LOW-DIMENSIONAL FILIFORM LIE SUPERALGEBRAS

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Abstract

The aim of this paper is to give a classification up to isomorphism of low dimension filiform Lie superalgebras.

1 Introduction

There exists a lot of work concerning Lie superalgebras. But less of them are interested in nilpotent Lie superalgebras, although, the case of nilpotent Lie algebras has been well studied, see for example [1]. This fact is a consequence of the development of deformation theory. In this paper, we will focus a particular class of nilpotent Lie superalgebras. We recall some definitions from [3]. We point out the definition of filiform Lie superalgebras. This definition is analogue to the definition of filiform Lie algebras given by Vranceanu [10] and Vergne [9]. The classification of this super-algebras is still a problem, as the classification of filiform Lie algebras over an algebraically closed field of characteristic zero is only done up to dimension 11 [4]. In this paper, we will give a first classification for filiform Lie superalgebras in low dimensions.

2 Filiform Lie superalgebras

A $\mathbb{Z}_2$-graded vector space $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ over an algebraically closed field is a Lie superalgebra if there exists a bilinear product $[,]$ over $\mathcal{G}$ such that

$$[G_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta \mod 2},$$
$$[g_\alpha, g_\beta] = -(-1)^{\alpha.\beta}[g_\beta, g_\alpha]$$

for all $g_\alpha \in \mathcal{G}_\alpha$ and $g_\beta \in \mathcal{G}_\beta$ and satisfying Jacobi's super-relation:

$$(-1)^{\gamma \cdot \alpha}[A, [B, C]] + (-1)^{\alpha \cdot \beta}[B, [C, A]] + (-1)^{\beta \cdot \gamma}[C, [A, B]] = 0$$

for all $A \in \mathcal{G}_\alpha$, $B \in \mathcal{G}_\beta$ and $C \in \mathcal{G}_\gamma$.

For such a Lie superalgebra we define the lower central series by

$$\begin{cases} C^0(\mathcal{G}) = \mathcal{G}, \\ C^{i+1}(\mathcal{G}) = [\mathcal{G}, C^i(\mathcal{G})] \end{cases}$$

A Lie superalgebra $\mathcal{G}$ is nilpotent if there exists an integer n such that $C^n(\mathcal{G}) = \{0\}$.

This definition is not easy to use. Therefore we define the following sequences for a Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$:

$$C^0(\mathcal{G}_0) = \mathcal{G}_0, \qquad C^{i+1}(\mathcal{G}_0) = [\mathcal{G}_0, C^i(\mathcal{G}_0)]$$

and

$$C^0(\mathcal{G}_1) = \mathcal{G}_1, \qquad C^{i+1}(\mathcal{G}_1) = [\mathcal{G}_0, C^i(\mathcal{G}_1)]$$

Theorem 2.1. *Let* $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ *be a Lie superalgebras. Then* $\mathcal{G}$ *is nilpotent if and only if there exist* (p, q) *such that* $C^p(\mathcal{G}_0) = \{0\}$ *and* $C^q(\mathcal{G}_1) = \{0\}$.

Proof. If the Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is nilpotent it is easy to prove that there exist (p, q) such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.

For the converse, we use the classical Engel's theorem. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebra. Assume that there exist (p, q) such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, then $ad(X)$ with $X \in \mathcal{G}_0$ is nilpotent. We have for $Y \in \mathcal{G}_1$:

$$ad(Y) \circ ad(Y) = \frac{1}{2} ad([Y, Y])$$

As $[Y, Y]$ is an element of $\mathcal{G}_0$, then $ad([Y, Y])$ is nilpotent. This implies that $ad(Y)$ is nilpotent for every $Y \in \mathcal{G}_1$. Hence $ad(X)$ and $ad(Y)$ is nilpotent, using Engel's theorem given in [6], $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is a nilpotent Lie superalgebra. ∎

Definition 2.1. *Let* $\mathcal{G}$ *be a nilpotent Lie superalgebra, the super-nilindex of* $\mathcal{G}$ *is the pair* (p, q) *such that:* $C^p(\mathcal{G}_0) = \{0\}$, $C^{p-1}(\mathcal{G}_0) \neq \{0\}$ *and* $C^q(\mathcal{G}_1) = \{0\}$, $C^{q-1}(\mathcal{G}_1) \neq \{0\}$. *It is and invariant up to isomorphism.*

Definition 2.2 (Filiform Lie superalgebras). *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a nilpotent Lie superalgebra with $\dim \mathcal{G}_0 = n + 1$ and $\dim \mathcal{G}_1 = m$. $\mathcal{G}$ is called filiform if it's super-nilindex is (n, m). We will note $\mathcal{F}_{n,m}$ the set of filiform Lie superalgebras.*

Let's define the set $\mathcal{N}_{n,m}^{p,q}$ of Lie superalgebras with $\dim \mathcal{G}_0 = n + 1$, $\dim \mathcal{G}_1 = m$ and with super-nilindex (k_0, k_1) such that $k_0 \leq p$ and $k_1 \leq q$. It is obvious that this set can be define by polynomial relations given by the Jacobi relations and the nilpotency relations. This prove that it is a Zariski-closed set of the algebraic variety of nilpotent Lie superalgebras denoted by $\mathcal{N}_{n,m}$.

The set $\mathcal{F}_{n,m}$ of filiform Lie superalgebras can be written as the complementary of a close set:

$$\mathcal{F}_{n,m} = \mathcal{N}_{n,m} \setminus (\mathcal{N}_{n,m}^{n-1,m} \cup \mathcal{N}_{n,m}^{n,m-1})$$

Hence the set of filiform Lie superalgebras is an open set of the variety of nilpotent Lie superalgebras (see [5]).

3 Classifications of filiforms over $\mathbb{C}$ in low dimensions

3.1 Adapted basis

Like for the filiform Lie algebras [9], there exists an adapted bases of a filiform Lie superalgebra:

Proposition 3.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra with $\dim \mathcal{G}_0 = n + 1$ and $\dim \mathcal{G}_1 = m$. Then there exists a bases $\{X_0, X_1, \ldots X_n, Y_1, Y_2, \ldots Y_m\}$ of $\mathcal{G}$ with $X_i \in \mathcal{G}_0$ and $Y_i \in \mathcal{G}_1$ such that:*

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-1, \quad [X_0, X_n] = 0; \\ [X_1, X_2] \in \mathbb{C}.X_4 + \mathbb{C}.X_5 + \cdots + \mathbb{C}.X_n; \\ [X_0, Y_i] = Y_{i+1} & 1 \leq i \leq m-1, \quad [X_0, Y_m] = 0. \end{cases}$$

Such basis are called adapted.

Proof. Let $gr\mathcal{G} \in \mathcal{F}_{n,m}$ be a graded filiform lie superalgebra. The lower central series implies the following graduations:

$$gr\mathcal{G}_0 = W_1^0 \oplus W_2^0 \oplus \cdots \oplus W_{n-1}^0$$
$$gr\mathcal{G}_1 = W_1^1 \oplus W_2^1 \oplus \cdots \oplus W_{m-1}^1$$

with $\dim W_1^0 = 2$, $\dim W_i^0 = 1$ for $2 \leq i \leq n-1$ and $\dim W_j^1 = 1$ for $1 \leq j \leq m-1$. Then we have $[W_1^0, W_i^0] = W_{i+1}^0$ and $[W_1^0, W_j^1] = W_{j+1}^1$.

Let $x_i \in W_i^0$ and $y_j \in W_j^1$ be non zero elements, then if $w \in W_1^0$,

$$[w, x_i] = \lambda_i^0(w)\ x_{i+1}$$
$$[w, y_j] = \lambda_j^1(w)\ y_{j+1}$$

the maps $w \to \lambda_i^0(w)$ and $w \to \lambda_j^1(w)$ are non zero linear forms on W_1^0, because $[W_1^0, W_i^0] = W_{i+1}^0$ and $[W_1^0, W_j^1] = W_{j+1}^1$.

As the field is $\mathbb{C}$, there exists x_0 in W_1^0 such that $\lambda_i^0(w) \neq 0$ for $1 \leq i \leq n-1$ and $\lambda_j^1(w) \neq 0$ for $1 \leq j \leq m-1$. Hence it exist a bases $X_0, X_1, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ with X_i (resp. Y_j) multiple of x_i (resp. y_j) such that:

$$[X_0, X_i] = X_{i+1} \text{ for } 1 \leq i \leq n-1$$
$$[X_0, Y_j] = Y_{j+1} \text{ for } 1 \leq j \leq m-1$$
$$[X_1, X_2] = a\ X_3 \text{ for } a \in \mathbb{C}$$

Substitute X_1 by $X_1 - a\ X_0$ we get the adapted bases of $gr\mathcal{G}$:

$$[X_0, X_i] = X_{i+1} \text{ for } 1 \leq i \leq n-1$$
$$[X_0, Y_j] = Y_{j+1} \text{ for } 1 \leq j \leq m-1$$
$$[X_1, X_2] = 0$$

If the Lie superalgebras is not graded, then we introduce a graded Lie superalgebra, as it was done for Lie algebras in [9]. ∎

3.2   Adapted changes of basis

Definition 3.1. *Let f be a graded change of bases of a filiform Lie superalgebra $\mathcal{G}$. f is an adapted changes of bases if f is Lie superalgebra homomorphism and if the image of an adapted bases of $\mathcal{G}$ is an adapted bases.*

Proposition 3.2. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra. Let $f = f_0 + f_1$ be an adapted change of basis of $\mathcal{G}$. Then f_0 is an adapted change of bases of the filiform Lie algebra $\mathcal{G}_0$ and f_1 satisfies:*

$$\begin{cases} f_1(Y_1) = d_1\ Y_1 + d_2\ Y_2 + \cdots + d_m\ Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

with the condition:

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

where the A_j's are defined by:

$$A_j = a_0 + \sum_{k=0}^{j-1} (-1)^k C_{j-1}^k \left(\sum_{i=1}^{n-k} a_i . r_{i+k,2+k} \right)$$

and the product of $\mathcal{G}$ is given in the adapted bases by

$$[X_i, Y_1] = \sum_{s=2}^{m} r_{i,s} \, Y_s \quad 1 \leq i \leq n$$

Proof. The proof can be found in [3]. We give a sketch of it.

First, we establish that f_0 is an adapted change of bases for Lie algebras as in [4]. The we set:

$$\begin{cases} f_0(X_0) = a_0 \, X_0 + a_1 \, X_1 + \ldots a_n \, X_n \\ f_1(Y_1) = d_1 \, Y_1 + d_2 \, Y_2 + \ldots d_m \, Y_m \end{cases}$$

and assuming $f(Y_t) = [f(X_0), f(Y_{t-1})]$ for $2 \leq t \leq m$ we prove by induction on t that:

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p . Y_t + \sum_{p \geq t+1} d_p . Y_p$$

This implies that $f_1(Y_m) = d_1 \prod_{p=1}^{m-1} A_p \, Y_m$. For this vector to be non zero, we must have $d_1 \prod_{p=1}^{m-1} \neq 0$ This prove that

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p . Y_t + \sum_{p \geq t+1} d_p . Y_p$$

start we a non zero componant on Y_t, and then the images of the vectors Y_t by f form a bases.

Conversely, let f_0 be an adapted change of bases of filiform Lie algebras. Then the map $f = f_0 + f_1$, with f_1 given by:

$$\begin{cases} f_1(Y_1) = d_1 \, Y_1 + d_2 \, Y_2 + \ldots d_m \, Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

and the condition:

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

is an adapted change of basis. ∎

3.3   Classification in low dimensions

We have a description of the products of the filiform Lie superalgebras on $\mathcal{F}_{1,m}$. There exists two types:

(1)

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, & 1 \leq i \leq m - 1 \\ [Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, & \text{if } i + j \text{ even and } 2 \leq i + j \leq m + 1 \end{cases}$$

the other products vanish.

(2)

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, & 1 \leq i \leq m - 1 \\ [X_1, Y_r] = \sum_{s=2}^{m} r_s Y_{s+r-1} \text{ with } s + r - 1 \leq m \end{cases}$$

the other products vanish.

Using adapted changes of bases, like it was done for Lie algebras in [4], we can eliminate the parameters $a_{\frac{i+j}{2}}$.

Theorem 3.1. *Every filiform Lie superalgebra of $\mathcal{F}_{1,m}$ which is not a Lie algebra is isomorphic to one of the following filiform Lie superalgebras:*

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m - 1$$
$$[Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \leq i \leq k$$

with $1 \leq k \leq z + 1$ if $m = 2z + 1$ and $1 \leq k \leq z$ if $m = 2z$.

Proof. We can assume that every non trivial filiform Lie superalgebra with $1 \leq p \leq k - 1$ is isomorphic to:

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m - 1$$
$$[Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \leq i \leq k$$
$$[Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, \text{ if } i + j \text{ even and } 2 \leq i + j \leq 2(k - p)$$

for $1 \leq p \leq k - 1$. The other products vanish.

Using this change of adapted bases:

$$
\begin{cases}
X_0^1 = X_0 \\
X_1^1 = X_1 \\
Y_1^1 = Y_1 - (-1)^p \dfrac{a_{k-1}}{2} Y_{2p+1} \\
\cdots \\
Y_{k-p}^1 = Y_{k-p} - (-1)^p \dfrac{a_{k-1}}{2} Y_{k+p} \\
Y_{k-p+s}^1 = Y_{k-p+s} - (-1)^p \dfrac{a_{k-1}}{2} Y_{k+p+s}
\end{cases}
$$

we can reduce to

$$
\begin{cases}
[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m - 1 \\
[Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \leq i \leq k \\
[Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, \text{ if } i+j \text{ even and } 2 \leq i+j \leq 2(k-p-1)
\end{cases}
$$

The other products vanish.

By induction on p, we have a complete classification. ∎

By using adapted changes of bases, we established the following classifications:

$\mathcal{F}_{1,2}$:

(1) $\begin{cases} [X_0, Y_1] = Y_2 \end{cases}$

(2) $\begin{cases} [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$

(3) $\begin{cases} [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1 \end{cases}$

$\mathcal{F}_{1,3}$:

(1) $\begin{cases} [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \end{cases}$

(2) $\begin{cases} [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1 \end{cases}$

(3) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [Y_2, Y_2] = X_1, & [Y_1, Y_3] = -X_1 \end{cases}$

(4) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, & [X_1, Y_2] = Y_3 \end{cases}$

(5) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \end{cases}$

(6) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2 + Y_3, & [X_1, Y_2] = Y_3 \end{cases}$

$\mathcal{F}_{1,4}$:

(1) $\{ \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4$

(2) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [Y_1, Y_1] = X_1 \end{cases}$

(3) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [Y_2, Y_2] = X_1, & [Y_1, Y_3] = -X_1 \end{cases}$

(4) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2, & [X_1, Y_2] = Y_3, \quad [X_1, Y_3] = Y_4 \end{cases}$

(5) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_3, & [X_1, Y_2] = Y_4 \end{cases}$

(6) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_4 \end{cases}$

(7) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2 + Y_3, & [X_1, Y_2] = Y_3 + Y_4, \quad [X_1, Y_3] = Y_4 \end{cases}$

(8) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2 + Y_4, & [X_1, Y_2] = Y_3, \quad [X_1, Y_3] = Y_4 \end{cases}$

(9) $\begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2 + Y_3 + 2\, Y_4, & [X_1, Y_2] = Y_3 + Y_4, \quad [X_1, Y_3] = Y_4 \end{cases}$

$\mathcal{F}_{2,2}$:

(1) $\{ \quad [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2$

(2) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1, \quad [Y_1, Y_2] = \dfrac{1}{2} X_2 \end{cases}$

(3) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2 \end{cases}$

(4) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$

(5) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_2 \end{cases}$

$\mathcal{F}_{2,3}$:

(1) $\{ \; [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3$

(2) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_1, Y_2] = Y_3 \end{cases}$

(3) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \end{cases}$

(4) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_2] = -Y_3, \quad [X_2, Y_1] = Y_3 \end{cases}$

(5) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2 + Y_3, \quad [X_1, Y_2] = Y_3 \end{cases}$

(6) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_2, Y_1] = Y_3 \end{cases}$

(7) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = 2Y_2, \quad [X_1, Y_2] = Y_3, \quad [X_2, Y_1] = Y_3 \end{cases}$

(8) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$

(9) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_1, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$

$$(10) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(11) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2 + Y_3, \quad [X_1, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(12) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_3] = -X_2, \quad [Y_2, Y_2] = X_2 \end{cases}$$

$$(13) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_1, Y_2] = Y_3 \\ [Y_1, Y_3] = -X_2, \quad [Y_2, Y_2] = X_2 \end{cases}$$

$$(14) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1, \quad [Y_1, Y_2] = \frac{1}{2}X_2 \end{cases}$$

$$(15) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_2] = -Y_3, \quad [X_2, Y_1] = Y_3 \\ [Y_1, Y_1] = X_1, \quad [Y_1, Y_2] = \frac{1}{2}X_2 \end{cases}$$

$$(16) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1, \quad [Y_1, Y_2] = \frac{1}{2}X_2, \quad [Y_1, Y_3] = -X_2, \quad [Y_2, Y_2] = X_2 \end{cases}$$

$\mathcal{F}_{3,2}$:

$$(1) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, Y_1] = Y_2 \end{cases}$$

$$(2) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1, \quad [Y_1, Y_2] = \frac{1}{2}X_2, \quad [Y_2, Y_2] = \frac{1}{2}X_3 \end{cases}$$

$$(3) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2, \quad [Y_1, Y_2] = \frac{1}{2}X_3 \end{cases}$$

$$(4) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_3 \end{cases}$$

$$(5) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$$

(6) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, & [Y_1, Y_1] = X_3 \end{cases}$

$\mathcal{F}_{4,2}$:

(1) $\{$ $[X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, Y_1] = Y_2$

(2) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2, & [Y_1, Y_2] = \dfrac{1}{2} X_3, & [Y_2, Y_2] = \dfrac{1}{2} X_4 \end{cases}$

(3) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_3, & [Y_1, Y_2] = \dfrac{1}{2} X_4 \end{cases}$

(4) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_4 \end{cases}$

(5) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$

(6) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, & [Y_1, Y_1] = X_4 \end{cases}$

(7) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 \end{cases}$

(8) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [Y_1, Y_1] = X_3, & [Y_1, Y_2] = \dfrac{1}{2} X_4 \end{cases}$

(9) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [Y_1, Y_1] = X_4 \end{cases}$

(10) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [X_1, Y_1] = Y_2 \end{cases}$

(11) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [X_1, Y_1] = Y_2, & [Y_1, Y_1] = X_4 \end{cases}$

$\mathcal{F}_{5,2}$:

(1) $\begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \end{cases}$

(2) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_3, \quad [Y_1, Y_2] = \dfrac{1}{2} X_4, \quad [Y_2, Y_2] = \dfrac{1}{2} X_5 \end{cases}$

(3) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \dfrac{1}{2} X_5 \end{cases}$

(4) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_5 \end{cases}$

(5) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$

(6) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$

(7) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5 \end{cases}$

(8) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [Y_1, Y_1] = X_3, \quad [Y_1, Y_2] = \dfrac{1}{2} X_4, \quad [Y_2, Y_2] = \dfrac{1}{2} X_5 \end{cases}$

(9) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \dfrac{1}{2} X_5 \end{cases}$

(10) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [Y_1, Y_1] = X_5 \end{cases}$

(11) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, Y_1] = Y_2 \end{cases}$

(12) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$

(13) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \end{cases}$

(14) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [Y_1, Y_1] = X_5 \end{cases}$

(15) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2 \end{cases}$

(16) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$

(17) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \end{cases}$

(18) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [Y_1, Y_1] = X_5 \end{cases}$

(19) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = Y_2 \end{cases}$

(20) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$

(21) $\begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = -Y_2 \end{cases}$

$$(22) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2, \quad [X_1, X_4] = -X_5 \\ [X_2, X_3] = X_5, \quad [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2} X_5 \end{cases}$$

$$(23) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$$

$$(24) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \end{cases}$$

$$(25) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [Y_1, Y_1] = X_5 \end{cases}$$

$$(26) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2 \end{cases}$$

$$(27) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$$

$$(28) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2 \end{cases}$$

$$(29) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2} X_5 \end{cases}$$

$$(30) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$$

$$(31) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 + X_5, \quad [X_1, X_3] = X_5, \quad [X_1, Y_1] = -2Y_2 \end{cases}$$

$$(32) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 + X_5, [X_1, X_3] = X_5, [X_1, Y_1] = -2Y_2, [Y_1, Y_1] = X_5 \end{cases}$$

$$(33) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2 \end{cases}$$

$$(34) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2}X_5 \end{cases}$$

$$(35) \begin{cases} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{cases}$$

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