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ASYMPTOTIC BEHAVIOUR FOR A PHASE FIELD MODEL IN HIGHER ORDER SOBOLEV SPACES*

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Abstract

In this paper we analyze the long time behavior of a phase–field model by showing the existence of global compact attractors in the strong norm of high order Sobolev spaces.

1 Introduction

In this paper we will study the asymptotic behavior of a coupled systems of evolutionary PDE's known as phase–field equations. The system includes two unknown functions, $u(t,x)$ and $\varphi(t,x)$, which respectively represent the temperature at the point x at time t of a substance which may appear in two different phases (liquid-solid, for example) and $\varphi(t,x)$ is the phase–field function, or order parameter, which represents a local phase average and so describes the current phase at the site x. Phase–field equations have been introduced to describe and analyze phase transitions and in particular, the motion of interfaces; see [8, 9, 10, 11] for some explanations on the physical relevance of such models. In this direction, the formation of layered patterns that evolve in time has been established. Moreover, metastability, that is solutions that evolve very slowly in time, apparently sitting on an equilibrium have also been constructed; see [11, 13, 14].

On the other hand, phase–field equations have been used as a general device to obtain several other well known models of phase transitions and/or motion of interfaces as singular limits, such as Stefan, Hele–Shaw

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and Cahn–Hilliard models; see for example [9, 10, 12, 19, 20, 22]. Finally other more sophisticated models have been also introduced and analyzed in [18, 23].

Here we will consider the following semilinear system

$$\left\{ \begin{array}{rcll} \tau\varphi_t & = & \xi^2\Delta\varphi - g(\varphi) + 2u & \text{in } \Omega\times\mathbb{R}^+ \\ u_t + \frac{l}{2}\varphi_t & = & k\Delta u & \text{in } \Omega\times\mathbb{R}^+ \end{array} \right. \tag{1.1}$$

where, as indicated above, $u(t,x)$ represents the local temperature of the melt while $\varphi(t,x)$ represents a local phase average, Ω is an open, smooth bounded set in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary Γ. The nonlinear function $g(\varphi)$ is costumarily taken to be $\frac{1}{2}(\varphi^3 - \varphi)$ but we consider a more general and sufficiently regular function; see [8].

We consider (1.1) under either one of the following boundary conditions

- (D) Dirichlet boundary conditions

$$u = \varphi = 0 \quad \text{on} \quad \Gamma\times\mathbb{R}^+. \tag{1.2}$$

  which have been considered in [4, 5, 6] and [8], among others.

- (N_e) Neumann boundary conditions, as can be found, for example, in [4, 5, 6] and [23]

$$\frac{\partial u}{\partial n} = \frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \Gamma\times\mathbb{R}^+ \tag{1.3}$$

  where n is the outward unit normal vector on Γ.

- (P) Periodic boundary conditions in $\Omega = \prod_{i=1}^{N}(0, L_i)$, with $L_i > 0$, which have been considered among others in [5] and [6],

$$\varphi|_{x_i=0} = \varphi|_{x_i=L_i}, \quad \frac{\partial\varphi}{\partial x_i}|_{x_i=0} = \frac{\partial\varphi}{\partial x_i}|_{x_i=L_i}, \quad i = 1,\dots,N \tag{1.4}$$

$$u|_{x_i=0} = u|_{x_i=L_i}, \quad \frac{\partial u}{\partial x_i}|_{x_i=0} = \frac{\partial u}{\partial x_i}|_{x_i=L_i}, \quad i = 1,\dots,N \tag{1.5}$$

i.e. u, φ and their derivatives are equal in opposite faces of Γ.

Finally, we consider an initial condition at $t = 0$,

$$\varphi(x,0) = \varphi_0(x), \quad u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.6}$$

The system above can be rewritten in an evolution form by means of the enthalpy function $v = u + \frac{l}{2}\varphi$, and the resulting system reads

$$\left\{ \begin{array}{lcll} \varphi_t & = & k_1 \Delta\varphi - h(\varphi) - b\varphi + av, & \varphi(0) = \varphi_0 \\ v_t & = & k_2 \Delta v - c\Delta\varphi, & v(0) = v_0 = u_0 + \frac{l}{2}\varphi_0 \end{array} \right. , \tag{1.7}$$

supplemented with one of the boundary conditions above, (1.2), (1.3), or (1.4)–(1.5) for φ and v and with

$$k_1 = \frac{\xi^2}{\tau} > 0, \quad k_2 = k > 0,$$

$$a = \frac{2}{\tau} > 0, \quad b = \frac{l}{\tau} > 0, \quad c = \frac{kl}{2} > 0, \quad h(\varphi) = \frac{1}{\tau} g(\varphi). \tag{1.8}$$

Several results are available on the asymptotic behavior in time of the phase field equations above; see [4, 5, 6, 7] for example. All these results use in an essential way the fact that there is a natural energy functional that plays the role of a Lyapunov functional for the solutions of (1.1), which gives enough information to control the L^2 norm of the gradient of the order parameter and the L^2 norm of the enthalpy function. With this information a global compact attractor, or some finite dimensional exponentially attracting attractor or even inertial sets and manifolds have been constructed; see the references above. Due to the gradient–like structure of the system, the global attractor is described as the unstable set of the equilibria and moreover, in a generic situation, it is given by the union of the unstable set of each equilibria, [15]. This in particular implies that the omega–limit set of each single solution is made up of equilibria and in a generic situation, each solution converges to a single equilibria.

Hence we address here the question of the asymptotic behavior of solutions for smoother initial data which are taken in higher order Sobolev spaces. In this case the energy estimate mentioned before gives no further information on the stronger norm of the function space in which the initial data lives. However one would like to have some control of the solution in this stronger norm. This problem can be also brought up as the regularity of the global attractor. For example one may ask about estimates of the size of the attractor in stronger norms and, more important, if the attractors attracts solutions in stronger norms; in particular this contains the question about if the equilibria attract solutions strongly.

When approaching this problem, one observes that the phase field model above lacks of maximum or comparison principles so one looses one of the strongest tools in analyzing nonlinear pde's. On the other hand energy estimates on the solutions rapidly become intractable, and therefore they become quite useless in controlling higher order derivatives of solutions. Therefore our approach consists in exploiting the smoothing effect of the solutions combined with the natural energy estimate obtained through the Lyapunov functional to prove global existence in higher order Sobolev spaces and to analyze the asymptotic behavior for large times.

The paper is organized as follows. First, in Section 2 we prove the local existence and regularity of solutions of system (1.1) when the initial data belongs to higher order Sobolev spaces. For this we will rely on the results in [2] from where regularity will be also obtained. In Section 3 we prove the local solution is globally defined in these spaces by suitably using the natural energy estimates and the regularity obtained in the previous section. Finally in Section 4 we study the dynamics in these spaces obtaining regularity results on the attractor and the attraction in stronger norms.

2 Local existence and regularity of solutions

In this section we show local existence, uniqueness and regularity of solutions of (1.7) for suitable classes of initial data. In what follows we introduce some notations that will be used throughout the paper. We denote by $-\Delta_D$ the Laplacian operator in $L^p(\Omega)$, $1 < p < \infty$, with Dirichlet boundary conditions (1.2), i.e. with domain $W^{2,p}_D = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. In the same way, $-\Delta_{N_e}$ represents the Laplacian in $L^p(\Omega)$, $1 < p < \infty$, with Neumann boundary conditions (1.3), i.e. with domain $W^{2,p}_{N_e} = \{u \in W^{2,p}(\Omega), \frac{\partial u}{\partial n} = 0\}$. Finally, $-\Delta_P$ represents the Laplacian with periodic boundary conditions (1.4)–(1.5), in $L^p_{per}(\Omega)$, $1 < p < \infty$, with domain $W^{2,p}_P = W^{2,p}_{per}(\Omega)$, and $\Omega = \prod_{i=1}^{N}(0, L_i)$, $L_i > 0$, where the subscript "per" refers to periodicity i.e. to functions in $\mathbb{R}^n$ such that $u(x + L_i e_i) = u(x)$, a.e. $x \in \mathbb{R}^N$, and for $i = 1, \dots, N$, where $\{e_i\}_{i=1}^N$ represents the canonical basis of $\mathbb{R}^N$.

To shorten the notation, we will consider the letter B to represent either D or N_e or P and then we will use the notation L^p_B, $W^{2,p}_B$ and $-\Delta_B$, so that the three types of boundary conditions can be considered simultaneously.

Also we will denote by $W^{1,p}_B$, $1 < p < \infty$, the spaces $W^{1,p}_D = W^{1,p}_0(\Omega)$, $W^{1,p}_{N_e} = \{v \in W^{1,p}(\Omega), \frac{\partial u}{\partial n} = 0\}$, $W^{1,p}_P = W^{1,p}_{per}(\Omega)$. Since $-\Delta_B$ is a sectorial operator in L^p_B, with domain $W^{2,p}_B$ and compact resolvent, we have an associated scale of interpolation spaces, as constructed in [1, 2, 16], that will be denoted X^α_B, for $\alpha \geq 0$, and satisfy $X^\alpha_B \hookrightarrow W^{2\alpha,p}(\Omega)$ where the latter space is the usual Sobolev space. In particular $W^{1,p}_B$, as defined above, coincides with $X^{\frac{1}{2}}_B$. In general we will denote by $W^{2\alpha,p}_B$ the space X^α_B. When $p = 2$ we will also denote $W^{\alpha,2}_B$ by H^α_B.

Moreover, the spaces

$$W^{-2\alpha,p}_B = (W^{2\alpha,p'}_B)'$$

with $p' = \frac{p}{p-1}$ and $\alpha > 0$, are well defined thanks to interpolation and extrapolation, see [1, 2]. In particular $W^{-1,p}_B \doteq (W^{1,p'}_B)'$.

With these notations, we can write (1.7), with one of the boundary conditions above (1.2), (1.3) or (1.4)–(1.5), as

$$U_t + A_B U = G(U)$$

where $U = (\varphi, v)$, and

$$A_B = \begin{pmatrix} -k_1\Delta_B & -aI \\ c\Delta_B & -k_2\Delta_B \end{pmatrix} \quad \text{and} \quad G(U) = \begin{pmatrix} -h(\varphi) - b\varphi \\ 0 \end{pmatrix}. \tag{2.1}$$

First, we prove that A_B is a sectorial operator on suitable spaces and thus $-A_B$ generates an analytic semigroup. The next result is based on elementary arguments of perturbation and regularity for linear equations, so we just sketch the proof. For this, we consider A_B as a perturbation of its diagonal part which is a sectorial operator, and then apply Theorem 1.3.2 page 19 in [16]; see also [1, 2]. The main difficulty is due to the therm $c\Delta_B\varphi$, which is of the same order than the diagonal part of A_B. Therefore, we consider the realization of A_B in the space $Y_B = W_B^{2\alpha,p} \times W_B^{2\beta,p}$ with different exponents α and β, so the different norms in the two components help to compensate the large size of the perturbation. Thus we get

Proposition 2.1.

For $1 < p < \infty$ and α, β such that $0 < \alpha - \beta < 1$, the operator

$$A_B = \begin{pmatrix} -k_1\Delta_B & -aI \\ c\Delta_B & -k_2\Delta_B \end{pmatrix}$$

$$\textit{with domain} \quad D(A_B) = W_B^{2(\alpha+1),p} \times W_B^{2(\beta+1),p} \tag{2.2}$$

is a sectorial operator in $Y_B = W_B^{2\alpha,p} \times W_B^{2\beta,p}$ with compact resolvent. Thus, $-A_B$ generates a compact analytic semigroup in $W_B^{2\alpha,p} \times W_B^{2\beta,p}$, denoted by $\{e^{-A_Bt}\}_{t\geq 0}$, and the associated scale of spaces is given by

$$Y_B^\epsilon = W_B^{2(\alpha+\epsilon),p} \times W_B^{2(\beta+\epsilon),p}$$

for $0 \leq \epsilon \leq 1$.

In particular, if $(\varphi_0, v_0) \in W_B^{2\alpha,p} \times W_B^{2\beta,p}$, there exists a unique solution of

$$\left\{ \begin{array}{lcll} \varphi_t - k_1\Delta_B\varphi - av & = & 0 & \textit{in} \quad \Omega \times \mathbb{R}^+ \\ v_t + c\Delta_B\varphi - k_2\Delta_B v & = & 0 & \textit{in} \quad \Omega \times \mathbb{R}^+ \end{array} \right. \tag{2.3}$$

which is given by $(\varphi(t), v(t)) = e^{-A_Bt}(\varphi_0, v_0)$, satisfies (2.3) as an equality in $W_B^{2\alpha,p} \times W_B^{2\beta,p}$ and $(\varphi, v) \in C^\omega((0,\infty), W_B^{2(\alpha+\gamma),p} \times W_B^{2(\beta+\gamma),p})$ for every $\gamma \in \mathbb{R}$. Moreover, $\varphi, v \in C^\infty((0,\infty) \times \overline{\Omega})$. ■

Now we turn to the local existence of solutions of (1.7).

Theorem 2.2. *For $1 < p < \infty$, α and β satisfying $0 < \alpha - \beta < 1$, assume $\epsilon \in [0, 1)$ is such that the mapping*

$$h : \varphi \in W_B^{2(\alpha+\epsilon),p} \to h(\varphi) \in W_B^{2\alpha,p} \tag{2.4}$$

is locally Lipschitz.

Then, for every $(\varphi_0, v_0) \in W_B^{2(\alpha+\epsilon),p} \times W_B^{2(\beta+\epsilon),p}$*, there exists a unique solution of (1.7) in* $[0, T)$*, with* $T = T(\varphi_0, v_0) > 0$*. Moreover, the solution satisfies*

$$(\varphi, v) \in C([0,T), W_B^{2(\alpha+\epsilon),p} \times W_B^{2(\beta+\epsilon),p}) \cap C((0,T), W_B^{2(\alpha+1),p} \times W_B^{2(\beta+1),p})$$

$$(\varphi_t, v_t) \in C((0,T), W_B^{2(\alpha+\theta),p} \times W_B^{2(\beta+\theta),p})$$

for every $0 \leq \theta < 1$ *and* (φ, v) *satisfies (1.7) as an equality in* $W_B^{2\alpha,p} \times W_B^{2\beta,p}$*.*

If h *maps bounded sets into bounded sets and we assume the solution* (φ, v) *has been extended to a maximal interval of time* $[0, T_{max})$*, we have that either* $T_{max} = +\infty$*, or the solution blows–up in the* $W_B^{2(\alpha+\epsilon),p} \times W_B^{2(\beta+\epsilon),p}$ *norm as* $t \to T_{max}$*.*

Proof. From Proposition 2.1, we have that A_B is a sectorial operator in $Y_B = W_B^{2\alpha,p} \times W_B^{2\beta,p}$, and from (2.4) we get that the mapping G, defined in (2.1), is locally Lipschitz from $Y_B^\epsilon = W_B^{2(\alpha+\epsilon),p} \times W_B^{2(\beta+\epsilon),p}$ into $Y_B = W_B^{2\alpha,p} \times W_B^{2\beta,p}$. From [1, 2, 16] we get the result. ■

Next, we prove the existence of solutions of (1.7) with initial data in $W_B^{n,p} \times W_B^{n-1,p}$ for some $n \in \mathbb{N}$. These solutions will satisfy the equations as an equality in the space $W_B^{n-1,p} \times W_B^{n-2,p}$. We start with $n = 1$, which, with the notations on Theorem 2.2, corresponds to $\alpha = 0$, $\beta = -\frac{1}{2}$ and $\epsilon = \frac{1}{2}$, and we must show that, under some growth and regularity conditions on h, that $h : W_B^{1,p} \to L_B^p$ is locally Lipschitz on bounded set.

Proposition 2.3. *For* $1 < p < \infty$ *assume that* h *satisfies one the following conditions*
i) $h \in C^1(\mathbb{R})$*, if* $N < p$*.*
ii) $h \in C^1(\mathbb{R})$ *satisfies*

$$|h(s)| \leq C(1 + |s|^r) \quad \text{and} \quad |h'(s)| \leq C(1 + |s|^{r-1}), \quad s \in \mathbb{R} \tag{2.5}$$

if $N \geq p$*, with* r *such that*

$$1 \leq r \begin{cases} < \infty & \text{if } N = p \\ \leq \frac{N}{N-p} & \text{if } N > p \end{cases}. \tag{2.6}$$

Then, given an initial condition $(\varphi_0, v_0) \in W_B^{1,p} \times L_B^p$ *there exists a unique local solution* $(\varphi, v) \in C([0,T), W_B^{1,p} \times L_B^p)$ *of (1.7) satisfying the system as an equality in* $L_B^p \times W_B^{-1,p}$*. Moreover, for every* $0 < \theta \leq 1$ *we have that*

$$(\varphi, v) \in C((0,T), W_B^{2,p} \times W_B^{1,p}) \quad \textit{and}$$

$$(\varphi_t, v_t) \in C((0,T), W_B^{2-\theta,p} \times W_B^{1-\theta,p}).$$

If T_{max} *is the maximal existence time then either* $T_{max} = \infty$ *or the solution blows–up in the* $W_B^{1,p} \times L_B^p$ *norm as* $t \to T_{max}$*.*

Proof. If we prove that $h : W_B^{1,p} \to L_B^p$ is locally Lipschitz and maps bounded sets into bounded sets, then Theorem 2.2 gives the result with $\alpha = 0$, $\beta = \frac{-1}{2}$ and $\epsilon = \frac{1}{2}$.

If $N < p$, from Sobolev embeddings we get $W_B^{1,p} \hookrightarrow C(\overline{\Omega})$. From this and since h is of class C^1, we conclude. On the other hand, if $N \geq p$, from (2.5) we have that $h : L_B^{pr} \to L_B^p$ is locally Lipschitz and maps bounded sets into bounded sets. Hence, if $N = p$, from Sobolev embedding, we get $W_B^{1,p} \hookrightarrow L_B^s$ for every $s \in (1, \infty)$, while if $N > p$, then $W_B^{1,p} \hookrightarrow L_B^s$ for $s \leq \frac{Np}{N-p}$. Thus taking $s = pr$ we obtain the result. ■

Next, we show that if h is of class C^2, we have local existence for (1.7), with initial data $(\varphi_0, v_0) \in W_B^{2,p} \times W_B^{1,p}$. For this, we apply Theorem 2.2 in the space $W_B^{1,p} \times L_B^p$. Note that $W_B^{2,p} \times W_B^{1,p}$ corresponds in Theorem 2.2 to $\alpha = \epsilon = \frac{1}{2}$ and $\beta = 0$. Hence, we have to prove that $h : W_B^{2,p} \to W_B^{1,p}$ is locally Lipschitz.

Proposition 2.4. *Let* $1 < p < \infty$ *and assume that* h *satisfies* $h(0) = 0$ *if* $B = D$ *and one of the following hypotheses:*
i) $h \in C^2(\mathbb{R})$ *if* $N < 2p$*.*
ii) $h \in C^2(\mathbb{R})$ *and satisfies*

$$|h(s)| \leq C(1 + |s|^r), \quad |h'(s)| \leq C(1 + |s|^{r-1}), \quad |h''(s)| \leq C(1 + |s|^{r-2}) \tag{2.7}$$

if $N/3 \leq p < N/2$*, with* r *such that*

$$2 \leq r \begin{cases} < \infty & \textit{if } N = 2p \\ \leq \frac{N-p}{N-2p} & \textit{if } \frac{N}{3} \leq p < \frac{N}{2} \end{cases}. \tag{2.8}$$

Then, for every $(\varphi_0, v_0) \in W^{2,p}_B \times W^{1,p}_B$ *there exists a local solution,* (φ, v)*, of (1.7) satisfying the equations as an equality in* $W^{1,p}_B \times L^p_B$*. The solution also satisfies*

$$(\varphi, v) \in C([0,T), W^{2,p}_B \times W^{1,p}_B) \cap C((0,T), W^{3,p}_B \times W^{2,p}_B)$$

$$(\varphi_t, v_t) \in C((0,T), W^{3-\theta,p}_B \times W^{2-\theta,p}_B)$$

for every $\theta \in (0,1]$*. Moreover, if* T_{max} *is the maximal existence time, then either* $T_{max} = \infty$ *or the solution blows–up in the* $W^{2,p}_B \times W^{1,p}_B$ *norm as* $t \to T_{max}$*.*

Proof. The result comes from Theorem 2.2 with $\alpha = \epsilon = \frac{1}{2}$ and $\beta = 0$ provided $h : W^{2,p}_B \to W^{1,p}_B$ is locally Lipschitz and maps bounded sets into bounded sets. Note that for this if $B = D$, from boundary conditions in $W^{1,p}_B$, we must have $h(0) = 0$.

Therefore, we consider a bounded set K in $W^{2,p}_B$ and show that for some constant $M_1 > 0$

$$\|h(\varphi_1) - h(\varphi_2)\|_{W^{1,p}_B} \leq M_1 \|\varphi_1 - \varphi_2\|_{W^{2,p}_B} \quad \text{for all} \quad \varphi_1, \varphi_2 \in K. \quad (2.9)$$

For this it is enough to prove that

$$h : W^{2,p}_B \to L^p_B \quad (2.10)$$

is Lipschitz on bounded sets and

$$\|\nabla(h(\varphi_1) - h(\varphi_2))\|_{L^p_B} \leq M_2 \|\varphi_1 - \varphi_2\|_{W^{2,p}_B}, \quad (2.11)$$

for every φ_1, φ_2 in K. Also note that

$$\|\nabla(h(\varphi_1) - h(\varphi_2))\|_{L^p_B} \leq$$

$$\leq \|h'(\varphi_1)(\nabla\varphi_1 - \nabla\varphi_2)\|_{L^p_B} + \|(h'(\varphi_1) - h'(\varphi_2))\nabla\varphi_2\|_{L^p_B}. \quad (2.12)$$

If $N < 2p$ from Sobolev embeddings $W^{2,p}_B \hookrightarrow C(\overline{\Omega})$, and therefore $\|\varphi\|_{L^\infty} \leq c_1$, for every $\varphi \in K$. Therefore,

$$\|h(\varphi_1) - h(\varphi_2)\|_{L^p_B} \leq c_2 \|\varphi_1 - \varphi_2\|_{L^p_B} \leq c_3 \|\varphi_1 - \varphi_2\|_{W^{2,p}_B}$$

for some $c_2, c_3 > 0$ and we obtain (2.10). Next, observe that $\|h'(\varphi_1)(\nabla\varphi_1 - \nabla\varphi_2)\|_{L^p_B} \leq \|h'(\varphi_1)\|_{L^\infty} \|(\nabla\varphi_1 - \nabla\varphi_2)\|_{L^p_B} \leq c_4 \|\varphi_1 - \varphi_2\|_{W^{2,p}_B}$.

Finally, since $\|(h'(\varphi_1)-h'(\varphi_2))\nabla\varphi_2\|_{L^p_B} \leq \|h'(\varphi_1)-h'(\varphi_2)\|_{L^\infty}\|\nabla\varphi_2\|_{L^p_B}$, again we get, using $h \in C^2(\mathbb{R})$,

$$\|h'(\varphi_1) - h'(\varphi_2)\|_{L^\infty} \leq c_5\|\varphi_1 - \varphi_2\|_{L^\infty} \leq c_6\|\varphi_1 - \varphi_2\|_{W^{2,p}_B}.$$

Thus, using (2.12) we obtain (2.11), (2.10) and so (2.9).

If $N \geq 2p$ and $h(s)$ satisfies (2.7), then $h : L^{pr}_B \to L^p_B$, is Lipschitz and bounded on bounded sets. Then, to prove (2.10) it suffices to show that $W^{2,p}_B \subset L^{pr}_B$. Indeed, if $N = 2p$, from Sobolev embeddings we have that $W^{2,p}_B \hookrightarrow L^s_B$ for every $s < \infty$ while if $N > 2p$, then $W^{2,p}_B \subset L^s_B$ for $s \leq \frac{Np}{N-2p}$ and we can take $s = pr$, since in (2.8) we have $r \leq \frac{N-p}{N-2p} \leq \frac{N}{N-2p}$.

To conclude, we prove (2.11). From (2.12), and using Hölder's inequality, we have that

$$\|\nabla(h(\varphi_1) - h(\varphi_2))\|_{L^p_B} \leq$$

$$\leq \|h'(\varphi_1)\|_{L^q_B}\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^{q'}_B} + \|h'(\varphi_1) - h'(\varphi_2)\|_{L^q_B}\|\nabla\varphi_2\|_{L^{q'}_B}$$

where $\frac{1}{q}+\frac{1}{q'} = \frac{1}{p}$ have to be chosen such that $h'(\varphi_i) \in L^q_B$ and $\nabla\varphi_i \in L^{q'}_B$. Now observe that this choice is possible since, when $\varphi \in W^{2,p}_B$, Sobolev embeddings give $\nabla\varphi \in W^{1,p}_B \hookrightarrow L^{q'}_B$ for $1 - \frac{N}{p} \geq -\frac{N}{q'}$ and then we can take $q' = \frac{Np}{N-p}$ and $q = N$.

Thus, it remains to prove that $h' : W^{2,p}_B \to L^N_B$, is Lipschitz and bounded on bounded sets. But from (2.7), $h' : L^{N(r-1)}_B \to L^N_B$ is Lipschitz on bounded sets, provided $r \geq 2$, so it suffices to have that $W^{2,p}_B \hookrightarrow L^{N(r-1)}_B$ and then we conclude. Indeed, Sobolev embeddings, as in the preceding paragraph, concludes when $N = 2p$ for any $r \geq 2$, while when $N > 2p$ we must have $r \leq \frac{N-p}{N-2p}$ which is compatible with $r \geq 2$ only if $N \leq 3p$. ∎

Observe that Theorem 2.2 may be used also if h is of class C^n, $n \geq 3$, to obtain a local solution of (1.7) with initial data in $W^{n,p}_B \times W^{n-1,p}_B$. With the notations of Theorem 2.2 this situation corresponds to $\alpha = \frac{n-1}{2}$, $\beta = \frac{n}{2} - 1$ and $\epsilon = \frac{1}{2}$. Therefore, one must show that $h : W^{n,p}_B \to W^{n-1,p}_B$ is locally Lipschitz.

Also note that if $h : W^{n,p}(\Omega) \to W^{n-1,p}(\Omega)$ is locally Lipschitz, and since the space $W^{n-1,p}_B$ may include some boundary conditions, then some conditions must be imposed on h, to obtain that h maps $W^{n,p}_B$ into $W^{n-1,p}_B$.

In particular in the case of periodic boundary conditions, if $h : W^{n,p}(\Omega) \to W^{n-1,p}(\Omega)$ then with no further requirements we will automatically have $h(W^{n,p}_P) \hookrightarrow W^{n-1,p}_P$.

In the case of Dirichlet boundary conditions, $u \in W^{n,p}_D$ satisfies that, on Γ, $(-\Delta)^j u = 0$, for $j = 0, 1, \ldots, k$, if $n = 2k+1$ or $n = 2k+2$, so we need to impose that some derivatives of h are zero at zero. In particular $h(0) = 0$ will be always required.

The case of Neumann boundary conditions is the more involved one since the boundary conditions in $W^{n,p}_{N_e}$ are of the form $\frac{\partial}{\partial n}((-\Delta)^j u) = 0$, for $j = 0, \ldots, k-1$ if $n = 2k$ or $n = 2k+1$ and then one must verify that $\frac{\partial}{\partial n}((-\Delta)^j (h(u))) = 0$, for $j = 0, \ldots, k-2$ if $n = 2k$ or for $j = 0, \ldots, k-1$ if $n = 2k+1$. When n is large checking this property becomes nontrivial. However note that $\frac{\partial}{\partial n} h(u) = h'(u)\frac{\partial u}{\partial n} = 0$, provided that the normal derivative of u vanishes on Γ; so the boundary condition for $j = 0$ will be always satisfied. Indeed it can be easily verified that for $n \leq 4$ no further assumptions are needed on h.

In fact we have

Lemma 2.5. *Assume that* $h : W^{n,p}(\Omega) \to W^{n-1,p}(\Omega)$ *is locally Lipschitz (respectively Lipschitz on bounded sets) and* $n \geq 1$.
i) If $B = P$, *then* $h : W^{n,p}_P \to W^{n-1,p}_P$ *is also locally Lipschitz (respectively Lipschitz on bounded sets).*
ii) If $B = D$, *we also assume that* h *satisfies* $h(0) = 0$ *and*

$$h^{j)}(0) = 0, \quad j = 2, 3, \ldots, k, \quad \text{if } n = 2k \text{ or } 2k+1 \quad \text{with } k \geq 2 \tag{2.13}$$

then $h : W^{n,p}_D \to W^{n-1,p}_D$ *is locally Lipschitz (respectively Lipschitz on bounded sets).*
iii) If $B = N_e$, *we assume that for* $n \geq 5$ *and for any* $u \in W^{n,p}_{N_e}$ *the following holds*

$$\frac{\partial}{\partial n}[(-\Delta)(h(u))] = \ldots = \frac{\partial}{\partial n}[(-\Delta)^{k-2}(h(u))] = 0, \ \text{on } \Gamma \quad \text{if } n = 2k \tag{2.14}$$

$$\frac{\partial}{\partial n}[(-\Delta)(h(u))] = \ldots = \frac{\partial}{\partial n}[(-\Delta)^{k-1}(h(u))] = 0, \ \text{on } \Gamma \quad \text{if } n = 2k+1. \tag{2.15}$$

Then $h : W^{n,p}_{N_e} \to W^{n-1,p}_{N_e}$ *is locally Lipschitz (respectively Lipschitz on bounded sets).* ■

Remark 2.6. The typical nonlinearity for the phase field model, $h(\varphi) = \frac{1}{2}(\varphi^3 - \varphi)$ satisfies the hypotheses of Lemma 2.5 part ii) only for $n \leq 5$, since $h(0) = 0$ and $h''(0) = 0$ but $h'''(0) \neq 0$. Also, (2.14)–(2.15) are satisfied for $n \leq 4$

Also note that it is enough to verify conditions (2.14) and (2.15) for smooth functions satisfying the boundary conditions in $W^{n,p}_{N_e}$.

We will also make use of the following result

Lemma 2.7. *We assume that* h *is of class* C^{n+1}, $n \geq 1$. *Then the function*

$$h : W^{n,p}(\Omega) \cap W^{1,\infty}(\Omega) \to W^{n,p}(\Omega) \cap W^{1,\infty}(\Omega)$$

is Lipschitz on bounded sets, i.e. for every bounded set $K \subset W^{n,p}(\Omega) \cap W^{1,\infty}(\Omega)$, *there exists a constant* $C = C(K)$ *such that for all* $\varphi \in K$ *we have*

$$\|h(\varphi_1) - h(\varphi_2)\|_{W^{n,p}} + \|h(\varphi_1) - h(\varphi_2)\|_{W^{1,\infty}} \leq$$
$$\leq C(\|\varphi_1 - \varphi_2\|_{W^{n,p}} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}}).$$

Proof. First, note that if K is bounded in $L^\infty(\Omega)$ and h is of class C^1 then there exists $c_1 > 0$ such that

$$\|h(\varphi_1) - h(\varphi_2)\|_{L^\infty} \leq c_1\|\varphi_1 - \varphi_2\|_{L^\infty} \tag{2.16}$$

for every $\varphi_1, \varphi_2 \in K$. Now, assume h is of class C^2 and K is bounded in $W^{1,\infty}$. Then we show that there exists $c_2 > 0$ such that

$$\|h(\varphi_1) - h(\varphi_2)\|_{W^{1,\infty}} \leq c_2\|\varphi_1 - \varphi_2\|_{W^{1,\infty}} \tag{2.17}$$

for every $\varphi_1, \varphi_2 \in K$. Since for any $i = 1, \dots, N$, we have $\frac{\partial}{\partial x_i}(h(\varphi_1) - h(\varphi_2)) = (h'(\varphi_1) - h'(\varphi_2))\frac{\partial \varphi_1}{\partial x_i} + h'(\varphi_2)(\frac{\partial \varphi_1}{\partial x_i} - \frac{\partial \varphi_2}{\partial x_i})$ then we get

$$\|\frac{\partial}{\partial x_i}(h(\varphi_1) - h(\varphi_2))\|_{L^\infty} \leq$$

$$\leq \|h'(\varphi_2)\|_{L^\infty}\|\frac{\partial \varphi_1}{\partial x_i} - \frac{\partial \varphi_2}{\partial x_i}\|_{L^\infty} + \|\frac{\partial \varphi_1}{\partial x_i}\|_{L^\infty}\|h'(\varphi_1) - h'(\varphi_2)\|_{L^\infty}.$$

Now, using (2.16) for h', we obtain $\|\nabla(h(\varphi_1) - h(\varphi_2))\|_{L^\infty} \le c_3\|\varphi_1 - \varphi_2\|_{W^{1,\infty}}$ and using again (2.16) we get (2.17).

Therefore, to prove the Lemma, it is enough to show there exists $C > 0$ such that for every $\varphi_1, \varphi_2 \in K$

$$\|h(\varphi_1) - h(\varphi_2)\|_{W^{n,p}} \le C(\|\varphi_1 - \varphi_2\|_{W^{n,p}} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}}).$$

Now we proceed by induction in n. First, observe that (2.17) gives the result for $n = 1$. Assume the result is true for a function of class C^n, then given h of class C^{n+1} we consider a bounded set $K \subset W^{n,p}(\Omega) \cap W^{1,\infty}(\Omega)$ and $\varphi_1, \varphi_2 \in K$. Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $|\alpha| = n$, and take i such that $\alpha_i \ge 1$. Then we can write $D^\alpha[h(\varphi_1) - h(\varphi_2)] = D^\beta[\frac{\partial}{\partial x_i}(h(\varphi_1) - h(\varphi_2))]$ with $|\beta| = n - 1$. Now, applying D^β to

$$\frac{\partial}{\partial x_i}(h(\varphi_1) - h(\varphi_2)) = \left(h'(\varphi_1) - h'(\varphi_2)\right)\frac{\partial \varphi_1}{\partial x_i} + h'(\varphi_2)\left(\frac{\partial \varphi_1}{\partial x_i} - \frac{\partial \varphi_2}{\partial x_i}\right)$$

we get

$$D^\alpha\left(h(\varphi_1) - h(\varphi_2)\right) = \sum_{\sigma,\delta} D^\sigma\left(h'(\varphi_2)\right) D^\delta\left(\frac{\partial \varphi_1}{\partial x_i} - \frac{\partial \varphi_2}{\partial x_i}\right) +$$

$$+ \sum_{\sigma,\delta} D^\sigma\left(h'(\varphi_1) - h'(\varphi_2)\right) D^\delta\left(\frac{\partial \varphi_1}{\partial x_i}\right) \tag{2.18}$$

with σ, δ, such that $|\beta| = |\sigma| + |\delta| = n - 1$. For such σ and δ, we define $q(\sigma) = \frac{(n-1)p}{|\sigma|} \ge p$ and $q'(\delta) = \frac{(n-1)p}{|\delta|} \ge p$ and applying Hölder's inequality with exponents q and q' in each term of (2.18), we obtain

$$\|D^\alpha(h(\varphi_1) - h(\varphi_2))\|_{L^p} \le \sum_{\sigma,\delta} \|D^\sigma\left(h'(\varphi_2)\right)\|_{L^q} \|D^\delta\left(\frac{\partial \varphi_1}{\partial x_i} - \frac{\partial \varphi_2}{\partial x_i}\right)\|_{L^{q'}} +$$

$$+ \sum_{\sigma,\delta} \|D^\sigma[h'(\varphi_1) - h'(\varphi_2)]\|_{L^q} \|D^\delta\left(\frac{\partial \varphi_1}{\partial x_i}\right)\|_{L^{q'}}, \tag{2.19}$$

since $\frac{1}{q} + \frac{1}{q'} = \frac{1}{p}$. Now, we prove that for some constant $C_1 > 0$

$$\|D^\delta\left(\frac{\partial \varphi_1}{\partial x_i} - \frac{\partial \varphi_2}{\partial x_i}\right)\|_{L^{q'}} \le C_1(\|\varphi_1 - \varphi_2\|_{W^{n,p}} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}}) \tag{2.20}$$

for $\varphi_1, \varphi_2 \in K$. Writing $\frac{1}{q'} = \frac{j}{(n-1)}(\frac{1}{p} - \frac{1}{r}) + \frac{1}{r}$ for $r = \infty$ and $j = |\delta|$ and applying a Gagliardo-Nirenberg type inequality in [17], we get

$$\|D^{\delta}\left(\frac{\partial\varphi_1}{\partial x_i} - \frac{\partial\varphi_2}{\partial x_i}\right)\|_{L^{q'}} \leq$$

$$\leq C_2 \sum_{|\delta^*|=n-1} \|D^{\delta^*}\left(\frac{\partial\varphi_1}{\partial x_i} - \frac{\partial\varphi_2}{\partial x_i}\right)\|_{L^p}^{\frac{p}{q'}} \|\frac{\partial\varphi_1}{\partial x_i} - \frac{\partial\varphi_2}{\partial x_i}\|_{L^\infty}^{1-\frac{p}{q'}}.$$

Then, since $\|D^{\delta^*}\left(\frac{\partial\varphi_1}{\partial x_i} - \frac{\partial\varphi_2}{\partial x_i}\right)\|_{L^p} \leq \|\varphi_1 - \varphi_2\|_{W^{n,p}}$ and $\|\frac{\partial\varphi_1}{\partial x_i} - \frac{\partial\varphi_2}{\partial x_i}\|_{L^\infty} \leq \|\varphi_1 - \varphi_2\|_{W^{1,\infty}}$ and applying Young's inequality with exponents $l = \frac{q'}{p}$ and $l' = \frac{q'}{q'-p}$, we obtain $\|D^{\delta}\left(\frac{\partial\varphi_1}{\partial x_i} - \frac{\partial\varphi_2}{\partial x_i}\right)\|_{L^{q'}} \leq C_3\|\varphi_1 - \varphi_2\|_{W^{n,p}} + C_4\|\varphi_1 - \varphi_2\|_{W^{1,\infty}}$ for some positive constants C_3, C_4, and we get (2.20). Analogously,

$$\|D^{\delta}\frac{\partial\varphi_1}{\partial x_i}\|_{L^{q'}} \leq C_5(\|\varphi_1\|_{W^{n,p}} + \|\varphi_1\|_{W^{1,\infty}}) \leq C_6(K). \tag{2.21}$$

On the other hand, using the interpolation inequality above, we also have

$$\|D^{\sigma}\left(h'(\varphi_1) - h'(\varphi_2)\right)\|_{L^q} \leq$$

$$\leq C_7 \sum_{|\sigma^*|=n-1} \|D^{\sigma^*}\left(h'(\varphi_1) - h'(\varphi_2)\right)\|_{L^p}^{\frac{p}{q}} \|h'(\varphi_1) - h'(\varphi_2)\|_{L^\infty}^{1-\frac{p}{q}}.$$

Since $\|D^{\sigma^*}\left(h'(\varphi_1) - h'(\varphi_2)\right)\|_{L^p} \leq \|h'(\varphi_1) - h'(\varphi_2)\|_{W^{n-1,p}}$ and using Young's inequality with exponents $l = \frac{q}{p}$ and $l' = \frac{q}{q-p}$, we get

$$\|D^{\sigma}\left(h'(\varphi_1) - h'(\varphi_2)\right)\|_{L^q} \leq$$

$$\leq C_8(\|h'(\varphi_1) - h'(\varphi_2)\|_{W^{n-1,p}} + \|h'(\varphi_1) - h'(\varphi_2)\|_{L^\infty}). \tag{2.22}$$

In a similar way we get $\|D^{\sigma}\left(h'(\varphi_2)\right)\|_{L^q} \leq C_9(\|h'(\varphi_2)\|_{W^{n-1,p}} + \|h'(\varphi_2)\|_{L^\infty})$. From the induction assumption, since h' is of class C^n, then $h' : W^{n-1,p}(\Omega) \cap W^{1,\infty}(\Omega) \to W^{n-1,p}(\Omega) \cap W^{1,\infty}(\Omega)$ is Lipschitz on bounded sets and therefore we have

$$\|D^{\sigma}[h'(\varphi_2)]\|_{L^q} \leq C_{10}(K) \tag{2.23}$$

and

$$\|h'(\varphi_1) - h'(\varphi_2)\|_{W^{n-1,p}} \le C_{11} \left(\|\varphi_1 - \varphi_2\|_{W^{n-1,p}} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}} \right).$$

Thus, using this and (2.16) for h' and replacing (2.20), (2.21), (2.22) and (2.23) into (2.19), we get

$$\|D^\alpha \left(h(\varphi_1) - h(\varphi_2) \right)\|_{L^p} \le C_{12} \left(\|\varphi_1 - \varphi_2\|_{W^{n,p}} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}} \right)$$

and we conclude. ∎

From the previous lemmas, we get the following result.

Proposition 2.8. *Assume that h is of class C^n, $n \ge 2$ and $N < (n-1)p$ and satisfies (2.13) if $B = D$ or (2.14)–(2.15) if $B = N_e$.*

Then, for every $(\varphi_0, v_0) \in W^{n,p}_B \times W^{n-1,p}_B$ there exists a unique local solution, (φ, v), in $W^{n-1,p}_B \times W^{n-2,p}_B$ of (1.7), which satisfies

$$(\varphi, v) \in C([0,T), W^{n,p}_B \times W^{n-1,p}_B) \cap C((0,T), W^{n+1,p}_B \times W^{n,p}_B)$$

$$(\varphi_t, v_t) \in C((0,T), W^{n+1-\theta,p}_B \times W^{n-\theta,p}_B)$$

for every $\theta \in (0,1]$. Moreover, if $T_{max} > 0$, is the maximal existence time then either $T_{max} = \infty$ or the solution blows–up in the $W^{n,p}_B \times W^{n-1,p}_B$ norm as $t \to T_{max}$.

Proof. Since $N < (n-1)p$, from Sobolev embeddings, we have $W^{n,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and therefore $W^{n,p}(\Omega) \hookrightarrow W^{n-1,p}(\Omega) \cap W^{1,\infty}(\Omega)$. From Lemma 2.7, we get that $h : W^{n,p}(\Omega) \to W^{n-1,p}(\Omega)$ is Lipschitz on bounded sets. Then, from Lemma 2.5, and the embeddings above, $h : W^{n,p}_B \to W^{n-1,p}_B$ is Lipschitz on bounded sets. Applying Theorem 2.2 with $\alpha = \frac{n-1}{2}$, $\beta = \frac{n}{2} - 1$ and $\epsilon = \frac{1}{2}$ we conclude. ∎

Note that the case $n = 2$ of the proposition complements the result in Proposition 2.4. Also note that in contrast to Propositions 2.3 and 2.4 no growth assumptions are needed for the previous result; indeed growth assumptions are replaced by smoothness of h and (2.13) if $B = D$ or (2.14)–(2.15) if $B = N_e$.

Next, we prove a regularity result for the solutions of (1.7) with initial data in $W^{1,p}_B \times L^p_B$, $1 < p < \infty$ given in Proposition 2.3. For this a key remark is that the growth assumptions in Proposition 2.3, namely the upper bound for r in (2.6), increases with p. Therefore if one can use that proposition in $W^{1,p}_B \times L^p_B$ it can also be used in $W^{1,s}_B \times L^s_B$ for any larger s.

Proposition 2.9.
i) Assume h is of class C^1 and satisfies (2.5) and (2.6) if $N \geq p$. Then, for every $(\varphi_0, v_0) \in W^{1,p}_B \times L^p_B$, the local solution of (1.7) given in Proposition 2.3 also satisfies

$$(\varphi, v) \in C((0,T), W^{2,s}_B \times W^{2,s}_B)$$

and

$$(\varphi_t, v_t) \in C((0,T), W^{2-\theta,s}_B \times W^{1-\theta,s}_B)$$

for every $s > 1$ and $0 < \theta \leq 1$. In particular, $(\varphi, v) \in C((0,T), C^1(\overline{\Omega}) \times C^1(\overline{\Omega}))$.

ii) Assume moreover that h is of class C^n, for $n \geq 2$, and assume conditions (2.13) if $B = D$ or (2.14)–(2.15) if $B = N_e$ are satisfied. Then, for every $(\varphi_0, v_0) \in W^{1,p}_B \times L^p_B$, the local solution of (1.7) given in Proposition 2.3 satisfies, besides the regularity in i),

$$(\varphi, v) \in C((0,T), W^{n+1,s}_B \times W^{n+1,s}_B)$$

and

$$(\varphi_t, v_t) \in C((0,T), W^{n+1-\theta,s}_B \times W^{n-\theta,s}_B)$$

for every $\theta \in (0,1]$ and $s > 1$. In particular, $(\varphi, v) \in C((0,T), C^n(\overline{\Omega}) \times C^n(\overline{\Omega}))$.

Proof.
i) First, we prove that for every $s > N$ and $\theta \in (0,1]$

$$\begin{aligned} &(\varphi, v) \in C((0,T), W^{2,s}_B \times W^{1,s}_B) \qquad \text{and} \\ &(\varphi_t, v_t) \in C((0,T), W^{2-\theta,s}_B \times W^{1-\theta,s}_B). \end{aligned} \tag{2.24}$$

Assume first $p \geq N$. Then the solution of (1.7) with initial data $(\varphi_0, v_0) \in W^{1,p}_B \times L^p_B$, given in Proposition 2.3, satisfies $(\varphi(t), v(t)) \in W^{2,p}_B \times W^{1,p}_B \hookrightarrow W^{1,s}_B \times L^s_B$, for every $s \in (1, \infty)$. Taking $s > N$, $t_1 > 0$ and initial data $(\varphi(t_1), v(t_1))$ in $W^{1,s}_B \times L^s_B$, from Proposition 2.3 part i) we obtain (2.24).

Now if $p < N$, we show that for $t_1 > 0$ and some $q \geq N$ we have $(\varphi(t_1), v(t_1)) \in W^{1,q}_B \times L^q_B$ and as before, we get (2.24). First from Proposition 2.3 and Sobolev embeddings, we have $(\varphi(t_1), v(t_1)) \in W^{1,q}_B \times L^q_B$ for $q = p^* = \frac{Np}{N-p} > p$. Now note that since h satisfies (2.5) with $r \leq \frac{N}{N-p}$ then for every $q > p$ we can apply Proposition 2.3 in $W^{1,q}_B \times L^q_B$. Thus, if $q \geq N$ we conclude as before. Otherwise we can repeat the argument above and in a finite number of steps we get (2.24).

Now, from (2.24), we have , $v_t, \Delta\varphi \in C((0,T), L^s_B)$ and since

$$k_2 \Delta_B v = v_t + c\Delta_B \varphi \tag{2.25}$$

using elliptic regularity results, we obtain that $v \in C((0,T), W^{2,s}_B)$. Now Sobolev embeddings give $(\varphi, v) \in C((0,T), C^1(\overline{\Omega}) \times C^1(\overline{\Omega}))$.
ii) We now show by induction in $1 \leq k \leq n$, that if $s > N$, then we have

$$(\varphi, v) \in C((0,T), W^{k+1,s}_B \times W^{k,s}_B) \text{ and}$$

$$(\varphi_t, v_t) \in C((0,T), W^{k+1-\theta,s}_B \times W^{k-\theta,s}_B), \tag{2.26}$$

for every $\theta \in (0, 1]$.

From part i) we have that (2.26) is true for $k = 1$. Now, we assume that (2.26) is true for some k and we prove that (2.26) is also true for $k + 1$, whenever $k + 1 \leq n$. Given $t_1 > 0$ from the induction hypothesis, we have that $(\varphi(t_1), v(t_1)) \in W^{k+1,s}_B \times W^{k,s}_B$. Thus, since $s > N$, we can apply Proposition 2.8 to get (2.26) for $k + 1$. Now, using $v_t, \Delta\varphi \in C((0,T), W^{n-1,s}_B)$, again from (2.25), we get $v \in C((0,T), W^{n+1,s}_B)$. Sobolev embeddings give the rest. ■

Analogously, for the solutions of (1.7) with initial data in $W^{2,p}_B \times W^{1,p}_B$, $\frac{N}{3} < p < \infty$ given in Proposition 2.4, we have

Proposition 2.10.
i) Assume h is of class C^2 and satisfies (2.7) and (2.8) if $N/3 \leq p < N/2$. Then, for every $(\varphi_0, v_0) \in W^{2,p}_B \times W^{1,p}_B$, the local solution of (1.7) given in Proposition 2.4 also satisfies

$$(\varphi, v) \in C((0,T), W^{3,s}_B \times W^{3,s}_B) \quad \textit{and}$$

$$(\varphi_t, v_t) \in C((0,T), W^{3-\theta,s}_B \times W^{2-\theta,s}_B)$$

for every $s > 1$ and $0 < \theta \leq 1$. In particular, $(\varphi, v) \in C((0,T), C^2(\overline{\Omega}) \times C^2(\overline{\Omega}))$.
ii) Assume moreover that h is of class C^n, for $n \geq 2$, and assume conditions (2.13) if $B = D$ or (2.14)–(2.15) if $B = N_e$ are satisfied. Then, for every $(\varphi_0, v_0) \in W^{2,p}_B \times W^{1,p}_B$, the local solution of (1.7) given in Proposition 2.4 satisfies, besides the regularity in i),

$$(\varphi, v) \in C((0,T), W^{n+1,s}_B \times W^{n+1,s}_B) \text{ and}$$

$$(\varphi_t, v_t) \in C((0,T), W^{n+1-\theta,s}_B \times W^{n-\theta,s}_B)$$

for every $\theta \in (0,1]$ and $s > 1$. In particular, $(\varphi, v) \in C((0,T), C^n(\overline{\Omega}) \times C^n(\overline{\Omega}))$. ■

By using the first part of both propositions above in a row, we get

Corollary 2.11. *Assume h is of class C^1 and satisfies (2.5) and (2.6) if $N \geq p$. Then, for every $(\varphi_0, v_0) \in W^{1,p}_B \times L^p_B$, the local solution of (1.7) given in Proposition 2.3 also satisfies*

$$(\varphi, v) \in C((0,T), W^{3,s}_B \times W^{3,s}_B) \quad \text{and} \quad (\varphi_t, v_t) \in C((0,T), W^{3-\theta,s}_B \times W^{2-\theta,s}_B)$$

for every $s > 1$ and $0 < \theta \leq 1$. In particular, $(\varphi, v) \in C((0,T), C^2(\overline{\Omega}) \times C^2(\overline{\Omega}))$. ■

3 Global existence

In this section we prove that, under suitable growth and sign conditions on the nonlinear term h, the solutions of (1.7) with initial data in $W^{n,p}_B \times W^{n-1,p}_B$, $n \geq 1$, given by Propositions 2.3, 2.4 or 2.8, are globally defined.

First we show that (1.7) has a natural Lyapunov function in $H^1_B \times L^2_B$, see also [4, 5, 6]. So we assume h satisfies the hypothesis in Proposition 2.3, (2.5), (2.6) for $p = 2$, i.e.

$$|h(s)| \leq C(1 + |s|^r), \quad |h'(s)| \leq C(1 + |s|^{r-1}),$$

$$1 \leq r \begin{cases} < \infty & \text{if } N = 1, 2 \\ \leq \frac{N}{N-2} & \text{if } N \geq 3 \end{cases} . \tag{3.1}$$

Then we have

Proposition 3.1. *If h satisfies (3.1), then*

$$\mathcal{F}(\varphi, v) = \frac{k_1}{2}\|\nabla\varphi\|^2 + \frac{k_2 a}{2c}\|v\|^2 + \frac{b}{2}\|\varphi\|^2 + \int_\Omega H(\varphi) - a\int_\Omega v\varphi \quad (3.2)$$

where $H(s) = \int_0^s h(z)dz$, which can be rewritten as

$$\mathcal{F}(\varphi, v) = \frac{k_1}{2}\|\nabla\varphi\|^2 + \frac{b}{2}\int_\Omega (\frac{a}{b}v - \varphi)^2 + \int_\Omega H(\varphi), \quad (3.3)$$

is a Lyapunov function for (1.7), i.e.
i) $\mathcal{F}(\varphi, v)$ is continuous in $H^1_B \times L^2_B$.
ii) $\frac{d}{dt}(\mathcal{F}(\varphi(t), v(t))) \leq 0$ for every solution of (1.7).
iii) (φ, v) is an equilibrium point of (1.7) if and only if $\frac{d}{dt}(\mathcal{F}(\varphi(t), v(t))) = 0$.

Proof. Property i) is standard and comes from (3.1). To prove ii) note that from Proposition 2.3 if (φ, v) is a solution of (1.7) with initial data $(\varphi_0, v_0) \in H^1_B \times L^2_B$, we have $(\varphi, v) \in C((0,T), H^2_B \times H^1_B)$, $(\varphi_t, v_t) \in C((0,T), H^1_B \times L^2_B)$. Therefore, we can multiply the first equation of (1.7) by $\frac{\partial\varphi}{\partial t}$ in L^2_B and, integrating by parts, we obtain

$$\|\frac{\partial\varphi}{\partial t}\|^2 + \frac{d}{dt}\left(\frac{k_1}{2}\|\nabla\varphi\|^2 + \int_\Omega (H(\varphi) + \frac{b}{2}\varphi^2)\right) = a\int_\Omega v\frac{\partial\varphi}{\partial t} \quad (3.4)$$

since $\int_\Gamma \frac{\partial\varphi}{\partial n}\frac{\partial\varphi}{\partial t} = 0$ with any of the boundary conditions (1.2)–(1.5).

On the other hand, $\int_\Omega v\frac{\partial\varphi}{\partial t} = \frac{d}{dt}[\int_\Omega v\varphi] - \int_\Omega \varphi\frac{\partial v}{\partial t}$. Therefore, from (3.4) we obtain

$$\|\frac{\partial\varphi}{\partial t}\|^2 + \frac{d}{dt}\left(\frac{k_1}{2}\|\nabla\varphi\|^2 + \int_\Omega (H(\varphi) + \frac{b}{2}\varphi^2 - av\varphi)\right) = -a\int_\Omega \varphi\frac{\partial v}{\partial t}. \quad (3.5)$$

Now, if $B = N_e$ or P, integrating the equation for v in Ω we get

$$0 = \int_\Omega v_t + c\int_\Gamma \frac{\partial\varphi}{\partial n} - k_2\int_\Gamma \frac{\partial v}{\partial n} = \int_\Omega v_t = \frac{d}{dt}(\int_\Omega v) = 0. \quad (3.6)$$

Thus, $\int_\Omega v(t) = \int_\Omega v_0$, i.e. the mass is conserved.

Now, we multiply the second equation in (1.7) by $\frac{a}{c}(-\Delta_B)^{-1}v_t$ in L^2_B. Note that with $B = D, (-\Delta_B)^{-1}$ is well defined, but if $B = N_e$ or $B = P$, then $-\Delta_B$ has a one dimensional kernel generated by constant functions. However, from (3.6), $(-\Delta_B)^{-1}v_t$ is also well defined as an element of H^2_B with zero average. Thus, we obtain

$$\frac{a}{c}\|\frac{\partial v}{\partial t}\|^2_{-1} + \frac{ak_2}{2c}\frac{d}{dt}\|v\|^2 = a\int_\Omega \varphi\frac{\partial v}{\partial t} \tag{3.7}$$

where $\|.\|_{-1}$ is the norm in H^{-1}_B. Adding (3.5) and (3.7) we get

$$\|\frac{\partial \varphi}{\partial t}\|^2 + \frac{a}{c}\|\frac{\partial v}{\partial t}\|^2_{-1} + \frac{d}{dt}\mathcal{F}(\varphi, v) = 0 \tag{3.8}$$

and ii) and iii) follow.

Moreover, from (1.8) we have $\frac{k_2}{c} = \frac{a}{b}$, which allows us to write $\mathcal{F}$ as in (3.3) since

$$\frac{b}{2}\int_\Omega(\frac{a}{b}v - \varphi)^2 = \frac{b}{2}\frac{a^2}{b^2}\|v\|^2 + \frac{b}{2}\|\varphi\|^2 - a\int_\Omega v\varphi.$$

The rest follows easily. ■

Next, we show that under a sign assumption on h, the local solution of (1.7) in $H^1_B \times L^2_B$ is globally defined.

Corollary 3.2. *Assume that h satisfies (3.1) and*

$$\liminf_{|s|\to\infty} \frac{h(s)}{s} > 0. \tag{3.9}$$

Then the solutions of (1.7) are global and bounded in $H^1_B \times L^2_B$. In particular we have a well defined nonlinear semigroup, $S(t)$ for $t \geq 0$, in $H^1_B \times L^2_B$, given by

$$S(t)(\varphi_0, v_0) = (\varphi(t), v(t)). \tag{3.10}$$

Moreover, if $K \subset H^1_B \times L^2_B$ is a bounded set, then its orbit, i.e. $\{S(t)K, t \geq 0\}$, is also bounded.

Proof. If h satisfies (3.9) then there exists $\delta > 0$ and $c(\delta) > 0$ such that $H(s) \geq \delta s^2 - c(\delta)$ for every $s \in \mathbb{R}$, and hence we have

$$\int_\Omega H(\varphi) \geq \delta\|\varphi\|^2 - c(\delta)|\Omega|. \tag{3.11}$$

On the other hand, we have $\mathcal{F}(\varphi(t), v(t)) \leq \mathcal{F}(\varphi(0), v(0))$ for $t > 0$. Thus, (3.3) and (3.11), imply

$$\frac{k_1}{2}\|\nabla\varphi\|^2 + \frac{b}{2}\int_\Omega (\frac{a}{b}v - \varphi)^2 + \delta\|\varphi\|^2 \leq c(\delta)|\Omega| + \mathcal{F}(\varphi_0, v_0) < \infty. \quad (3.12)$$

Hence, $\|\nabla\varphi\|^2$, $\|\varphi\|^2$ and $\|\frac{a}{b}v - \varphi\|^2$ remain bounded on finite time intervals. Therefore the solution remains bounded in $H^1_B \times L^2_B$ and, from Proposition 2.3, we get that the solution is global. Moreover, from (3.12), we also have

$$\|(\varphi(t), v(t))\|^2_{H^1_B \times L^2_B} \leq c_1 + c_2 \left(\|\nabla\varphi_0\|^2 + \|\varphi_0\|^2 + \|v_0\|^2 + \int_\Omega |H(\varphi_0)| \right).$$

But from (3.1) we get $\int_\Omega |H(\varphi_0)| \leq c_3\big(1 + \int_\Omega |\varphi_0|^{r+1}\big) \leq c_4\Big(1 + \|\varphi_0\|^{r+1}_{H^1_B}\Big)$, since from the hypothesis on r we have $H^1_B \hookrightarrow L^{r+1}(\Omega)$. Thus

$$\|(\varphi(t), v(t))\|_{H^1_B \times L^2_B} \leq c_6 \left(1 + \|(\varphi_0, v_0)\|_{H^1_B \times L^2_B} + \|\varphi_0\|^{\frac{r+1}{2}}_{H^1_B} \right) \quad (3.13)$$

and we conclude. ■

Now we extend this result to $p \neq 2$. Observe that the energy (3.2) can only give information on the solutions in the norm of $H^1_B \times L^2_B$; therefore our goal is to use this information to control different norms for the case $p \neq 2$. Also note that the growth assumption (3.1) will be always required for this. This will impose some extra restrictions on the growth of nonlinear terms than those needed just for local existence purposes, when we work on stronger norms than that of $H^1_B \times L^2_B$.

Then we obtain the following global existence result for the solutions of (1.7) given in Propositions 2.3, 2.4 and 2.8.

Proposition 3.3. *Let $1 < p < \infty$ and h be of class C^1 satisfying (3.9). i) Assume moreover that h satisfies*

$$|h(s)| \leq C(1 + |s|^r), \quad |h'(s)| \leq C(1 + |s|^{r-1})$$

with r such that

$$1 \leq r \begin{cases} < \infty & \text{if } N = 1 \\ \leq \frac{N}{N-p} & \text{if } N \geq 2 \text{ and } 1 < p < 2 \\ \leq \frac{N}{N-2} & \text{if } N \geq 2 \text{ and } 2 \leq p \end{cases}.$$

Then for every $(\varphi_0, v_0) \in W^{1,p}_B \times L^p_B$ *the solution of (1.7) constructed in Proposition 2.3,* $(\varphi(t), v(t))$*, is globally defined. Thus (3.10) also defines a nonlinear semigroup in* $W^{1,p}_B \times L^p_B$*.*
ii) Assume that h *is of class* C^2*,* $N \leq 4$*, and* h *satisfies*

$$|h(s)| \leq C(1 + |s|^r), \quad |h'(s)| \leq C(1 + |s|^{r-1}), \quad |h''(s)| \leq C(1 + |s|^{r-2})$$

with r *such that*

$$2 \leq r \begin{cases} < \infty & \text{if } N = 1,2 \text{ and } 1 < p < \infty \\ \leq \frac{N-p}{N-2p} & \text{if } N = 3,4 \text{ and } \frac{N}{3} \leq p < \frac{2N}{N+2} \\ \leq \frac{N}{N-2} & \text{if } N = 3,4 \text{ and } \frac{2N}{N+2} \leq p \end{cases}.$$

Then, for every $(\varphi_0, v_0) \in W^{2,p}_B \times W^{1,p}_B$ *the solution of (1.7) constructed in Proposition 2.4,* $(\varphi(t), v(t))$*, is globally defined and (3.10) also defines a nonlinear semigroup in* $W^{2,p}_B \times W^{1,p}_B$*.*
iii) Assume that h *is of class* C^n*, for* $n \geq 2$*, with* $N < (n-1)p$*, is as in Proposition 2.8 and satisfies (3.1), that is,*

$$|h(s)| \leq C(1 + |s|^r), \quad |h'(s)| \leq C(1 + |s|^{r-1})$$

with r *such that*

$$1 \leq r \leq \frac{N}{N-2}.$$

Then for every $(\varphi_0, v_0) \in W^{n,p}_B \times W^{n-1,p}_B$ *the solution of (1.7) constructed in Proposition 2.8,* $(\varphi(t), v(t))$*, is globally defined and (3.10) also defines a nonlinear semigroup in* $W^{n,p}_B \times W^{n-1,p}_B$*.*

Proof. Let $X = W^{n,p}_B \times W^{n-1,p}_B$, for $n \geq 1$, be the space for the initial data in each of the cases of the statement. Note that in each of these cases, h satisfies, respectively, the assumptions of Proposition 2.3 if $n = 1$, Proposition 2.4 if $n = 2$ or Proposition 2.8 if $n \geq 2$. Therefore, we have local existence of solutions of (1.7) with initial data $(\varphi_0, v_0) \in X$. At the same time, in all cases, h also satisfies (3.1) which implies local existence in $H^1_B \times L^2_B$. Also, since (3.9) is satisfied, then Corollary 3.2 applies in the latter space.

First, we note that, from Sobolev embeddings, if $n = 1$ and $p \geq 2$, $n = 2$ and $p \geq \frac{2N}{N+2}$ or $n \geq 2$ and $N < (n-1)p$, then $X \hookrightarrow H^1_B \times L^2_B$. On the other hand if $n = 1$ and $1 < p < 2$, we have $H^1_B \times L^2_B \hookrightarrow X$.

Finally, in the case $n = 2$ and $1 < p < \frac{2N}{N+2}$, neither space X or $H^1_B \times L^2_B$ contains the other.

Now, assume that $X \hookrightarrow H^1_B \times L^2_B$, as above, and let $(\varphi_0, v_0) \in X$. Then from Proposition 3.1 the solution of (1.7) with initial data (φ_0, v_0) is globally defined in $H^1_B \times L^2_B$. From the regularity result in i) of Proposition 2.9, for $p = 2$, the solution in $H^1_B \times L^2_B$ is in X, for every $t > 0$, in the cases i) and ii). On the other hand, for case iii), the regularity result in ii) of Proposition 2.9, for $p = 2$, the solution in $H^1_B \times L^2_B$ is in X, for every $t > 0$. Thus the solution is global in X.

If $H^1_B \times L^2_B \hookrightarrow X$, again from Proposition 2.9, the local solution of (1.7) with initial data in X, is in $H^1_B \times L^2_B$ for every $t > 0$. From Proposition 3.1 this solution is globally defined in $H^1_B \times L^2_B$ and thus in X.

When neither space X or $H^1_B \times L^2_B$ contains the other, again Proposition 2.9 concludes as before. ∎

4 Asymptotic behavior of solutions

In this section we analyze the asymptotic behavior of solutions of (1.7) in different function spaces. First, we show, by means of the Lyapunov function constructed before, that (1.7) has a global attractor in $H^1_B \times L^2_B$ when h satisfies (3.1) and (3.9). Later, we will consider the case of initial data in $W^{n,p}_B \times W^{n-1,p}_B$, $n \geq 1$, as in Propositions 2.3, 2.4 or 2.8 and prove that the attractor in $H^1_B \times L^2_B$ is still the attractor in these different phase spaces. To obtain these results a key argument will be to transfer good uniform in time estimates on the solutions available in the $H^1_B \times L^2_B$ setting by means of the Lyapunov function, to the stronger norms of the spaces $W^{n,p}_B \times W^{n-1,p}_B$.

4.1 Global attractor in $H^1_B \times L^2_B$.

Note that if $B = N_e$ or $B = P$, there is not a global attractor in $H^1_B \times L^2_B$ in the usual sense, since from (3.6) we have that $\frac{d}{dt}(\int_\Omega v) = 0$. Therefore for each $m \in \mathbb{R}$ the affine hyperplane $Z(m) = H^1_B \times \{v \in L^2_B,\ \int_\Omega v = m\}$ is invariant. However, we will show below that there exists a global attractor $\mathcal{A}_m$ in $Y_m = H^1_B \times \{v \in L^2_B,\ |\int_\Omega v| \leq m\}$ for each $m \in \mathbb{R}^+$. Therefore, for every $|m_0| \leq m$ there exists a global attractor in $Z(m_0)$

given by $\mathcal{A}(m_0) = \mathcal{A}_m \cap Z(m_0)$.

Proposition 4.1. *Assume that h satisfies (3.1) and (3.9), then we have i) If $B = D$ there exists a global compact and connected attractor, $\mathcal{A}$, in $Y = H^1_0(\Omega) \times L^2(\Omega)$ for the semigroup $S(t)$, which can be described as $\mathcal{A} = W^u(E)$, that is, the unstable set of the equilibrium points, E .*

Moreover, if E is a discrete set, then $\mathcal{A} = \cup_{(\varphi_0,v_0)\in E} W^u((\varphi_0, v_0))$, and for each solution, $(\varphi(t), v(t))$, of (1.7), there exists an equilibrium point $(\varphi_0, v_0) \in E$, such that

$$(\varphi(t), v(t)) \to (\varphi_0, v_0) \text{ in } H^1_0(\Omega) \times L^2(\Omega) \text{ as } t \to \infty.$$

ii) If $B = N_e$ or $B = P$, for each $m \geq 0$, there exists a global compact and connected attractor, $\mathcal{A}_m$, in $Y_m = H^1_B \times \{v \in L^2_B,\ |\int_\Omega v| \leq m\}$ for the semigroup $S(t)$, which can be described as $\mathcal{A}_m = W^u(E_m)$, that is, the unstable set of the equilibrium points in Y_m, $E_m = E \cap Y_m$.

Moreover, for each m_0 such that $|m_0| \leq m$ if the set of equilibria, $E(m_0) = E \cap \{(\varphi, v), \int_\Omega v = m_0\}$ is discrete in $Z(m_0)$, then the attractor in $Z(m_0)$, $\mathcal{A}(m_0) = \mathcal{A}_m \cap Z(m_0)$ is given by

$$\mathcal{A}(m_0) = \cup_{(\varphi_0,v_0)\in E(m_0)} W^u((\varphi_0, v_0)),$$

and for each solution, $(\varphi(t), v(t))$, of (1.7) with $\int_\Omega v_0 = m_0$, there exists a equilibrium point $(\varphi_0, v_0) \in E(m_0)$, such that

$$(\varphi(t), v(t)) \to (\varphi_0, v_0) \text{ in } H^1_B \times L^2_B \text{ as } t \to \infty.$$

Proof. From Proposition 3.1 and Corollary 3.2 the semigroup $S(t)$ satisfies that orbits of bounded sets are bounded. Since the resolvent of A_B is compact, see Proposition 2.1, then Theorem 4.2.2 in [15] gives that $S(t)$ is compact for $t > 0$ and in particular it is an asymptotically smooth gradient system in $H^1_B \times L^2_B$; see Corollary 3.2.2 of [15]. Moreover $B = N_e$ or $B = P$ then Y_m is invariant for $S(t)$. Thus, for the case $B = D$, if the set of equilibrium points, E, (respectively $E_m = E \cap Y_m$ if $B = N_e$ or $B = P$) is bounded, then the semigroup $S(t)$ is point dissipative and from Theorem 3.4.6 of [15] we get the existence of the global attractor. Finally, if E (respectively E_{m_0}) is a discrete set, from [15, 16, 21], we conclude.

Observe that the equilibrium points $(\varphi_0, v_0) \in H^1_B \times L^2_B$ satisfy

$$\left\{ \begin{array}{rcl} -k_1\Delta_B\varphi_0 + h(\varphi_0) + b\varphi_0 - av_0 &=& 0 \\ -k_2\Delta_B v_0 + c\Delta_B\varphi_0 &=& 0 \end{array} \right. \tag{4.1}$$

Thus, when $B = D$, using the precise relationship between coefficients given in (1.8), we get

$$-k_1\Delta_D\varphi_0 + h(\varphi_0) = 0 \tag{4.2}$$

and $b\varphi_0 = av_0$. Now, multiplying (4.2) by φ_0 and integrating by parts, we obtain

$$k_1 \int_\Omega |\nabla\varphi_0|^2 + \int_\Omega h(\varphi_0)\varphi_0 = 0.$$

Using (3.9), there exists $\delta > 0$, $c(\delta) > 0$ such that $h(s)s \geq \delta s^2 - c(\delta)$ for every $s \in \mathbb{R}$ and then we have

$$\int_\Omega h(\varphi_0)\varphi_0 \geq \delta\|\varphi_0\|^2 - c(\delta)|\Omega|. \tag{4.3}$$

Therefore there exists $R > 0$ such that $\|\varphi_0\|_{H^1_0(\Omega)} \leq R$ and $\|v_0\|_{H^1_0(\Omega)} \leq \frac{b}{a}R$ and the set of equilibria is bounded in $H^1_0(\Omega) \times H^1_0(\Omega)$ and so in $H^1_0(\Omega) \times L^2(\Omega)$.

On the other hand, if $B = N_e$ or $B = P$, the equilibria satisfy

$$\left\{ \begin{array}{rcl} -k_1\Delta_B\varphi_0 + h(\varphi_0) + \lambda &=& 0 \\ b\varphi_0 - av_0 &=& \lambda \end{array} \right. \tag{4.4}$$

since $\lambda \in \mathbb{R}$ is a free parameter, the set of equilibrium is not bounded in $H^1_B \times L^2_B$. Nevertheless, given m if we consider only equilibrium points (φ_0, v_0) in Z_m i.e., with $|\int_\Omega v_0| \leq m$, then from (4.4) we have

$$b\int_\Omega \varphi_0 = \lambda|\Omega| + a\int_\Omega v_0 \leq \lambda|\Omega| + am. \tag{4.5}$$

Thus using (4.5) and proceeding as in i), for $B = D$, the set E_m is bounded in Y_m. ∎

4.2 Global attractor in $W_B^{n,p} \times W_B^{n-1,p}$, $n \geq 1$.

Now we work in the space $X = W_B^{n,p} \times W_B^{n-1,p}$, where h, p and n are as in cases i), ii) or iii) of Proposition 3.3.

First, we consider the case $B = D$ and we prove that the global attractor in $H_0^1(\Omega) \times L^2(\Omega)$ given in Proposition 4.1, $\mathcal{A}$, is also the global attractor in X. Note that for this, one must show then that bounded sets of X are attracted by $\mathcal{A}$ in the norm of X and not only in the norm of $H_0^1(\Omega) \times L^2(\Omega)$. Also, as a consequence we will obtain that $\mathcal{A}$ attracts bounded sets of $H_0^1(\Omega) \times L^2(\Omega)$ in the stronger norm of the space X. The idea for this is then to obtain estimates of solutions in the norm of X, and in fact in stronger norms, from estimates in $H_0^1(\Omega) \times L^2(\Omega)$. To get these estimates, we use similar arguments as in Proposition 2.9 and the variation of constants formula.

Analogously, for $B = N_e$ or $B = P$, we will prove that for fixed $m \in \mathbb{R}^+$, the global attractor in $Y_m \subset H_B^1 \times L_B^2$ given by Proposition 4.1, $\mathcal{A}_m$, is the global attractor in

$$X_m = \{(\varphi, v) \in X, \ |\int_\Omega v| \leq m\}.$$

First we prove the following result for solutions of (1.7) with initial data in $W_B^{1,p} \times L_B^p$ as in Proposition 2.3.

Proposition 4.2. *Let $1 < p < \infty$ and assume h satisfies the assumptions of point i) in Proposition 3.3. Assume also that the solution of (1.7) with initial data $(\varphi_0, v_0) \in W_B^{1,p} \times L_B^p$, satisfies*

$$sup_{t \geq 0} \|(\varphi(t), v(t))\|_{W_B^{1,p} \times L_B^p} < \infty.$$

i) Then, for every $s > 1$ and $\tau > 0$ we have that

$$sup_{t \geq \tau} \|(\varphi(t), v(t))\|_{W_B^{2,s} \times W_B^{1,s}} \leq C(\tau, sup_{t \geq 0} \|(\varphi(t), v(t))\|_{W_B^{1,p} \times L_B^p}).$$

ii) If moreover h is of class C^n for $n \geq 2$ and is as in Proposition 2.8, then for every $s > 1$ and $\tau > 0$, we have

$$sup_{t \geq \tau} \|(\varphi(t), v(t))\|_{W_B^{n+1,s} \times W_B^{n,s}} \leq C(\tau, sup_{t \geq 0} \|(\varphi(t), v(t))\|_{W_B^{1,p} \times L_B^p}).$$

Proof. Note that the solution of (1.7) with initial data $(\varphi(\tau), v(\tau))$ is given for $t \geq \tau$, by

$$(\varphi(t), v(t)) = e^{-A_B(t-\tau)}(\varphi(\tau), v(\tau)) + \int_\tau^t e^{-A_B(t-s)} G(\varphi(s), v(s))ds \tag{4.6}$$

and that by adding a term λI with $\lambda > 0$ sufficiently large to A_B and G, in (2.1) we can always assume that $Re(\sigma(A_B)) > \mu > 0$.
i) From Proposition 2.9, $(\varphi(t), v(t))$ belongs to $W^{1,q}_B \times L^q_B$ for every $q > 1$ and $t > 0$. Considering the scale of interpolation spaces associated to A_B in $Y_q = L^q_B \times W^{-1,q}_B$, then $Y_q^{\frac{1}{2}} = W^{1,q}_B \times L^q_B$, $Y^1_q = W^{2,q}_B \times W^{1,q}_B$ and $Y^{1-\epsilon}_q = W^{2(1-\epsilon),q}_B \times W^{1-2\epsilon,q}_B$, for $\epsilon \in [0, 1]$. Thus from (4.6) we have

$$\|(\varphi, v)(t)\|_{Y^{1-\epsilon}_q} \leq M \frac{e^{-\mu(t-\tau)}}{(t-\tau)^{\frac{1}{2}-\epsilon}} \|(\varphi(\tau), v(\tau))\|_{Y_q^{\frac{1}{2}}} +$$

$$+ sup_{t\geq\tau} \|G(\varphi, v)(t)\|_{Y_q} \int_\tau^t M \frac{e^{-\mu(t-s)}}{(t-s)^{1-\epsilon}} ds \tag{4.7}$$

for $0 < \epsilon \leq \frac{1}{2}$, and for every $t > \tau$. Moreover, if $q \geq p$ then, from the proof of Proposition 2.3, $G : Y_q^{\frac{1}{2}} \to Y_q$ is Lipschitz and bounded on bounded sets. Therefore, if

$$sup_{t\geq\tau} \|(\varphi, v)(t)\|_{Y_q^{\frac{1}{2}}} \leq c^1_q < \infty \tag{4.8}$$

we get that $sup_{t\geq\tau}\|G(\varphi, v)(t)\|_{Y_q} \leq c^2_q < \infty$. Consequently, given $\tau^* > \tau$, using (4.7), we get that (4.8) implies

$$sup_{t\geq\tau^*} \|(\varphi, v)(t)\|_{Y^{1-\epsilon}_q} \leq c^3_q < \infty. \tag{4.9}$$

Now we proceed as in the proof of Theorem 3.5.2 in [16], see also the proof of Theorem 4.6 in [3], to obtain that (φ_t, v_t) is uniformly bounded, for $t \geq \tau^*$, in Y_q (in fact this is even true in some Y^ϵ_q spaces). Therefore, from (1.7), using that A_B in (2.1) is sectorial in Y_q with domain Y^1_q and that $Re(\sigma(A_B)) > \mu > 0$, we obtain

$$sup_{t\geq\tau^*} \|(\varphi, v)(t)\|_{Y^1_q} \leq c^4_q < \infty. \tag{4.10}$$

Next, we are going to apply this general argument for some choices of q and τ. First, applying (4.8)–(4.10) for $q = p$ and $\tau = 0$, we have that, for every $t_1 > 0$

$$sup_{t \geq t_1} \|(\varphi, v)(t)\|_{Y^1_p} \leq c_1 < \infty. \tag{4.11}$$

If $p > N$, then from Sobolev embeddings, we obtain $Y^1_p = W^{2,p}_B \times W^{1,p}_B \hookrightarrow Y^{\frac{1}{2}}_s = W^{1,s}_B \times L^s_B$ for every s, and thus from (4.11) we get $sup_{t \geq t_1} \|(\varphi, v)(t)\|_{Y^{\frac{1}{2}}_s} \leq c_2$. Therefore, we can apply again (4.8)–(4.10) now with $q = s$ and $\tau = t_1 > 0$, to obtain

$$sup_{t \geq t_2} \|(\varphi, v)(t)\|_{Y^1_s} \leq c_3$$

with $t_2 > t_1 > 0$ and $Y^1_s = W^{2,s}_B \times W^{1,s}_B$, and we get the result.

If $p \leq N$, again from Sobolev embeddings we obtain $Y^1_p = W^{2,p}_B \times W^{1,p}_B \hookrightarrow Y^{\frac{1}{2}}_s = W^{1,s}_B \times L^s_B$ for every $s \leq p_1 = \frac{Np}{N-p}$. Thus, we have that $sup_{t \geq t_1} \|(\varphi, v)(t)\|_{Y^{\frac{1}{2}}_{p_1}} < \infty$. Since $p < p_1$ repeating this argument a finite number of steps we obtain that

$$sup_{t \geq \bar{t}} \|(\varphi, v)\|_{Y^{\frac{1}{2}}_{\bar{p}}} \leq C < \infty,$$

for every $\bar{t} > 0$ and some $\bar{p} > N$ and we conclude as above.

ii) If h is of class C^n, $n \geq 2$, we consider $s > N$ and we apply an induction argument in n, like in Proposition 2.9, to prove: For every $1 \leq k \leq n$ and $\tau > 0$, there exists a constant $c_k = c_k(\tau) > 0$, such that

$$sup_{t \geq \tau} \|(\varphi, v)(t)\|_{Y_s^{\frac{k+1}{2}}} \leq c_k \tag{4.12}$$

with $Y_s^{\frac{k+1}{2}} = W^{k+1,s}_B \times W^{k,s}_B$.

Note that, from i), we have that (4.12) is true for $k = 1$. Assume now (4.12) is true for $1 \leq k \leq n$ and then we prove that (4.12) is also true for $k+1$, whenever $k+1 \leq n$. For this, also note that for each $k \geq 1$, since $h \in C^{k+1}$ and $N < ks$, from Proposition 2.8, there exists solution with initial data in $W^{k+1,s}_B \times W^{k,s}_B$.

Therefore, we consider a solution such that

$$sup_{t \geq t_1} \|(\varphi, v)(t)\|_{Y_s^{\frac{k+1}{2}}} \leq c_k < \infty$$

for any $t_1 > 0$ and some $c_k > 0$. Consequently, from Lemma 2.7, we have

$$sup_{t\geq t_1}\|G(\varphi,v)(t)\|_{Y_s^{\frac{k}{2}}} \leq c_k^*$$

with c_k^* a positive constant only depending on c_k. Therefore, similarly to (4.8)–(4.10), working now in the scale of spaces associated to A_B in $Z_s = Y_s^{\frac{k}{2}} = W_B^{k,s} \times W_B^{k-1,s}$, for which we have $Z_s^{\frac{1}{2}} = Y_s^{\frac{k+1}{2}} = W_B^{k+1,s} \times W_B^{k,s}$ and $Z_s^1 = Y_s^{\frac{k}{2}+1} = W_B^{k+2,s} \times W_B^{k+1,s}$, we get $sup_{t\geq t_2}\|(\varphi,v)(t)\|_{Y_s^{\frac{k}{2}+1}} \leq c_{k+1}$ for $t_2 > t_1 > 0$, and we conclude. ■

Analogously, for the solutions of (1.7) with initial data in $W_B^{2,p} \times W_B^{1,p}$, $\frac{N}{3} < p < \infty$ given in Proposition 2.4, we have

Proposition 4.3. *Let $1 < p < \infty$ and assume h satisfies the assumptions of point ii) in Proposition 3.3. Assume also that the solution of (1.7) with initial data $(\varphi_0, v_0) \in W_B^{2,p} \times W_B^{1,p}$, satisfies*

$$sup_{t\geq 0}\|(\varphi(t), v(t))\|_{W_B^{2,p}\times W_B^{1,p}} < \infty.$$

i) Then, for every $s > 1$ and $\tau > 0$ we have

$$sup_{t\geq \tau}\|(\varphi(t), v(t))\|_{W_B^{3,s}\times W_B^{2,s}} \leq C(\tau, sup_{t\geq 0}\|(\varphi(t), v(t))\|_{W_B^{2,p}\times W_B^{1,p}}).$$

ii) If moreover h is of class C^n for $n \geq 2$ and is as in Proposition 2.8, then for every $s > 1$ and $\tau > 0$, we have

$$sup_{t\geq \tau}\|(\varphi(t), v(t))\|_{W_B^{n+1,s}\times W_B^{n,s}} \leq C(\tau, sup_{t\geq 0}\|(\varphi(t), v(t))\|_{W_B^{2,p}\times W_B^{1,p}}).$$

■

By using the first part of both propositions above in a row, we get

Corollary 4.4. *Let $1 < p < \infty$ and assume h satisfies the assumptions of point i) in Proposition 3.3. Assume also that the solution of (1.7) with initial data $(\varphi_0, v_0) \in W_B^{1,p} \times L_B^p$, satisfies*

$$sup_{t\geq 0}\|(\varphi(t), v(t))\|_{W_B^{1,p}\times L_B^p} < \infty.$$

Then, for every $s > 1$ and $\tau > 0$ we have

$$sup_{t\geq \tau}\|(\varphi(t), v(t))\|_{W_B^{3,s}\times W_B^{2,s}} \leq C(\tau, sup_{t\geq 0}\|(\varphi(t), v(t))\|_{W_B^{2,p}\times W_B^{1,p}}).$$ ■

As a consequence, we obtain

Corollary 4.5. *Assume h is of class* C^n, $n \geq 1$, *satisfies (3.1) and (3.9). Then we have*
i) For every $(\varphi_0, v_0) \in H^1_B \times L^2_B$, *the solution of (1.7),* $(\varphi(t), v(t))$, *is uniformly bounded in* $W^{3,s}_B \times W^{2,s}_B$ *for every* $s > 1$ *and* $t \geq \tau > 0$.

If moreover h is as in Proposition 2.8, then the solution is uniformly bounded in $W^{n+1,s}_B \times W^{n,s}_B$ *for every* $s > 1$ *and* $t \geq \tau > 0$.
ii) If $K \subset H^1_B \times L^2_B$ *is a bounded set, then* $\{S(t)K,\ t \geq \tau > 0\}$ *is bounded in* $W^{3,s}_B \times W^{2,s}_B$ *for every* $s > 1$ *and* $\tau > 0$.

If moreover h is as in Proposition 2.8, then $\{S(t)K,\ t \geq \tau > 0\}$ *is uniformly bounded in* $W^{n+1,s}_B \times W^{n,s}_B$ *for every* $s > 1$ *and* $\tau > 0$. ■

Next, we prove a regularity result for the global attractor for the semigroup in $H^1_B \times L^2_B$ given in Proposition 4.1, $\mathcal{A}$ for $B = D$ or $\mathcal{A}_m$ for $B = N_e, P$.

Corollary 4.6. *Assume that h satisfies (3.1) and (3.9), and consider the global attractor in* $H^1_B \times L^2_B$, $\mathcal{A}$, *if* $B = D$, *or* $\mathcal{A}_m$ *if* $B = N_e$ *or* P, *given in Proposition 4.1.*

Then, for every $s > 1$ *we have* $\mathcal{A} \subset W^{2,s}_D \times W^{1,s}_0$ *if* $B = D$ *and* $\mathcal{A}_m \subset W^{2,s}_B \times W^{1,s}_B$ *if* $B = N_e$ *or* P *and is compact, connected and invariant in this space.*

Moreover, if h is of class C^n *with* $n \geq 2$ *and is as in Proposition 2.8, we have for every* $s > 1$, $\mathcal{A} \subset W^{n,s}_D \times W^{n-1,s}_D$ *if* $B = D$ *and* $\mathcal{A}_m \subset W^{n,s}_B \times W^{n-1,s}_B$ *if* $B = N_e$ *or* P, *and is compact, connected and invariant in this space.*

Proof. We prove the case $B = D$, since the cases $B = N_e$ and $B = P$ can be proved analogously. We know that $\mathcal{A} \subset H^1_0(\Omega) \times L^2(\Omega)$ is compact and for every $t \geq 0$, $S(t)\mathcal{A} = \mathcal{A}$.

Now, from Corollary 4.5, we obtain that $\mathcal{A} = S(t)\mathcal{A}$ is bounded in $W^{3,s}_D \times W^{2,s}_D$ for $s > 1$. If moreover h is of class C^n with $n \geq 2$ and is as in Proposition 2.8, then again from Corollary 4.5, $\mathcal{A}$ is bounded in $W^{n+1,s}_D \times W^{n,s}_D$ for $s > 1$.

Finally, using the compact embeddings of $W^{3,s}_D \times W^{2,s}_D$ and $W^{n+1,s}_D \times W^{n,s}_D$ into $W^{2,s}_D(\Omega) \times W^{1,s}_0(\Omega)$ and $W^{n,s}_D \times W^{n-1,s}_D$, respectively, we get that $\mathcal{A}$ is a compact and connected set in $W^{2,s}_D(\Omega) \times W^{1,s}_0(\Omega)$ or in $W^{n,s}_D \times W^{n-1,s}_D$. ■

Now we show that the attractor $\mathcal{A}$, for $B = D$, or respectively $\mathcal{A}_m$, for $B = N_e$ or P, constructed in Proposition 4.1, attracts in stronger norms.

Corollary 4.7. *Assume that h is of class C^n, $n \geq 1$, satisfies (3.1), (3.9) and consider the global attractor in $H^1_B \times L^2_B$, $\mathcal{A}$, if $B = D$, or $\mathcal{A}_m$ if $B = N_e$ or P, given in Proposition 4.1.*

Let K be a bounded set in X if $B = D$, or in $X_m = \{(\varphi, v) \in X,\ |\int_\Omega v| \leq m\}$ if $B = N_e$ or P, where

$$X = \begin{cases} W^{1,p}_B \times L^p_B, & \text{with } p \geq 2 \text{ if } n = 1 \\ W^{2,p}_B \times W^{1,p}_B, & \text{with } p \geq \frac{2N}{N+2} \text{ if } n = 2 \\ W^{n,p}_B \times W^{n-1,p}_B, & \text{with } N < (n-1)p \text{ if } n \geq 2 \end{cases}$$

where, in the latter case, we assume moreover that h is as in Proposition 2.8.

Then, denoting $Y = W^{2,s}_B \times W^{1,s}_B$ for any $s > 1$, we get that

$$dist_Y(S(t)K, \mathcal{A}) \to 0$$

as $t \to \infty$, if $B = D$ (respectively $dist_Y(S(t)K, \mathcal{A}_m) \to 0$ as $t \to \infty$, if $B = N_e$ or P), i.e., $\mathcal{A}$ (respectively $\mathcal{A}_m$), attracts bounded sets of X (respectively of X_m) in the norm of Y.

If moreover $n \geq 2$ and h is as in Proposition 2.8, then we can take above $Y = W^{n,s}_B \times W^{n-1,s}_B$ for any $s > 1$.

Proof. We consider the case $B = D$, the case $B = N_e$ or P being similar. Observe that in all cases of the statement h satisfies the assumptions of Proposition 3.3 and so we have global existence of solutions in X. Also, in all cases we have $X \hookrightarrow H^1_0(\Omega) \times L^2(\Omega)$. Therefore, if K is bounded in X, from Corollary 4.5, we have that $\{S(t)K,\ t \geq 0\}$ is a bounded set in $W^{3,s}_D \times W^{2,s}_D$ for every $s > 1$.

Therefore, using the compact embedding of this space into $Y = W^{2,s}_D \times W^{1,s}_D$, we get the existence the omega limit set of K in the latter space, $w(K)$, [15, 21]. Thus $w(K) \subset Y \hookrightarrow H^1_0(\Omega) \times L^2(\Omega)$ is compact, invariant and $dist_Y(S(t)K, w(K)) \to 0$ as $t \to \infty$. From the maximality of $\mathcal{A}$ we get $w(K) \subset \mathcal{A}$. Also, from Corollary 4.6, we get $\mathcal{A} \subset Y$. Consequently, we obtain that $dist_Y(S(t)K, \mathcal{A}) \to 0$ as $t \to \infty$.

If moreover $n \geq 2$ and h is as in Proposition 2.8 the argument above can be performed with $Y = W_D^{n,s} \times W_D^{n-1,s}$, for every $s > 1$, since from Corollary 4.5, $\{S(t)K,\ t \geq \tau > 0\}$ is also a bounded set in $W_D^{n+1,s} \times W_D^{n,s}$, for $s > 1$ and we get the result. ■

By using all previous results we obtain

Theorem 4.8. *With the notations and hypotheses of Corollary 4.7, consider the semigroup $S(t)$ defined by (1.7) in*

$$X = \begin{cases} W_B^{1,p} \times L_B^p, & \text{with } p \geq 2 \text{ if } n = 1 \\ W_B^{2,p} \times W_B^{1,p}, & \text{with } p \geq \frac{2N}{N+2} \text{ if } n = 2 \\ W_B^{n,p} \times W_B^{n-1,p}, & \text{with } N < (n-1)p \text{ if } n \geq 2 \end{cases}$$

where, in the latter case, we assume moreover that h is as in Proposition 2.8. Also, denote $Y = W_B^{2,s} \times W_B^{1,s}$ for any $s > 1$.

Then if $B = D$, $S(t)$ has a compact and connected global attractor which attracts bounded sets of X in the norm of Y. Moreover, the global attractor in X coincides with the global attractor in $H_0^1(\Omega) \times L^2(\Omega)$ given by Proposition 4.1, which can be described as $\mathcal{A} = W^u(E)$, that is, the unstable set of the equilibrium points, E. In particular if E is a discrete set, for every solution of (1.7), $(\varphi(t), v(t))$, there exists an equilibrium point $(\varphi_0, v_0) \in E$, such that

$$(\varphi(t), v(t)) \to (\varphi_0, v_0) \quad \text{in} \quad Y \quad \text{as} \quad t \to \infty.$$

If $B = N_e$ or $B = P$, given $m > 0$, then $S(t)$ has a compact and connected global attractor in $X_m = \{(\varphi, v) \in X,\ |\int_\Omega v| \leq m\}$ which attracts bounded sets of X_m in the norm of Y. Moreover, this attractor coincides with the global attractor $\mathcal{A}_m$, in $H_B^1 \times \{v \in L_B^2,\ |\int_\Omega v| \leq m\}$ given Proposition 4.1 which can be described as $\mathcal{A}_m = W^u(E_m)$, that is, the unstable set of the equilibrium points in X_m, $E_m = E \cap X_m$.

Moreover, for each m_0 such that $|m_0| \leq m$, if the set of equilibria, $E(m_0) = E \cap Z(m_0)$, with $Z(m_0) = \{(\varphi, v),\ \int_\Omega v = m_0\}$ is discrete in $Z(m_0)$, then the attractor in $Z(m_0)$, $\mathcal{A}(m_0) = \mathcal{A}_m \cap Z(m_0)$ is given by $\mathcal{A}(m_0) = \cup_{(\varphi_0, v_0) \in E(m_0)} W^u((\varphi_0, v_0))$, and for each solution, $(\varphi(t), v(t))$, of (1.7) with $\int_\Omega v_0 = m_0$, there exists a equilibrium point $(\varphi_0, v_0) \in$

$E(m_0)$, such that

$$(\varphi(t), v(t)) \to (\varphi_0, v_0) \quad in \quad Y \quad as \quad t \to \infty.$$

In either case for B above, if moreover $n \geq 2$ and h is as in Proposition 2.8, then we can take above $Y = W^{n,s}_B \times W^{n-1,s}_B$, for every $s > 1$.

Proof. As before we will consider the case $B = D$. From Corollary 4.6, $\mathcal{A}$ is invariant, compact and connected in X. Moreover, from Corollary 4.7, $\mathcal{A}$ attracts bounded sets of X in the norm of Y.

Let $K \subset X$ be a compact and invariant set in X, i.e. $S(t)K = K$. Since in all cases we have $X = W^{n,p}_D \times W^{n-1,p}_D \hookrightarrow H^1_0(\Omega) \times L^2(\Omega)$, then K is also compact and invariant in $H^1_0(\Omega) \times L^2(\Omega)$. Consequently $K \subset \mathcal{A}$ and $\mathcal{A}$ is the attractor of $S(t)$ in X.

On the other hand, from Proposition 4.1, if E is discrete then the omega limit set of each single trajectory is an element of E. Since this trajectory is also compact in Y we conclude. ∎

Next, we analyze the behavior of solutions for the cases not included in the theorem, that is, for initial data in $X = W^{n,p}_B \times W^{n-1,p}_B$, for $n = 1$ and $1 < p < 2$ or $n = 2$ and $1 < p < \frac{2N}{N+2}$.

Proposition 4.9. *Define $X = W^{n,p}_B \times W^{n-1,p}_B$, $n = 1, 2$ and assume h is of class C^n and satisfies (3.9). Assume moreover that either one of the conditions holds:*
i) $n = 1$, $1 < p < 2$ and h satisfies the growth assumptions in point i) of Proposition 3.3.
ii) $n = 2$, $1 < p < \frac{2N}{N+2}$ and h satisfies the hypotheses of point ii) of Proposition 3.3.

Then consider a bounded set K in X satisfying one of the following conditions

a) There exists $\tau > 0$, such that $S(\tau)K \subset H^1_B \times L^2_B$, is bounded.

b) There exits $\tau > 0$, such that the set $\{S(t)K,\ 0 \leq t \leq \tau\}$ is bounded in X.

Then the global attractor in $H^1_B \times L^2_B$, $\mathcal{A}$, if $B = D$, or $\mathcal{A}_m$ if $B = N_e$ or P, given in Proposition 4.1, attracts K in the topology of $Y = W^{2,s}_B \times W^{1,s}_B$ for any $s > 1$.

Furthermore, $\mathcal{A}$, if $B = D$, or $\mathcal{A}_m$ if $B = N_e$ or P, attracts compact sets of X (respectively X_m), in the topology of Y and is the maximal compact, connected and invariant set with this property.

Proof. As before, we consider only the case $B = D$ and note that, from the proofs of Proposition 4.2 or 4.3, assumption b) above implies a). Therefore we prove the result for this case.

Now choose $q \geq p$ such that the space $Z = W_D^{n,q} \times W_D^{n-1,q}$ satisfies the conditions of Corollary 4.7 and $Z \hookrightarrow X$. For this it suffices to take $q > 2$ if $n = 1$ of $q \geq \frac{2N}{N+2}$ if $n = 2$.

From assumption a), define $K_1 = S(\tau)K \subset H_0^1(\Omega) \times L^2(\Omega)$ and again, from Corollary 4.5, $K_2 = S(\tau)K_1 = S(2\tau)K$ is a bounded set in Y and in Z. Thus, from Corollary 4.7, we have that $dist_Y(S(t)K_2, \mathcal{A}) \to 0$ as $t \to \infty$.

Furthermore, now we show that if $K \subset X$ is a compact set, then K satisfies assumption b) of the statement. Indeed if K does not satisfies b), then for every $\tau > 0$ and $n \in \mathbb{N}$, there exist $s < \tau$ and $u \in K$, such that $\|S(s)u\|_X \geq n$. Therefore, we can take $s_n \to 0$, $n \to \infty$ and $u_n = u_n(s_n, n) \in K$, such that $\|S(s_n)u_n\|_X \geq n$. Now, since $S : \mathbb{R}^+ \times X \to X$ is a continuous semigroup and K is compact there exists $u \in K$, such that that $u_n \to u$ and $S(s_n)u_n \to S(0)u = u$, and then $\|u\|_X = \infty$, which is absurd. ■

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