technique and the multiblock algorithm. Clearly we can deduce that both methods are quite similar and produce satisfactory results, in the sense of being quite similar to the sequential problem.

Table 2: Comparison of parallel metrics for the Schur complement (upper values) and the multiblock (lower values) technique. The number of processors has been fixed to 4 while varying the number of degrees of freedom (in bold types).

NDF:	**20.000**	**40.000**	**60.000**	**80.000**	**100.000**	**120.000**
T_1	136.4	273.98	408.5	567.2	769.5	870.17
T_p	68.3	129.3	184.94	255.40	356.55	456.4
	55.01	108.01	163.94	229.95	328.98	424.65
S_p	2.0	2.11	2.2	2.2	2.2	1.9
	2.5	2.53	2.5	2.46	2.33	2.0
E_p	0.5	0.52	0.55	0.55	0.55	0.475
	0.625	0.63	0.625	0.61	0.60	0.5

Table 3: Relative errors for the Schur complement (upper values) and the multiblock (lower values) algorithms using four processors (the four boxes are ordered from the top to the bottom following the MPI communicator). The time for the wave propagation is remarked with bold types.

Δ **t**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
Ω_1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Ω_2	$1.18e^{-23}$	$6.40e^{-34}$	0.00	0.00	0.00	0.00	0.00
	$3.76e^{-23}$	$3.87e^{-35}$	0.00	0.00	0.00	0.00	0.00
Ω_3	$9.46e^{-18}$	$8.78e^{-19}$	$2.20e^{-22}$	$5.27e^{-24}$	0.00	0.00	0.00
	$6.12e^{-18}$	$5.90e^{-19}$	$7.43e^{-22}$	$5.16e^{-24}$	0.00	0.00	0.00
Ω_4	–	$6.15e^{-09}$	$2.07e^{-16}$	$1.45e^{-18}$	$2.94e^{-20}$	$2.07e^{-23}$	$6.41e^{-27}$
	–	$7.71e^{-13}$	$4.88e^{-16}$	$5.91e^{-18}$	$3.35e^{-21}$	$1.03e^{-23}$	$1.02e^{-28}$

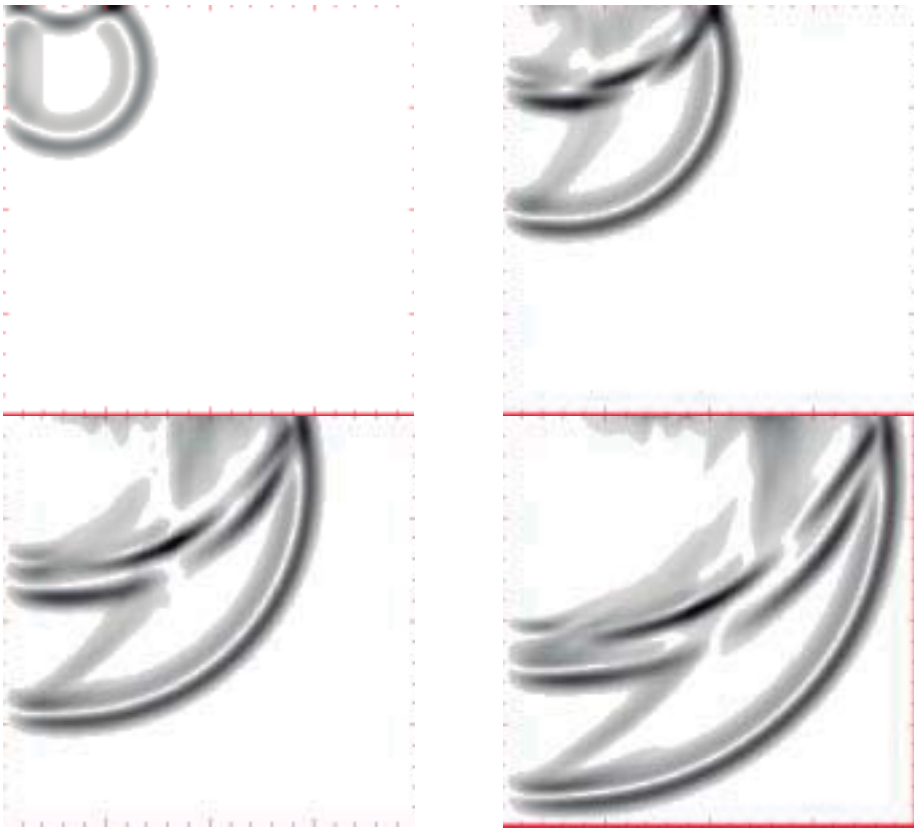

Figure 2: Snapshots for an elastic wave propagating through a homogeneous medium partitioned into four subdomains like a chess board. The size of the model is $2,000 \times 2,000\ m^2$ and the snaps have been taken at a simulation time of 0.2, 0.4, 0.6 and 0.8 seconds.

5 Concluding remarks

In this paper we have described a non iterative domain decomposition method for the elastic wave propagation problem. The depicted technique allows a very efficient parallel implementation, as regarded from the numerical examples and comparisons with other well–known methodologies. The fundamentals of our algorithm, based on Huygen's principle, provides a full physical meaning to the solution.

References

[1] M. B. Abbott, *An introduction to the method of characteristics*, Elsevier, 1966.

[2] K. Aki and P. Richards, *Quantitative Seismology. Theory and Methods*, W. H. Freeman & Co., San Francisco, 1980.

[3] G. Bachman and L. Narici, *Functional analysis*, Academic Press, New York, 1966.

[4] A. Bamberger, G. Chavent and P. Lailly, *Etude de schémas numériques pour les equations de l'elastodynamique linéaire*, Internal Report, INRIA, Paris, 1980.

[5] P. E. Bjorstad and O. B. Widlund, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, SIAM J. Numer. Anal. (1986), 1097–1120.

[6] C. Canuto, M. Hussaini, A. Quarteroni and T. A. Zang, *Spectral methods in fluid dynamics*, Springer Verlag, Berlin, 1988.

[7] J. Carcione, *Wave propagation in anisotropic linear viscoelastic media: theory and simulated wavefields*, Geophys. Int. (1990), **101**, 739–750.

[8] J. Carcione, *Domain decomposition for wave propagation problems*, J. Sci. Comput. (1991), **6(4)**, 453–472.

[9] P. G. Ciarlet, *The finite element method for elliptic problems*, North–Holland Publishing Co., 1978.

[10] R. Cottle, *Manifestations of the Schur complement*, Linear Algebra Appl. 8, 189–211, 1974.

[11] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. **II**: Partial differential equations. Interscience Publishers, 1962.

[12] C. N. Dawson and T. F. Dupont *Noniterative domain decomposition for second order hyperbolic problems*, Contemporary Mathematics (1994), **157**, 45–52.

[13] M. Dryja, *A finite–element capacitance matrix method for the elliptic problem*, SIAM J. Numer. Anal. (1983), **20**, 671–680.

[14] G. H. Golub and C. Van Loan, *Matrix Computations*, Johns Hopkins University Press, London, 1989.

[15] G. Gottlieb, M. Gunzburger and E. Turkel, *On numerical treatment for hyperbolic systems*, SIAM J. Numer. Anal. (1982), **19**, 671–682.

[16] M. Kern, *Parallel solution of the wave equation using higher order finite elements*, in F. J. Serón and F. J. Sabadell (eds.) "High Performance Computing in Seismic Modelling" (1997), Zaragoza.

[17] F. Maggio and C. Bagaini, *Parallel spectral elements for 3D linear elastodynamics*, in F. J. Serón and F. J. Sabadell (eds.) "High Performance Computing in Seismic Modelling" (1997), Zaragoza.

[18] F. Maggio and A. Quarteroni, *Acoustic wave simulation by spectral methods*, East–West J. Num. Math. (1994), **2**, 129–150.

[19] D. Marini and A. Quarteroni, *A relaxation procedure for domain decomposition methods using finite elements*, Num. Math. (1989), **55**, 575–598.

[20] J. A. Nitsche, *On Korn's second inequality*, Analyse Numérique (1981), **15(3)**, 237–248.

[21] A. Quarteroni, *Domain decomposition methods for wave propagation problems*, in D. E. Keyes, Y. Saad and D. G. Truhlar (eds.) "Domain based parallelism and problem decomposition methods in computational science and engineering", SIAM, Philadelphia, 1995.

[22] P. A. Raviart and J. M. Thomas, *Introduction a l' Analyse Numerique des Équations aux Dérivées Partielles*, Masson, Paris, 1983.

[23] D. Rodrigues, *Simulation de la propagation des ondes sismiques sur une machine massivement parallele*, Doctoral Thesis, Ecole Centrale, Paris, 1993 (in french).

[24] F. J. Sabadell, *Computational Seismic Modeling: solution with finite elements on parallel computing environments*, Doctoral Thesis, UPC, Barcelona, 1997 (in spanish)

[25] U. Schendel, *Introduction to numerical methods for parallel computers*, Ellis Horwood, Chichester, 1984.

[26] F. J. Serón, F. J. Sanz and M. Kindelán, *Elastic wave propagation with the finite element method*, Internal Report ICE–0028, IBM, Roma, 1989.

[27] F. J. Serón, J. Badal and F. J. Sabadell, *A numerical laboratory for simulation and visualization of seismic wavefields*, Geophysical Prospecting (1996), **44**, 603–642.

[28] B. F. Smith, *Domain Decomposition Algorithms for the Partial Differential Equations of Linear Elasticity*, Technical Report 517, Courant Institute of Mathematical Sciences, New York University, 1990.

[29] J. Sochacki, R. Kibicheck, J. George, W. Fletcher and S. Smithson, *Absorbing boundary conditions and surface waves*, Geophysics (1987), **52**, 60–71.

[30] E. Tessmer, D. Kessler, D. Kosloff and A. Behle, *Multi domain Chebyshev Fourier Method for the solution of the equations of motion of dynamic elasticity*, J. Comp. Phys. (1992), **100**, 355–363.

[31] E. Zampieri, *A multidomain spectral collocation solver for the elasticity problem*, in T. F. Chan et al. (eds.) "Domain Decomposition Methods for partial differential equation" (1992), SIAM, Philadelphia.

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