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EFFECTIVE ALGEBRAIC GEOMETRY AND NORMAL FORMS OF REVERSIBLE MAPPINGS

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Abstract

We present a new method to compute normal forms, applied to the germs of reversible mappings. We translate the classification problem of these germs to the theory of ideals in the space of the coefficients of their jets. Integral factorization coupled with Gröbner basis construction is the key factor that makes the process efficient. We also show that a language with typed objects like AXIOM is very convenient to solve these kinds of problems.

1 Introduction

The problem we deal with comes from the theory of discrete dynamical systems (iterations of diffeomorphisms) of $\mathbb{R}^3$. Let ϕ be a germ of a diffeomorphism at the origin and V be a small neighborhood of 0. We call *trajectory* of a point $x \in V$ the discrete set of iterates $V \cap \{\phi^n(x) \mid n \in \mathbb{Z}\}$. A general concern of the theory is to characterize the behavior of the trajectories near fixed points, and the stability of this behavior under perturbations.

Here we restrict ourselves to a class of germs of diffeomorphisms which involve symmetries, the reversible mappings (see Definition 1 below). Birkhoff (see [B]) introduced reversibility in his study of the restricted problem of three bodies. Reversible dynamical systems (with continuous and discrete time) appear in many branches of physics, and there is a growing literature about such systems. We suggest the reader to consult [L1, L2, RQ, M] for further references and connections with other problems. There were some attempts to use computers for the study of such systems (refer to [M]), via floating point arithmetic. But

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these methods are useless when a system has some degree of degeneracy because of their intrinsic lack of accuracy. Around a singular system (like the ones we consider here) ones needs to use appropriate analytic tools to understand its dynamic, among them are the normal forms. We propose here a new strategy to get (formal) normal forms using tools coming from effective algebraic geometry and symbolic computations. This strategy is really efficient in providing an automatic classification of a large class of reversible systems. In fact, this is the first step in the study of the local dynamics of finite determined reversible mappings (see [JT3]). It may be extended to other types of mappings satisfying some symmetric constraints, and we think that such techniques are also applicable to equivariant systems.

2 Preliminaries and statement of results

2.1 Reversible mappings in $\mathbb{R}^3$

Let Γ be the space of $\mathcal{C}^\infty$ involutions $f : (\mathbb{R}^3, 0) \to \mathbb{R}^3$ (satisfying $f^2 = Id$) which are *non-singular* at 0: it means that $\dim(Ker(f'(0) - Id)) = 2$ (for each $f \in \Gamma$, $Fix(f)$ is a codimension-one submanifold). It is well known (Theorem of Montgomery-Bochner, in [MZ]) that such an involution is $\mathcal{C}^\infty$ -conjugated to the germ of the *linear* involution $f_0 : (x, y, z) \mapsto (x, y, -z)$ (the *standard involution* in these coordinates).

Definition 1. *We say that a mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is reversible if $\phi = g \circ f$ with $(g, f) \in \Gamma^2$. We then say that (g, f) is a reversible system for ϕ.*

In the following, no distinction is made between a germ at 0 and any one of its representatives. We also say that two reversible mappings are *equivalent* if they are $\mathcal{C}^r$-conjugate for some $r \geq 1$. Of course, if ϕ is reversible, then ϕ is equivalent to a product of involutions $g_0 \circ f_0$, f_0 being the standard involution in some coordinates.
We denote by:

- Ξ the space of all reversible mappings, and $\Xi_p = \{\phi \in \Xi \mid \phi(p) = p\}$;
- $\ell_p(f)$ (*resp* $j^k_p(f)$, for $k \geq 2$) the linear part (*resp* the k-jet) at p of any map f ;

- $\mathrm{Trace}_0(\phi)$ the trace of $\ell_0(\phi)$ (for $\phi \in \Xi_0$).

2.2 The classification process via algebraic geometry

Our concern is the study of reversible mappings near a given fixed point (assumed without loss of generality to be the origin). We mention that there are in [L1, L2] discussions about some constructions of normal forms of reversible mappings. Our new approach follows a systematic method, briefly outlined in what follows. We translate as much as possible questions from the $\mathcal{C}^\infty$ point of view to the algebraic point of view, dealing with k-jets at the origin of reversible mappings. Thus we have to deal with real algebraic geometry problems in the space of the coefficients of the k-jets. We point out that there are at our disposal effective algorithms such as the Gröbner bases construction and the integral polynomial factorization. We improve the basic algorithms to make them more efficient in this particular problem (see section **3** for details). As a matter of fact, we make extensive use of these computational tools. All computing was performed via the AXIOM software. The fact that all AXIOM objects are given a precise type improves the efficiency in this kind of classification problems.

Our classification procedure is the following:

- We begin by the classification of the (reversible) linear part of the reversible mappings. Introducing here computer algebra methods, we obtain an algebraic classification of the 1-jet. We first focus on mappings $\phi \in \Xi_0$ such that $\ell_0(\phi)$ has distinct eigenvalues. We briefly proceed to a mathematical analysis of these mappings, considered as *generic*. We then consider the remaining mappings in Ξ_0, called *special mappings*. In particular, their 1-jets satisfy additional algebraic constraints, and that makes normal forms for their 1-jets easy to find. However these normal forms for the 1-jet are not *sufficient* to characterize the mappings from a dynamical viewpoint.

- We precise the study of these special mappings, searching normal forms of their 2-jets. We prove that every 2-jet of a special involution is $\mathcal{C}^\infty$ -conjugated to a quadratic involution, and give the complete list of special involutions which are quadratic. We then deduce normal forms for special involutions. We mention that the 2-jets of special reversible mappings are reversible. Moreover, the proof of Theorem 5 can be read as an algorithm to get both the desired normal forms and the conjugations which provide the normal forms. This process can be extended to higher order jets: this extension is far from obvious but it is straightforward (refer to [JT1, JT2, JT3]).

2.3 Results

Our aim is to give normal forms, that is analytic formulas which characterize in a simple way the type of the origin as a fixed point of a mapping $\phi \in \Xi_0$. If $\phi \in \Xi_p$, we say that ϕ is *semi-elliptic* at p (*resp. semi-hyperbolic*) if $\ell_p(\phi)$ has eigenvalues $1, \lambda, \lambda^{-1}$ with $\lambda \in S^1 \setminus \{-1, 1\}$ (*resp.* with $\lambda \in \mathbb{R} \setminus \{-1, 1\}$). We denote by G_0 the set of $\phi \in \Xi_0$ such that the eigenvalues of $\ell_0(\phi)$ are all distinct. The main results are summarized in the following theorems:

Theorem 1.

1. *G_0 is open in Ξ_0 for the $\mathcal{C}^r$-topology ($r \geq 1$).*

2. $\phi \in G_0 \iff |\mathit{Trace}_0(\phi) - 1| \neq 2$

3. *there is a decomposition $G_0 = G_s \cup G_e$ such that for all $\phi \in G$, there exists a neighborhood V of $0 \in \mathbb{R}^3$ such that $Fix(\phi) \cap V$ is a smooth curve of points, and,*

   - *if $\phi \in G_s$, for all $p \in Fix(\phi) \cap V$, ϕ is semi-hyperbolic at p.*
   - *if $\phi \in G_e$, for all $p \in Fix(\phi) \cap V$, ϕ is semi-elliptic at p.*

4. *a normal form for $\phi \in G_0$ is a linear mapping which has a matrix:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & Re(\lambda) & Im(\lambda^{-1}) \\ 0 & Im(\lambda) & Re(\lambda^{-1}) \end{bmatrix}$$

   in some coordinates, where $\lambda \neq 1$ is an eigenvalue of $\ell_0(\phi)$.

Remark. Concerning the last classification, in the semi-hyperbolic (*resp.* semi-elliptic) case, we can derive immediately from the Center Manifold Theorem that it is represented by a one-parameter family of reversible planar hyperbolic saddles (*resp.* planar elliptic points). Let us mention that the dynamics of a mapping $\phi \in G_e$ has been discussed in [S] (Theorem 2.10).

Observe that $\phi \mapsto \mathrm{Trace}_0(\phi)$ is onto $\mathbb{R}$. It is clear that taking $\mathrm{Trace}_0(\phi)$ as a parameter, the orbit structure of ϕ can change drastically at the (bifurcation) values -1 and 3 . The value -1 corresponds to a doubling period phenomenon (refer to [L1, RL, M]) and the corresponding normal forms are given in [RL, JT1]. To illustrate our algorithm we just deal here with the bifurcation value 3. In order to localize the problem we consider

$$\Xi_0^+ = \{\phi \in \Xi_0 \text{ such that } \mathrm{Trace}_0(\phi) > 0\}$$

We also put $d_0(\phi) = \dim Fix(\ell_0(\phi))$, and define by $\mathfrak{M}_3 =< x, y, z >\subset \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ the (maximal) ideal generated by the coordinate functions. We use the notation $[x, y, z]^n$ for functions in $\mathfrak{M}_3^n$.

Theorem 2. *Any reversible mapping $\phi \in \Xi_0^+ \setminus G_0$ presents in some coordinates one of the following types. Each normal form is obtained via a conjugation which is polynomial in the coefficients of $j_0^2(\phi)$.*

1. *Case* $d_0(\phi) = 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - z \\ y \\ y + z + c_{xx}\,(x-z)^2 + [x,y,z]^3 \end{pmatrix} \tag{1}$$

2. *Case* $d_0(\phi) = 2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - z \\ y \\ z + c_{xx}\,(x-z)^2 + c_{xy}\,(x-z)\,y + c_{yy}\,y^2 + [x,y,z]^3 \end{pmatrix} \tag{2}$$

3. Case $d_0(\phi) = 3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z + c_{xx}\, x^2 + c_{xy}\, x\, y + c_{yy}\, y^2 + [x, y, z]^3 \end{pmatrix} \tag{3}$$

with constants $c_{xx}, c_{xy}, c_{yy} \in \mathbb{R}$.

Remark. This result provides a formal classification of reversible mappings from the shape of their 2-jets. At this stage, one cannot insure that this classification is from a dynamical point of view the finest one: other tools are necessary to study this question (see [JT3]).

2.4 Conclusion, further prospects

Our method consists of an alternative mechanism to study symmetric singularities of some reversible mappings. Of course, this is only the first step in the study of their local behavior near their fixed points. It appears that the 2-jet is not always *sufficient* to characterize these mappings. To insure that the 2-jet is sufficient (in other words, the system is 2-determined), one has to impose extra conditions. For instance, let us roughly examine the case $d_0(\phi) = 2$ for a reversible mapping $\phi \in \Xi_0^+ \setminus G_0$ (see section **5**). There exists a representative system (g, f) such that the manifolds $Fix(g)$ and $Fix(f)$ are tangent at 0. In this case the order of contact between $Fix(g)$ and $Fix(f)$ is of particular importance: according to this, the topology of the fixed points of ϕ can be drastically different. We develop these ideas and study some special mappings in [JT3]. In particular we characterize the degree of degeneracy of jets via analytic invariants involving the contact between $Fix(g)$ and $Fix(f)$ as well as the local behavior of the trace function. Then, the *combination* of our analytic characterization of degeneracy and our systematic method for finding normal forms (presented here) is applied to provide a complete classification of some reversible mappings (see [JT3]). Observe that reversible mappings are not always finite determined; see for example the normal forms of generic reversible mappings (saddle or elliptical point) given in [M].

3 Reversibility and effective algebraic geometry

Considering finite jets (that is, truncated systems), reversibility problems can be treated as algebraic problems. For instance, searching necessary conditions for a jet of a mapping to be involutive leads to a system of polynomial equations in the space of coefficients (see sections **4** and **6**). So the problem is carried over to study algebraic varieties in some space of coefficients. Obviously, the study of an algebraic variety, even a real one, becomes easier if we know the structure of the associated ideal. For that study we dispose of effective well-known algorithms such as the construction of Gröbner bases and the division in a polynomial ring. Let us discuss in the following a variant of the first algorithm.

Let $\mathcal{I}$ be a reduced ideal generated by polynomials in $\mathbb{Z}\,[X_1, X_2, \ldots, X_n]$ and suppose that a generator f can be factorized in $\mathbb{Z}\,[X_1, X_2, \ldots, X_n]$ as a product $f'\,f''$. Then, $\mathcal{I}$ has the same zero set in $\mathbb{R}^n$ as the intersection $\mathcal{I}' \cap \mathcal{I}''$, replacing the generator f by these factors. It is well known that this idea (see [D, MMN, GL]) can be applied during the construction of a Gröbner basis for the ideal $\mathcal{I}$. Let us recall that this construction is done by recursively adding a new polynomial (a syzygy polynomial) to the current set of generators obtained (we refer to [CLO, LJ] for the detailed algorithms). If this new polynomial is factorizable in $\mathbb{Z}\,[X_1, X_2, \ldots, X_n]$, we apply the remark above. Hence it remains to compute separately Gröbner bases for the new ideals so created, and forget the construction of a Gröbner basis for $\mathcal{I}$. The initial problem of constructing a Gröbner basis is replaced by the problem of constructing a list of Gröbner bases for each ideal corresponding to each factor found in the factorization. Moreover, we can avoid multiplicity just by taking the squarefree parts of the involved polynomials.

Obviously, since each factor is in general much more simple than the syzygy polynomial, this method often appears very efficient provided the original ideal $\mathcal{I}$ is not prime. Remember that complexity bounds for the construction of Gröbner bases are of order 2^{2^d}, where d is a bound for the degree of the generating polynomials (see [CLO]).

At the end of the process we get a list of Gröbner bases defining each new ideal created. Observe that in the theory of reversible mappings the presence of symmetries has consequences on the shape of polynomials we have to deal with. As a matter of fact several integral factorizations occur.

However this process may lead to a great number of ideals in the case $\mathcal{I}$ has many primary components. Since our aim is to reduce the classification as few cases as possible, we have to clean up redundant cases. That means that we have to eliminate ideals already containing others. This elimination can be done algorithmically: suppose that two ideals $\mathcal{I}'$, $\mathcal{I}''$ were in the list obtained by the process above. Denote by G' and G'' the associated Gröbner bases. If, for each element of G'' its normal form with respect to G' is zero, then the ideal $\mathcal{I}'$ can be forgotten, since it contains $\mathcal{I}''$. In practice, due to the great number of symmetries of our problem, the ideals we met (see sections **4** and **6**) appeared as the intersection of a lot of ideals (with integral coefficients). Hence the reduction step is very useful.

In the future we will refer to this algorithm in two steps:

- creating by factorization over the integers a list of ideals whose intersection gives the same zero set as the zero set of $\mathcal{I}$
- eliminating redundant ideals from the list

by the name CFR (for Construction by Factorization and Reduction) algorithm. Note that the first part (factorization over the integers while constructing Gröbner bases) is usual in many computer algebra systems, and that the code for the second part is straightforward (the reader may find details in the appendix).

Observe that this algorithm does not guarantee a primary decomposition of ideals (see [CLO, GTZ, EHV] for such algorithms), but gives sufficiently precise answers on our type of problems (see sections **4** and **6**). As often when one computes Gröbner bases, the choice of this ordering is the key point for a great diminution of the complexity. And, although the number of variables is sometimes important, all the computings were made in a quite short time.

4 Generic reversible mappings

Let $\phi \in \Xi$. As we deal with local behavior near a fixed point, we assume from now that $\phi(0) = 0$. We fix the coordinates and denote by f_0 the standard involution in these coordinates. We put $g = \phi \circ f_0$. The aim of this section is to characterize the type of fixed point of ϕ from $\ell_0(\phi)$.

4.1 Algebraic conditions on the linear part

Fix the coordinate system (x, y, z) in $(\mathbb{R}^3, 0)$, and let $g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be given such that $\ell_0(g)$ is defined by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} a_x\ x + a_y\ y + a_z\ z \\ b_x\ x + b_y\ y + b_z\ z \\ c_x\ x + c_y\ y + c_z\ z \end{pmatrix}$$

with $(a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y, c_z) \in \mathbb{R}^9$. Denote by:

$$\begin{array}{cccc} \Theta : & GL_3(\mathbb{R}) & \longrightarrow & \mathbb{R}^9 \\ & g & \mapsto & (a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y, c_z) \end{array}$$

We look for linear involutions $\ell_0(g)$ having a hyperplane of fixed points. The conditions on $\ell_0(g)$ to be an involution correspond to the zero set in $\mathbb{R}^9$ of an ideal $\mathcal{G}$. This ideal is generated by the following polynomials:

$$\begin{array}{lll} a_x^2 + a_y\ b_x + a_z\ c_x - 1 & a_y\ a_x + a_y\ b_y + a_z\ c_y & a_z\ a_x + a_y\ b_z + a_z\ c_z \\ a_x\ b_x + b_y\ b_x + b_z\ c_x & b_y^2 + a_y\ b_x + b_z\ c_y - 1 & b_z\ b_y + a_z\ b_x + b_z\ c_z \\ b_x\ c_y + a_x\ c_x + c_z\ c_x & b_y\ c_y + c_z\ c_y + a_y\ c_x & c_z^2 + b_z\ c_y + a_z\ c_x - 1 \end{array}$$

Proceeding with the CFR algorithm with graded lexicographic ordering on the variables, we realize $Z(\mathcal{G})$ as the union of 4 algebraic varieties. The process takes a few minutes on a computer. Note that if the reduction step of our algorithm was not present, we would have to deal with 58 different cases (which becomes difficult to handle). We put aside the case corresponding to $g = Id$. We then filter the other cases by adding new algebraic conditions: let us denote by $\mathcal{J}_0(g)$ the Jacobian matrix of g at 0 (which is here $\mathcal{J}_0(g) = \ell_0(g)$); $Fix(\ell_0(g))$ must be of dimension 2; hence the characteristic polynomial of $\mathcal{J}_0(g)$ has $X = 1$ as double root; the nullity of the characteristic polynomial and its first derivative give us two algebraic conditions in terms of the coefficients.
Adding these conditions to each case, and recomputing the basis, we arrive to just on one ideal $\mathcal{G}^*$ spanned by:

$$
\begin{array}{ll}
(b_y - 1)\ c_x - b_x\ c_y & b_z\ c_x + (b_y + a_x)\ b_x \\
a_y\ c_x + (1 - a_x)\ c_y & a_z\ c_x + (a_x - 1)\ (b_y + a_x) \\
b_z\ c_y + (b_y + a_x - 1)\ b_y - a_x & a_z\ c_y + a_y\ (b_y + a_x) \\
c_z + b_y + a_x - 1 & a_y\ b_x + (1 - a_x)\ (b_y - 1) \\
a_z\ b_x + (1 - a_x)\ b_z & a_z\ (b_y - 1) - a_y\ b_z
\end{array}
\tag{4}
$$

It is easy to check (again via symbolic computation, using the complete division algorithm corresponding to a Gröbner basis) that 1 is not triple eigenvalue of $\mathcal{J}_0(g)$, and that all the 2-minors of the matrix $\mathcal{J}_0(g) - Id$ vanish modulo the ideal $\mathcal{G}^*$. Observe that it is not possible to give a parametrization of the irreducible components of $Z(\mathcal{G}^*)$. However we can establish an analysis on the eigenvalues of any reversible mapping $g \circ f_0$ (where f_0 is the involution $(x, y, z) \mapsto (x, y, -z)$) such that $\Theta(g) \in Z(\mathcal{G}^*)$: the characteristic polynomial $\Delta(X)$ of the Jacobian matrix at 0 of $g \circ f_0$ belongs to $\mathbb{R}[a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y, c_z][X]$; using again the division algorithm corresponding to a Gröbner basis for $\mathcal{G}^*$, we can perform the reduction modulo $\mathcal{G}^*$ of the coefficients of $\Delta(X)$.

4.2 Proof of Theorem 1

Under the conditions (4) above, the characteristic polynomial Δ of the Jacobian matrix $\mathcal{J}_0(g \circ f_0) = \ell_0(g) \circ f_0$ always admits 1 as root:

$$\Delta(X) = (1 - X)\ \left(X^2 + (2 - 2\ b_y - 2\ a_x)\ X + 1\right) = \ (1 - X)\ \bar{\Delta}(X)$$

The discriminant of $\bar{\Delta}(X)$ is:

$$4\ (b_y + a_x - 2)\ (b_y + a_x)$$

Observe that:

$$\mathrm{Trace}_0(g \circ f_0) = -1 + 2\ \ (b_y + a_x)$$

This gives the following classification:

1. if $(b_y + a_x - 2)\ (b_y + a_x) < 0$, then the Jacobian matrix at 0 has two distinct complex eigenvalues, and the eigenvalue 1. So there exists a neighborhood V of $0 \in \mathbb{R}^3$ such that $Fix(\phi) \cap V$ is a smooth

curve. Moreover, V can be chosen so that for each $p \in Fix(\phi) \cap V$, $\ell_p(\phi)$ has still the same configuration of eigenvalues. In fact, the stability of this configuration of eigenvalues implies that the type of 0 as a fixed point of ϕ is determined by $\ell_0(\phi)$ (in other words, by the matrix given in Theorem 1).

2. if $(b_y + a_x - 2)\ (b_y + a_x) > 0$, then the Jacobian matrix at 0 has three distinct real eigenvalues (1 is one of them). The situation is the same as above, replacing elliptic by hyperbolic.

3. if $b_y + a_x = 0$ or $b_y + a_x = 2$, then $\mathcal{J}_0(g \circ f_0)$ has multiple eigenvalues. Observe that if λ is an eigenvalue, then λ^{-1} also is. Hence multiple eigenvalues are 1 or -1.

We remark that all the conditions above depend upon the value of $\mathrm{Trace}_0(\phi)$, so they do not depend upon the choice of a representative reversible system for ϕ.
Let us define $G_0 = G_e \cup G_s$ where:

1. $G_e = \{\phi \in \Xi_0 \text{ s.t. } \mathrm{Trace}_0(\phi) \in]-1, 3[\ \}$
2. $G_s = \{\phi \in \Xi_0 \text{ s.t. } \mathrm{Trace}_0(\phi) \notin [-1, 3]\ \}$

Observe that G_0 is an open set in the space Ξ_0. ∎

From now, we shall only consider the reversible mappings $\phi \in \Xi_0^+ \setminus G_0$. Notice that if $\phi \in \Xi_0^+ \setminus G_0$ and $\mathrm{Trace}_0(\phi) = -1$, then $\mathrm{Trace}_0(\phi^2) = 3$. As said in the introduction, we suggest the reader to refer to [JT1], and we focus here just to the mappings ϕ with $\mathrm{Trace}_0(\phi) = 3$.

4.3 Study of case $\mathrm{Trace}_0(\phi) = 3$

Assume that the eigenvalues of $\mathcal{J}_0(g \circ f_0)$ are $1, 1, 1$. We add the polynomial $a_x + b_y + c_z - 3$ to $\mathcal{G}^*$. After re-computing a Gröbner basis with lexicographic ordering on the variables

$$[c_z, b_y, b_x, a_y, a_x, a_z, c_x, b_z, c_y]$$

we get an ideal $\mathcal{G}_1$ which has the following presentation:

$$\begin{array}{ll} c_z + 1 & b_z\ c_y + 2\ b_y - 2 \\ b_z\ c_x + 2\ b_x & a_z\ c_y + 2\ a_y \\ b_z\ c_y - 2\ a_x + 2 & a_z\ c_x + b_z\ c_y \end{array} \tag{5}$$

Remark. These mappings correspond to an algebraic variety T_3 of dimension 3 in $\mathbb{R}^9$ and non smooth at the point $\Theta(f_0) = (1, 0, 0, 0, 1, 0, 0, 0, -1)$. This gives insight of the local structure of such involutions near f_0.

5 Normal forms of linear special mappings

In this section we study the mappings $\phi \in \Xi_0^+ \setminus G_0$, called *special mappings*. We classify these mappings by the value $d_0(\phi) = \dim(Fix(\ell_0(\phi)))$.

Proposition 3. *Let $\phi \in \Xi_0^+ \setminus G_0$. Then, ϕ is equivalent to a decomposition $g \circ f_0$, where f_0 is the standard involution in some coordinates, and $\ell_0(g)$ has a matrix of the following type in these coordinates:*

$d_0(\phi)$	*1*	*2*	*3*
matrix of $\ell_0(g)$	$M_1 = \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$	$M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Proof.

- Assume that $\phi \in \Xi_0^+ \setminus G_0$, $\phi = g \circ f_0$ (f_0 being the standard involution in these coordinates) and $d_0(\phi) = 2$. Following equations (5) the matrix of $\ell_0(\phi) - Id$ is:

$$\begin{bmatrix} \frac{1}{2}\, b_z\, c_y & -\frac{1}{2}\, a_z\, c_y & -a_z \\ -\frac{1}{2}\, b_z\, c_x & -\frac{1}{2}\, b_z\, c_y & -b_z \\ c_x & c_y & 0 \end{bmatrix}$$

  This matrix has rank 1. This implies that either $c_x = c_y = 0$, or $a_z = b_z = 0$. Hence the matrix of $\ell_0(g)$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_x & c_y & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & a_z \\ 0 & 1 & b_z \\ 0 & 0 & -1 \end{bmatrix}$$

the parameters being not both zero. Let us examine the first case. We may suppose $c_y \neq 0$. Let χ be the linear change of coordinates $(x, y, z) \mapsto (c_y\, x, -c_x\, x + z, y)$. The matrix of $\chi^{-1} \circ \ell_0(\phi) \circ \chi$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c_y \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

So we claim that $\ell_0(\phi)$ is equivalent to is a product $g \circ f_0'$ where g presents the form of the second case, and f_0' is the standard involution in these new coordinates. Hence, up to a good change of coordinates, we can always assume that $\ell_0(g)$ has a matrix of the form:

$$\begin{bmatrix} 1 & 0 & a_z \\ 0 & 1 & b_z \\ 0 & 0 & -1 \end{bmatrix}$$

with $(a_z, b_z) \neq (0, 0)$. Assume that $a_z \neq 0$ (if not, then swap x and y). Then, the change of coordinates χ' : $(x, y, z) \mapsto (a_z\, x, b_z\, x + y, z)$ commutes with f_0 and $\ell_0({\chi'}^{-1} \circ g \circ \chi')$ has a matrix M_2 .

- If $d_0(\phi) = 1$, the matrix of $\ell_0(\phi \circ f_0)$ is such that $a_z \neq 0$ and $c_y \neq 0$. Then, the linear conjugation $(x, y, z) \mapsto (a_z\, x, b_z\, x + {c_y}^{-1}\, y, z)$ yields the matrix M_1 .
- The case $d_0(\phi) = 3$ is obvious. ■

6 Quadratic reversible mappings and normal forms

Definition 2. *We say that a change of variables χ is $\mathcal{R}$-admissible if $\chi^{-1} \circ f_0 \circ \chi = f_0$.*

Observe that if $\phi = g \circ f_0$, and χ is $\mathcal{R}$-admissible, then $\chi^{-1} \circ \phi \circ \chi = \chi^{-1} \circ g \circ \chi \circ f_0$.

In this section we show how to classify involutions (here, the special involutions) up to $\mathcal{R}$-admissible conjugation. Furthermore we show that non-linear involutions of each class do exist, by exhibiting examples of quadratic involutions. We give the normal forms for such involutions, and conclude by giving normal forms for the corresponding reversible mappings.

6.1 Necessary conditions on 2-jets

The aim of this subsection is to give conditions on the 2-jet for an involution fixing the origin be in Γ. We illustrate the method with an example of a mapping g such that $2 \leq d_0(g \circ f_0) \leq 3$. As before, we translate the problem into an algebraic one, and then use algorithmic methods.

Assume that $g \in \Gamma$. First of all we consider conditions under which the homogeneous part of degree 2 of the 2-jet of $g \circ g = Id$ is exactly zero. This leads to an ideal $\mathcal{H}_2$ generated by 15 polynomials. Constructing a Gröbner basis with lexicographic order on:

$$[a_{xx},\ a_{xy},\ a_{yy},\ a_{zz},\ b_{xx},\ b_{xy},\ b_{yy},\ b_{zz},\ c_{xz},\ c_{yz},\ a_z,\ b_z]$$

we reduce the problem to only 10 polynomials. They are:

$$a_z\ c_{xx} + 2\ a_{xx} \qquad a_z\ c_{xy} + 2\ a_{xy} \qquad a_z\ c_{yy} + 2\ a_{yy}$$
$$b_z\ c_{xx} + 2\ b_{xx} \qquad b_z\ c_{xy} + 2\ b_{xy} \qquad b_z\ c_{yy} + 2\ b_{yy}$$
$$2\ a_z\ c_{xx} + b_z\ c_{xy} - 2\ c_{xz} \qquad a_z\ c_{xy} + 2\ b_z\ c_{yy} - 2\ c_{yz}$$
$$a_z^3\ c_{xx} + a_z^2\ b_z\ c_{xy} + a_z\ b_z^2\ c_{yy} - 2\ a_z\ c_{zz} + 2\ a_{yz}\ b_z + 2\ a_z\ a_{xz} - 4\ a_{zz}$$
$$a_z^2\ b_z\ c_{xx} + a_z\ b_z^2\ c_{xy} + b_z^3\ c_{yy} - 2\ b_z\ c_{zz} + 2\ a_z\ b_{xz} + 2\ b_z\ b_{yz} - 4\ b_{zz}$$

But $\dim(Ker(g'(0) - Id)) = 2$ implies that $Fix(g)$ is defined by only one equation of degree 2, and this condition consists in a linear dependence of 3 vectors (the coefficients of the coordinates of g considered as polynomials in x, y, z). The ideal $\mathcal{H}_2^{'}$ corresponding to all these necessary conditions on the 2-jet of g is then generated by 14 polynomials, the 10 for $\mathcal{H}_2$ and the additional:

$$2\ a_z^2\ c_{xx} + a_z\ b_z\ c_{xy} + 4\ a_{xz} \qquad a_z^2\ c_{xy} + 2\ a_z\ b_z\ c_{yy} + 4\ a_{yz}$$
$$2\ a_z\ b_z\ c_{xx} + b_z^2\ c_{xy} + 4\ b_{xz} \qquad a_z\ b_z\ c_{xy} + 2\ b_z^2\ c_{yy} + 4\ b_{yz}$$

For these conditions, an elimination ordering provides the explicit solution:

$$\begin{cases} a_{xx} = -\dfrac{a_z\ c_{xx}}{2} & a_{xy} = -\dfrac{a_z\ c_{xy}}{2} & a_{yy} = -\dfrac{a_z\ c_{yy}}{2} \\ b_{xx} = -\dfrac{b_z\ c_{xx}}{2} & b_{xy} = -\dfrac{b_z\ c_{xy}}{2} & b_{yy} = -\dfrac{b_z\ c_{yy}}{2} \\ a_{xz} = \dfrac{-2\ a_z^2\ c_{xx} - a_z\ b_z\ c_{xy}}{4} & a_{yz} = \dfrac{-a_z^2\ c_{xy} - 2\ a_z\ b_z\ c_{yy}}{4} \\ b_{xz} = \dfrac{-2\ a_z\ b_z\ c_{xx} - b_z^2\ c_{xy}}{4} & b_{yz} = \dfrac{-a_z\ b_z\ c_{xy} - 2\ b_z^2\ c_{yy}}{4} \\ c_{xz} = \dfrac{2\ a_z\ c_{xx} + b_z\ c_{xy}}{2} & c_{yz} = \dfrac{a_z\ c_{xy} + 2\ b_z\ c_{yy}}{2} \\ a_{zz} = -\dfrac{a_z\ c_{zz}}{2} & b_{zz} = -\dfrac{b_z\ c_{zz}}{2} \end{cases}$$

These necessary conditions correspond to the following 4-parameters family of 2-jets for g:

$$j_0^2(g) \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + a_z \left(z - \dfrac{1}{2}\ \psi(x,y,z) \right) \\ y + b_z \left(z - \dfrac{1}{2}\ \psi(x,y,z) \right) \\ -z + \psi(x,y,z) \end{pmatrix} \tag{6}$$

with

$$\begin{cases} \psi(x,y,z) = \varphi(x,y) + \dfrac{1}{2}\ \tilde{a}_z\ x\ z + \dfrac{1}{2}\ \tilde{b}_z\ y\ z + c_{zz}\ z^2 \\ \varphi(x,y) = c_{xx}\ x^2 + c_{xy}\ x\ y + c_{yy}\ y^2 \\ \tilde{a}_z = 2\ a_z\ c_{xx} + b_z\ c_{xy} \\ \tilde{b}_z = a_z\ c_{xy} + 2\ b_z\ c_{yy} \end{cases}$$

It remains to analyze the question of the realizability of non-trivial involutions (that is non-linear involutions) having such 2-jets. This question is solved in the following subsection.

6.2 Quadratic involutions

We are searching here the existence of involutions of the type studied in 6.1 which are strictly of degree 2. Our method gives a complete answer.

Proposition 4. *Quadratic involutions g such that $d_0(g \circ f_0) \geq 2$ are given by the following 3-parameters family of involutions:*

$$g^* \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + a_z \left(z - \frac{1}{2} \, \psi^*(x,y,z) \right) \\ y + b_z \left(z - \frac{1}{2} \, \psi^*(x,y,z) \right) \\ -z + \psi^*(x,y,z) \end{pmatrix} \tag{7}$$

with

$$\begin{cases} \psi^*(x,y,z) = \varphi(x,y) + \frac{1}{2} \, \tilde{a}_z \ x \ z + \frac{1}{2} \, \tilde{b}_z \ y \ z + \frac{1}{4} \, \varphi(a_z, b_z) \ z^2 \\ \varphi(x,y) = c_{xx} \ x^2 + c_{xy} \ x \ y + c_{yy} \ y^2 \\ \tilde{a}_z = 2 \ a_z \ c_{xx} + b_z \ c_{xy} \\ \tilde{b}_z = a_z \ c_{xy} + 2 \ b_z \ c_{yy} \end{cases}$$

Proof. As before let us compute the ideals $\mathcal{H}_3$ and $\mathcal{H}_4$ corresponding to the nullity of the homogeneous parts of degree 3 and 4 of $g \circ g$. Applying the CFR algorithm to $\mathcal{H}_2' + \mathcal{H}_3$ yields an ideal presented by 15 polynomials, and then, it is straightforward to observe that the ideal $\mathcal{H}_4$ already contains $\mathcal{H}_2' + \mathcal{H}_3$. ■

In what follows, we discuss other types of quadratic involutions. Their classification is performed the same.

6.3 Normal forms of special involutions

We denote by Ω the mapping $\mathbb{R}^4 \to \mathbb{R}^3$ defined by:

$$(u,x,y,z) \mapsto \Omega(u,x,y,z) = \left(-\frac{1}{2} \, u, y, u \right)$$

and by $U\,{}^t(x,y,z)$ the product of a matrix U by a vector of coordinates (x,y,z). In what follows, U will be either M_1 or M_2 (refer to Proposition 3).

Let $\phi \in \Xi_0^+ \setminus G_0$. We fix coordinates and we put $g = \phi \circ f_0$. We then say that g is the *special involution* associated to ϕ in these coordinates.

Theorem 5. *Let $g \in \Gamma$ such that $(g \circ f_0) \in \Xi_0^+ \setminus G_0$. There exists (up to conjugation by an $\mathcal{R}$-admissible mapping) only one normal form for each type described in Proposition 3. The 2-jets of these normal forms are quadratic involutions. Moreover, for each type, the normal form is obtained via a conjugation which is a polynomial in the coefficients of $j_0^2(g)$. These normal forms are:*

1. *Case $d_0(g \circ f_0) = 1$:*

$$(x,y,z) \mapsto {M_2}\,{}^t(x,y,z) + \Omega\left(\psi(x,y,z),x,y,z\right) \tag{8}$$

with

$$\psi(x,y,z) = c_{xx}\left(x + \frac{1}{2}\,z\right)^2 + c_{xy}\left(x + \frac{1}{2}\,z\right)\,y + c_{yy}\,y^2 + [x,y,z]^3$$

2. *Case $d_0(g \circ f_0) = 2$:*

$$(x,y,z) \mapsto {M_1}\,{}^t(x,y,z) + \Omega\left(c_{xx}\left(x + \frac{1}{2}\,z\right)^2 + [x,y,z]^3, x,y,z\right) \tag{9}$$

3. *Case $d_0(g \circ f_0) = 3$:*

$$(x,y,z) \mapsto \left(x, y, -z + \left(c_{xx}\,x^2 + c_{xy}\,x\,y + c_{yy}\,y^2 + [x,y,z]^3\right)\right) \tag{10}$$

with $c_{xx}, c_{xy}, c_{yy} \in \mathbb{R}$.

The classification is performed in three steps. First we use the classification of the linear parts (see section above). We fix coordinates, that is we fix f_0. Of course, from now on, we only deal with $\mathcal{R}$-admissible changes of coordinates. The second step consists in giving the general form of the 2-jet for each type of linear normal form. We proceed in

the same way as in **6.1**. Moreover, this process gives also the conditions of involutivity of the 2-jet. Then, in a third step, we conjugate it by a quadratic $\mathcal{R}$-admissible mapping in order to get a quadratic involution in the conjugacy class. For each step we provide explicit formulas, so this classification process gives an algorithm to get the normal form of any special involution.
Remark. This algorithm does not guarantee that no further reduction is possible. The dynamics (see [JT3]) allow us to show that no class is reducible to another.

6.3.1 $\mathcal{R}$-admissible conjugations on the 2-jets

We intend to use conjugations involving second order terms in the third step of our classification (once a linear normal form has been found). So we just have to determine the quadratic mappings ξ_λ such that $\ell_0(\xi_\lambda) = Id$ and ${\xi_\lambda}^{-1} \circ f_0 \circ \xi_\lambda$ has the same 2-jet as f_0. Although ξ_λ^{-1} is not algebraic, its 2-jet is very easy to compute, just by changing signs. Moreover, the conditions upon the coefficients coming from the 2-jet of ${\xi_\lambda}^{-1} \circ f_0 \circ \xi_\lambda$ imply that six of those coefficients have to be zero. Thus we get the following result:

Lemma 6. *Quadratic $\mathcal{R}$-admissible mappings ξ_λ such that $\ell_0(\xi_\lambda) = Id$ are exactly quadratic mappings of type:*

$$\xi_\lambda \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + \alpha_{xx}\ x^2 + \alpha_{xy}\ x\ y + \alpha_{yy}\ y^2 + \alpha_{zz}\ z^2 \\ y + \beta_{xx}\ x^2 + \beta_{xy}\ x\ y + \beta_{yy}\ y^2 + \beta_{zz}\ z^2 \\ z + \gamma_{xz}\ x\ z + \gamma_{yz}\ y\ z \end{pmatrix}$$

6.3.2 Type 2 involutions

Using Proposition 3 and (6), $j_0^2(\chi^{-1} \circ g \circ \chi)$ has the form:

$$(x, y, z) \mapsto M_2\,{}^t(x, y, z) + \Omega(\psi'(x, y, z), x, y, z)$$

with

$$\begin{cases} \psi'(x, y, z) = \varphi'(x, y) + c'_{xx}\, x\, z + \dfrac{1}{2}\, c'_{xy}\, y\, z + c_{zz}\, z^2 \\ \varphi'(x, y) = c'_{xx}\, x^2 + c'_{xy}\, x\, y + c'_{yy}\, y^2 \end{cases}$$

Let us consider the $\mathcal{R}$-admissible transformation:

$$\theta \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + \dfrac{1}{8}\,(c_{xx} - 4\;c_{zz})\;x^2 \\ y \\ z + \dfrac{1}{4}\,(\;c_{xx} - 4\;c_{zz})\;x\;z \end{pmatrix} \tag{11}$$

Conjugating by θ, we get the 2-jet of (8). But $(g-Id)^{-1}(0)$ is defined by only one equation, and this implies the relationship between the order 3 terms in (8). Observe that the conjugation $\xi = \chi \circ \theta$ depends smoothly on $j_0^2(g)$.
The case $a_z = 0$ is treated via a swap between x and y. ∎

6.3.3 Type 1 involutions

Applying the same process as in **6.1**, we obtain the following normal form for the 2-jet:

$$j_0^2(g)(x,y,z) = (x,y,z) \mapsto M_1{}^t(x,y,z) + \Omega(\psi(x,y,z),x,y,z)$$

with

$$\begin{cases} \psi(x,y,z) = \varphi(x,y) + c_{xx}\,x\,z + \dfrac{c_{xx} + 2\,c_{xy} - 4\,c_{zz}}{4}\;y\;z + c_{zz}\,z^2 \\ \varphi(x,y) = c_{xx}\,x^2 + c_{xy}\,x\,y + c_{yy}\,y^2 \end{cases}$$

Moreover, this 2-jet is a quadratic involution if and only if $c_{xx} = 4\,c_{zz}$. We then consider the $\mathcal{R}$-admissible transformation:

$$\theta' \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + \alpha_{xx}\;x^2 \\ y + \beta_{xy}\;x\;y + \beta_{yy}\;y^2 + \beta_{zz}\;z^2 \\ z + \gamma_{xz}\;x\;z + \gamma_{yz}\;y\;z \end{pmatrix}$$

with:

$$\begin{cases} \alpha_{xx} = \dfrac{1}{2}\;\gamma_{xz} = 2\;\gamma_{yz} = \dfrac{1}{12}\;(c_{xx} + 2\;c_{xy} - 4\;c_{zz}) \\ \beta_{xy} = 2\;\beta_{zz} = \dfrac{1}{6}\;(c_{xx} - 4\;c_{xy} - 4\;c_{zz}) \\ \beta_{yy} = -\gamma_{yz} - 24\;c_{yy} \end{cases}$$

Conjugating by θ', we get the 2-jet of normal form (9). By similar considerations as above, we get the relationship between the order 3 terms as well as the regularity of $\chi' \circ \theta'$. The case $a_z = 0$ is treated via a swap between x and y. ∎

6.3.4 Type 3 involutions

If $\ell_0(\phi) = Id$, the results of **6.1** lead directly to the 2-jet of (10). We conclude the same as above.

6.4 Normal forms of special reversible mappings

Following Theorem 5, the special reversible mappings are $\mathcal{C}^\infty$ -conjugated to a product $g \circ f$ where $j_0^2(g)$ is a quadratic involution, and $j_0^2(f) = f_0$. In these normal forms, one can see at once symmetries of the mapping. If we do not impose one symmetry of the mapping to be associated to the coordinate z, we can give more concise normal forms. We allow here change of coordinates that are not $\mathcal{R}$-admissible ones. Of course these normal forms are less convenient in terms of showing symmetries of the mapping.
In both cases, we conjugate it by the linear mapping $(x, y, z) \mapsto (x - \frac{1}{2}\, z, y, z)$, and we get the forms given in Theorem 2.

6.5 A global program which provides normal forms

The different steps which illustrate our method in the case of 2-jets constitute a program which provides automatically normal forms, even for higher order jets. We suggest the reader to refer to [JT1, JT2, JT3] to see an efficient application of that principles to high order jets. Complete Axiom programs are available from the authors.

A Appendix: AXIOM programs

In this appendix we give some of the more illustrating AXIOM programs we have used to solve our problem. The fact that AXIOM objects have a type is essential in some parts: by giving a type we choose in fact particular classes of algorithms. Also, it is a way to make effective the conceptual difference between parameters and variables.

A.1 CFR algorithm

The CFR algorithm we explained in **3** is very simple to program. The order on variables is given by the type of the polynomials in the list `I` (for instance: `I:List DMP([x,y,z],Integer)`).

```
CFR(I)==
  local L
  L:=groebnerFactorize(I)
  L:=[[squareFreePart(p) for p in Lp] for Lp in L]
  ReductListGroebnerBases(L)

ReductListGroebnerBases(L)==
  local Lr,reducible?
  Lr:=remove([1],L)
  for l in L repeat
    if not(member?(l,Lr)) then iterate
    for m in Lr | not(m=l) repeat
      reducible?:=true
      for i in l repeat
        if not(zero? normalForm(i,m))
                     then (reducible?:=false;leave)
      if reducible? then Lr:=remove(m,Lr)
  Lr
```

A.2 Reduced 2×2 minors modulo an ideal

Again in this example, giving a precise type simplifies the programmation. The aim is to give the 2×2 minors of a matrix whose coefficients satisfy constraints corresponding to an ideal $\mathcal{I}$. First, the formal 2×2 minors can be computed (up to the sign) from the *formal* inverse of a 3×3 matrix, that is, considered as a matrix on the polynomials. Multiplying by the determinant gives an object in the lists of rational fractions, convertible to a list of polynomials in the coefficients of the matrix. Then, the formal minors can be reduced modulo the ideal via the division algorithm. In the following `I` is assumed to be a Gröbner basis.

```
RN:=Fraction Integer
```

```
PRN:=Polynomial RN

reducedTwoTwoMinors(M:Matrix PRN,I:List PRN):List PRN==
  local m:Matrix PRN
  local K:Segment Positive Integer
  K:=1..3
  m:=matrix([[m[i,j] for j in K] for i in K])::Matrix PRN
  FormalMinors:=parts((determinant(m)*inverse(m)))::List PRN
  subs:=reduce(concat,[[m(i,j)=M(i,j) for j in K] for i in K])
  TwoTwoMinors:=[eval(i,subs) for i in FormalMinors]::List PRN
  [normalForm(i,I) for i in TwoTwoMinors]
```

A.3 Finding the 2-jets of special involutions

To finish, we give here the beginning of the research of general 2-jets performed in **6.1**.
We generate any homogeneous part of degree n with the auxiliary function:

```
ser(S,n)==reduce(+,
          [reduce(+,
            [S[n-j-k,j,k]*x**(n-j-k)*y**j*z**k
                       for j in 0..n-k]) for k in 0..n])
```

Then, we generate the 2-jet corresponding to a linear part equal to Id or an involution of type 2:

```
g1:=[x+Alpg[0,0,1],y+Betg[0,0,1],-z]::List PRN
g2:=[ser('a,2),ser('b,2),ser('c,2)]::List PRN
g:=[g1.i+g2.i for i in 1..3]
```

We compute $g^2 - Id$:

```
G2:=[eval(i,[x=g.1,y=g.2,z=g.3]) for i in g]::List PRN
G2r:=[G2.1-x,G2.2-y,G2.3-z]::List DMP([x,y,z],PRN)
```

Note the type of `G2`: we can distinguish with the type what are the parameters and what are the main variables. Then, we select with a function `homog`, of signature:
`DMP([x,y,z],PRN),Integer)-> DMP([x,y,z],PRN)`
the homogeneous part of degree 2. And we get a representation of the ideal $\mathcal{H}_2$, just by taking the coefficients (which are of type `PRN`):

```
h2:=[homog(i,2) for i in G2]
H2:=reduce(concat,[coefficients(i) for i in h2])
```

We begin to solve. Note that the choice of solution variables could be programmed if necessary, by considerations on the rank at the origin. To simplify, let us give them directly, and solve:

```
V:=[a[2,0,0],a[1,1,0],a[0,2,0],a[0,0,2],b[2,0,0],b[1,1,0],
    b[0,2,0],b[0,0,2],c[1,0,1],c[0,1,1],a[0,0,1],b[0,0,1]]
HH2:=lexGroebner(H2,V)
```

Then, we compute the corresponding g:

```
g:=[normalForm(i,HH2) for i in g]
```

and add the conditions for smoothness (obtained easily by writing the linear dependence of the coefficients of the 3 coordinates of g). The complete solving is finished as indicated in **6.1**.

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