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ON THE STRUCTURE OF LINKED 3-FOLDS*

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Abstract

The structure of 3-folds in $\mathbb{P}^6$ which are generally linked via a complete intersection (f_1, f_2, f_3) to 3-folds in $\mathbb{P}^6$ of degree $d \leq 5$ is determined.

We also give three new examples of smooth 3-folds in $\mathbb{P}^6$ of degree 11 and genus 9. These examples are obtained via liaison. The first two are 3-folds linked via a complete intersection $(2, 3, 3)$ to 3-folds in $\mathbb{P}^6$ of degree 7: (i) the hyperquadric fibration over $\mathbb{P}^1$ and (ii) the scroll over $\mathbb{P}^2$. The third example is Pfaffian linked to a 3-dimensional quadric in $\mathbb{P}^6$.

1 Introduction

Submanifolds of $\mathbb{P}^N$ of codimension 3 are object of study of various authors. If $N \geq 10$ we are in the Hartshorne's range, that is, every codimension 3 submanifold of $\mathbb{P}^N$ with $N \geq 10$ is conjectured to be a complete intersection. Hence of particular interest are those contained in $\mathbb{P}^N$ with $N = 6, 7, 8, 9$. We will concentrate on $N = 6$.

This paper grew out as an attempt to determine the structure of linked 3-folds in $\mathbb{P}^6$, as well as, to construct new examples of smooth 3-folds in $\mathbb{P}^6$ of degree 11.

The reason for being interested in such 3-folds is on the one hand the paper by the first author and Besana [8], where one had to deal with the effectiveness of the lists given in there. On the other hand we know that the construction of examples plays a major role in classification problems.

The main tool for determining the structure of linked 3-folds is a combination of formulas for the blow up of Chern classes, [18], and

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adjunction theoretic methods, [19], [6]. As a byproduct we also get an alternative way to determine the structure of some smooth 3-folds in $\mathbb{P}^6$ of degree 9 and 10, see 3.6, 3.7. These two manifolds were obtained in [10], [11] with different methods.

In the second part of the paper we give new examples of 3-folds in $\mathbb{P}^6$. In fact with the help of the computer algebra system Macaulay we construct smooth 3-folds in $\mathbb{P}^6$ of degree 11 and genus 9. These examples are obtained via liaison. The first two are linked via a complete intersection $(2,3,3)$ to 3-folds in $\mathbb{P}^6$ of degree 7: (i) the hyperquadric fibration over $\mathbb{P}^1$ and (ii) the scroll over $\mathbb{P}^2$.

The third example is a 3-fold which is Pfaffian linked to a 3-dimensional quadric in $\mathbb{P}^6$.

Since Bertini-type criterion could not be applied in such cases we had to use the computer algebra system Macaulay to prove the smoothness of our examples.

The paper is structured as follows. In section 2. we fix our notation and give preliminary results that will be needed later on in the paper. Section 3. is devoted to study the structure of the linked 3-folds.

In the last section we construct new examples of 3-folds in $\mathbb{P}^6$ of degree 11.

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2 Notations and Preliminaries

Throughout this article, unless otherwise specified, X denotes a smooth connected projective 3-fold in $\mathbb{P}^6$ defined over the complex field $\mathbb{C}$. Its structure sheaf is denoted by $\mathcal{O}_X$. For any coherent sheaf $\Im$ on X, $h^i(\Im)$ is the complex dimension of $H^i(X,\Im)$ and $\chi(\mathcal{O}_X) = \sum_i(-1)^i h^i(\mathcal{O}_X)$. The following notation is used:
X, smooth 3-fold in $\mathbb{P}^6$;
$H \in |\mathcal{O}_{\mathbb{P}^6}(1)|$;
K = class of canonical bundle of X;
$c_i(X)$ = Chern classes of X;
T_X = tangent bundle of X;
$N_{X/\mathbb{P}}$ = normal bundle of X in $\mathbb{P}$;
I_X = ideal sheaf of X.

For the convenience of the reader we recall the following theorem which will be used throughout the paper. We state it only for 3-folds since those are what we are interested in.

Theorem 2.1. *([14], [19]) Let $\tilde{X}$ be a complex projective manifold of dimension 3 and let $\tilde{L}$ be a very ample line bundle over $\tilde{X}$. Assume that $\kappa(K_{\tilde{X}} + 2\tilde{L}) = 3$ and let (Y, L) be the first reduction of $(\tilde{X}, \tilde{L})$. Then $K_Y + L$ is nef and big unless either:*

i) (Y,L) $\cong (P^3, \mathcal{O}_{\mathbb{P}^3}(3))$;

ii) (Y,L) $\cong (Q, \mathcal{O}_Q(2))$, where Q is a hyperquadric in $\mathbb{P}^4$;

iii) there is a surjective morphism $\phi : Y \longrightarrow Z$ onto a smooth curve Z, whose general fibre is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and 2K+3L $\approx \phi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z;

iv) (Y,L) is a Fano variety of coindex 3;

v) (Y,L) is a Del Pezzo fibration over a smooth curve Z and K+L $\approx \psi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z;

vi) (Y,L) is a conic bundle over a surface Z and K+L $\approx \psi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z.

Lemma 2.2. *([4] §1) Let X be a smooth 3-fold embedded by $|L|$ in $\mathbb{P}^N$. We have the following:*

a) $c_2(X) \cdot L = e(S) - K_S \cdot L_S = 12\chi(\mathcal{O}_S) - K_S \cdot K_S + 2(1-g) + d$

b) $K \cdot K \cdot L = K_S \cdot K_S - 2K_S \cdot L_S + L_S \cdot L_S = K_S \cdot K_S + 4(1-g) + 3d$

c) $K \cdot L \cdot L = K_S \cdot L_S - d = 2(g - 1 - d)$.

The following facts about blowing up Chern classes are known, see [18], [12] for details.

Let P be a smooth projective variety, X a subvariety of P of codimension r. Let $\tilde{P}$ be the blow up of P along X, and let $E \cong \mathbb{P}(N^*)$ the exceptional divisor, where N denotes the normal bundle of X in P.

Let ξ be the tautological line bundle of E. Denote $c(T_X)$ with $c(X)$. We have the following blow up diagram.

$$\begin{array}{ccc} E & \xrightarrow{j} & \tilde{P} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & P \end{array}$$

For the reader's convenience we recall the following theorem due to Porteous ([18], Theorem 2) which relates the Chern classes of P with those of $\tilde{P}$, the blow up of P along a smooth subvariety X of codimension r. The notation used will be that in ([12], Theorem 15.4).

Theorem 2.3. *([18]) With the above notation we have*

$$c(\tilde{P}) - f^*c(P) = j_*(g^*c(X) \cdot \alpha) \tag{1}$$

where

$$\alpha \quad = \quad \frac{1}{\xi}\Big[\sum_{i=0}^{r} g^*c_{r-i}(N) - (1-\xi)\ \mathit{sum}_{i=0}^{r}(1+\xi)^i g^*c_{r-i}(N)\Big]$$

For our purpose we need only to know the degree $1, 2$ and 3 terms in (1). They are:

$$c_1(\tilde{P}) - f^*c_1(P) = j_*(1-r) = (1-r)[E] \tag{2}$$

$$\begin{aligned} c_2(\tilde{P}) - f^*c_2(P) \quad &= \quad -j_*\Big[(r-1)g^*c_1(X) + \frac{r(r-3)}{2}\xi \\ +(r-2)g^*c_1(N)\Big] & \end{aligned} \tag{3}$$

$$\begin{aligned} c_3(\tilde{P}) - f^*c_3(P) \quad = \quad & -j_*\Big[(r-1)g^*c_2(X) \\ & +(r-2)g^*c_1(X)g^*c_1(N) \\ & +\frac{(r-1)(r-4)}{2}\xi g^*c_1(N) + \frac{r(r-3)}{2}\xi g^*c_1(X) \\ & +\frac{r(r-1)(r-5)}{6}\xi^2 + (r-3)g^*c_2(N)\Big] \end{aligned} \tag{4}$$

For the terms of degree 1 and 2 we refer to ([12], Example 15.4.3).

We also recall the following proposition which is a Bertini-type criterion for varieties which are generally linked via complete intersections.

Proposition 2.4. *([17], Proposition pg.423) Let $X \subset P$ be a smooth subvariety of dimension n in a projective N-fold P defined by the ideal I_X. Suppose that $L_i \in Pic(P)$ are line bundles such that $L_i \otimes I_X$ is globally generated for each i = 1,...N-n, and let X' be linked to X via a general section $s \in H^0(P, \bigoplus_{i=1}^{N-n} L_i \otimes I_X)$. If $n < 4$, then X' is smooth.*

3 On the structure of some linked 3-folds in $\mathbb{P}^6$

Okonek in ([17], Examples 1 through 5) has constructed 3-folds $X' \subset \mathbb{P}^6$ which are generally linked via complete intersections ($\underline{s}$) of multidegree $\underline{d}$ to simple known examples of 3-folds in $\mathbb{P}^6$ of degree $d \leq 5$. The aim of this section is to determine the structure of such 3-folds.

This will be accomplished by combining formulas for the blow up of Chern classes and adjunction theoretic methods.

We will treat just one case for each of the tables giving in ([17], Examples 1 through 5) being the remaining cases analogous to the ones considered.

We start with the case in which the known 3-fold X is $\mathbb{P}^3 \subset \mathbb{P}^6$ and the multidegree of the complete intersection is $(2, 2, 3)$. Thus the $L_i \in Pic(P)$ we are considering are $L_1, L_2 = 2H, L_3 = 3H$. We will see that the following proposition holds.

Proposition 3.1. *Let $\mathbb{P}^3, X' \subset \mathbb{P}^6$ be linked via a complete intersection (2,2,3). Then X' is a conic bundle over $\mathbb{P}^2$ with $d = 11, g = 10, \chi(\mathcal{O}_{X'}) = 1, \chi(\mathcal{O}_{S'}) = 4, K_{S'}^2 = 2, e(X') = -48$.*

The proof will be done in several steps. We fix at first some notation.

Notation. Going through the proof in ([17], Proposition pg.423), one sees that X' is gotten as follows. Let $f : \tilde{P} \longrightarrow P$ be the blow up of P along X and denote the restriction of f to the exceptional divisor

E by g. The sections $s_i \in H^0(P, L_i \otimes I_X)$ correspond to sections $\tilde{s}_i \in H^0(\tilde{P}, \mathcal{O}_{\tilde{P}}(-E) \otimes f^*L_i)$. Let $\tilde{X}$ be the complete intersection $\tilde{X} = (\underline{\tilde{s}})$, where $\underline{\tilde{s}} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$. Note that $(\underline{s}) = (s_1, s_2, s_3) = X \cup X'$, with $X' = f(\tilde{X})$ and that $\tilde{X}$ is isomorphic to X'.

Being X' isomorphic to $\tilde{X}$ we can work with $\tilde{X}$ over which some of the computations are easy to handle.

Lemma 3.2. *Let $\tilde{X}$ be as above and let $\tilde{L} = f^*(H)_{|_{\tilde{X}}}$. Then $deg\tilde{X} = 11, g(\tilde{L}) = 10, \chi(\mathcal{O}_{\tilde{X}}) = 1, e(\tilde{X}) = -48, \chi(\mathcal{O}_{\tilde{S}}) = 4, K^2_{\tilde{S}} = 2$, where $\tilde{S}$ is a smooth member of $|\tilde{L}|$.*

Proof. Consider the following short exact sequence

$$0 \longrightarrow T_{\tilde{X}} \longrightarrow T_{\tilde{P}|\tilde{X}} \longrightarrow N_{\tilde{X}/\tilde{P}} \longrightarrow 0 \tag{5}$$

where $T_{\tilde{X}}$ and $N_{\tilde{X}/\tilde{P}}$ denote, respectively, the tangent bundle of $\tilde{X}$ and the normal bundle of $\tilde{X}$ in $\tilde{P}$. Using (5) we get that

$$c_t(\tilde{X}) = \frac{c_t(\tilde{P})}{c_t(N_{\tilde{X}/\tilde{P}})} \tag{6}$$

Note that the terms of degree $1, 2$ and 3 in (6) are respectively

$$c_1(\tilde{X}) = c_1(\tilde{P})_{|_{\tilde{X}}} - c_1(N_{\tilde{X}/\tilde{P}}) \tag{7}$$

$$c_2(\tilde{X}) = c_2(\tilde{P})_{|_{\tilde{X}}} - c_1(\tilde{P})_{|_{\tilde{X}}} c_1(\tilde{X}) + c_1(\tilde{X})^2 - c_2(N_{\tilde{X}/\tilde{P}}) \tag{8}$$

$$\begin{aligned} c_3(\tilde{X}) \quad = \quad & c_3(\tilde{P})_{|_{\tilde{X}}} - c_2(\tilde{P})_{|_{\tilde{X}}} c_1(\tilde{X}) + c_1(\tilde{P})_{|_{\tilde{X}}} c_1(\tilde{X})^2 \\ & -c_1(\tilde{P})_{|_{\tilde{X}}} c_2(\tilde{X}) - c_1(\tilde{X})^3 + 2c_1(\tilde{X})c_2(\tilde{X}) - c_3(N_{\tilde{X}/\tilde{P}}) \end{aligned} \tag{9}$$

Moreover if in (1) we let $P = \mathbb{P}^6, X = \mathbb{P}^3$, that is $r = 3$, and $N = N_{P^3/P^6}$, we see that (2), (3) and (4) become, respectively,

$$c_1(\tilde{P}) = 7f^*H - 2E \tag{10}$$

$$\begin{aligned} c_2(\tilde{P}) &= 21f^*H^2 - j_*(2g^*c_1(\mathbb{P}^3) + g^*c_1(N)) \\ &= 21f^*H^2 - j_*(11g^*i^*H) = 21f^*H^2 - 11(f^*H)\cdot E \end{aligned} \tag{11}$$

$$\begin{aligned} c_3(\tilde{P}) &= 35f^*H^3 - j_*(2g^*c_2(\mathbb{P}^3) + g^*c_1(N)g^*c_1(\mathbb{P}^3) - \xi g^*c_1(N) \\ &\quad -2\xi^2) = f^*(35H^3) - j_*(g^*i^*(24H) + g^*i^*(3H)j^*E - 2j^*(E^2)) \\ &= 35f^*H^3 - 24(f^*H^2)\cdot E - 3(f^*H)\cdot E^2 + 2E^3 \end{aligned} \tag{12}$$

In order to write down explicitly $c_1(\tilde{X})$, $c_2(\tilde{X})$ and $c_3(\tilde{X})$ we need to know $c_1(N_{\tilde{X}/\tilde{P}})$, $c_2(N_{\tilde{X}/\tilde{P}}), c_3(N_{\tilde{X}/\tilde{P}})$. Since $\tilde{X} = Z(\underline{\tilde{s}})$, with $\tilde{s}_i \in H^0(\tilde{P}, \mathcal{O}_{\tilde{P}}(-E) \otimes f^*L_i)$, where $L_1, L_2 = 2H, L_3 = 3H$ we see that

$$N_{\tilde{X}/\tilde{P}} = ((2f^*H \otimes [-E]) \oplus (2f^*H \otimes [-E]) \oplus (3f^*H \otimes [-E]))_{|_{\tilde{X}}} \tag{13}$$

Hence
$c_1(N_{\tilde{X}/\tilde{P}}) = (7f^*H - 3E)_{|_{\tilde{X}}}$,
$c_2(N_{\tilde{X}/\tilde{P}}) = (16f^*H^2 - 14f^*H\cdot E + 3E^2)_{|_{\tilde{X}}}$,
$c_3(N_{\tilde{X}/\tilde{P}}) = (12f^*H^3 - 16f^*H^2\cdot E + 7f^*H\cdot E^2 - E^3)_{|_{\tilde{X}}}$.

Plugging the above relations in (7), (8), (9) we get that

$$c_1(\tilde{X}) = E_{|_{\tilde{X}}} \tag{14}$$

$$c_2(\tilde{X}) = (5f^*H^2 - 4f^*H\cdot E)_{|_{\tilde{X}}} \tag{15}$$

$$c_3(\tilde{X}) = (-12f^*H^3 + 19f^*H^2\cdot E - 8f^*H\cdot E^2)_{|_{\tilde{X}}} \tag{16}$$

By the Riemann Roch theorem we have

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}}) &= \frac{1}{24}c_1(\tilde{X})c_2(\tilde{X}) = \frac{5}{2}f^*H^5\cdot E - \frac{16}{3}f^*H^4\cdot E^2 \\ &\quad + \frac{33}{8}f^*H^3\cdot E^3 - \frac{11}{8}f^*H^2\cdot E^4 + \frac{1}{6}f^*H\cdot E^5 \end{aligned} \tag{17}$$

To get the explicit value of $\chi(\mathcal{O}_{\tilde{X}})$ we write down the multiplication table in $E \cong \mathbb{P}(N^*)$.

In the cohomology ring of $E \cong \mathbb{P}(N^*)$ the following Wu-Chern equation holds

$$\xi^3 + 3\xi^2 \cdot g^*i^*H + 3\xi \cdot g^*i^*H^2 + g^*i^*H^3 = 0 \tag{18}$$

Intersecting such equation with $\xi \cdot g^*i^*H$, $g^*i^*H^2$, ξ^2, respectively and noting that $g^*i^*H^5 = 0, \xi \cdot g^*i^*H^4 = 0, \xi^2 \cdot g^*i^*H^3 = 1$ we see that the multiplication table is:

$$\begin{gathered} g^*i^*H^5 = 0, \quad \xi \cdot g^*i^*H^4 = 0, \quad \xi^2 \cdot g^*i^*H^3 = 1 \\ \xi^3 \cdot g^*i^*H^2 = -3, \quad \xi^4 \cdot g^*i^*H = 6, \quad \xi^5 = -10 \end{gathered} \tag{19}$$

Since $\xi = -E_{|E}$, the relations in (19) are equivalent to

$$\begin{gathered} g^*i^*H^5 = 0, \quad g^*i^*H^4 \cdot E_{|E} = 0, \quad g^*i^*H^3 \cdot E^2_{|E} = 1 \\ g^*i^*H^2 \cdot E^3_{|E} = 3, \quad g^*i^*H \cdot E^4_{|E} = 6, \quad E^5_{|E} = 10 \end{gathered} \tag{20}$$

Since $(g^*i^*H^k) \cdot E^{5-k}_{|E} = (f^*H^k) \cdot E^{6-k}$, $1 \leq k \leq 5$, substituting these values in (17) we get that $\chi(\mathcal{O}_{\tilde{X}}) = 1$.

Let $\tilde{L} = (f^*H)_{|_{\tilde{X}}}$ and let $\tilde{K}$ be the canonical bundle of $\tilde{X}$. Since $\tilde{X}$ is isomorphic to X' we will work over $\tilde{X}$.

Note that $deg\tilde{X} = (f^*H)^3 \cdot (2f^*H - E)^2 \cdot (3f^*H - E)$ and $e(\tilde{X}) = c_3(\tilde{X})$. Thus using (20) it follows that $deg\tilde{X} = 11$ and $e(\tilde{X}) = -48$.

Again using (20), as well as (2), we get that $g = 10, K^2_{\tilde{S}} = 2, \chi(\mathcal{O}_{\tilde{S}}) = 4$, where $\tilde{S}$ is a smooth member of $|\tilde{L}|$.

■

To determine the structure of $(\tilde{X}, \tilde{L})$ we will use adjuction theory. Some preliminary results are needed.

Lemma 3.3. *$\tilde{K} + 2\tilde{L}$ is spanned by its global sections.*

Proof. Note that $\tilde{K} + 2\tilde{L} = (2f^*H - E)_{|_{\tilde{X}}}$. Moreover, being $\mathcal{O}_{P^6}(2) \otimes I_{P^3}$ spanned by global sections it follows that $2f^*H - E$ is spanned by its global sections and thus the same is true for its restriction to $\tilde{X}$.

Let $\Phi : \tilde{X} \longrightarrow \mathbb{P}^M$ be the adjunction map given by $\Gamma(m(\tilde{K}+2\tilde{L}))$ and let $\Phi = s{\circ}r$ be the Remmert Stein factorization of Φ, where $r : \tilde{X} \longrightarrow Y$ has connected fibres and $s : Y \longrightarrow \mathbb{P}^M$ is finite to one.

In the following lemma we prove that $dim\Phi(\tilde{X}) = 3$.

Lemma 3.4. *Let $\Phi : \tilde{X} \longrightarrow \mathbb{P}^M$ be the adjunction map given by $\Gamma(\tilde{K} + 2\tilde{L})$. Then $dim\Phi(\tilde{X}) = 3$.*

Proof. Since $\tilde{K}+2\tilde{L}$ is nef it's enough to check that $(\tilde{K}+2\tilde{L})^3 > 0$. But $(\tilde{K} + 2\tilde{L})^3 = (2f^*H - E)^3_{|\tilde{X}} = (2f^*H - E)^5 \cdot (3f^*H - E) = 96f^*H^6 - 272(f^*H^5) \cdot E + 320(f^*H^4) \cdot E^2 - 200(f^*H^3) \cdot E^3 + 70(f^*H^2) \cdot E^4 - 13(f^*H) \cdot E^5 + E^6$. Now use (20) to get that $(\tilde{K} + 2\tilde{L})^3 = 38$.

∎

Since $dim\Phi(\tilde{X}) = 3$ we have that $r : \tilde{X} \longrightarrow Y$ is the blow up of a smooth 3-fold Y in a finite number of points, say γ. The pair (Y, L) is the first reduction of $(\tilde{X}, \tilde{L})$.

Our aim is to show that $(Y, L) = (\tilde{X}, \tilde{L})$ and that it is a conic bundle over P^2.

Lemma 3.5. Let (Y,L) be as above. Then K+L is nef and not big, where $K = K_Y$. Moreover $(Y, L) = (\tilde{X}, \tilde{L})$ and Y is a conic bundle over P^2.

Proof. By [14], [19], $K + L$ is nef and big unless:

i) $(Y, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$;

ii) $(Y, L) \cong (Q, \mathcal{O}_Q(2))$, where Q is a hyperquadric in $\mathbb{P}^4$;

iii) there is a surjective morphism $\phi : Y \longrightarrow Z$ onto a smooth curve Z, whose general fibre is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and $2K+3L \approx \phi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z;

iv) (Y, L) is a Fano variety of coindex 3;

v) (Y, L) is a Del Pezzo fibration $\psi : Y \longrightarrow Z$ over a smooth curve Z and $K + L \approx \psi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z;

vi) (Y, L) is a conic bundle $\psi : Y \longrightarrow Z$ over a surface Z and $K + L \approx \psi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z.

We will show that case vi) is the only possible one.

Suppose that $(Y, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. Let $S \in |L|$. Note that $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\tilde{S}}) = 4$. On the other hand since $L = \mathcal{O}_{\mathbb{P}^3}(3)$ it follows that $\chi(\mathcal{O}_S) = 1$, and this is impossible. Similarly we rule out the case $(Y, L) \cong (Q, \mathcal{O}_Q(2))$.

Suppose that (Y, L) is as in case iii). Reasoning as in ([4], Theorem 4.3, Case iii)) it follows that 4 divides $2g(L) - 2$. This is impossible being $2g(L) - 2 = 18$.

Suppose that (Y, L) is as in case iv). Then $K_S^2 = 0$, with $S \in |L|$. On the other hand $K_S^2 = K_{\tilde{S}}^2 + \gamma$, that is, $0 = K_{\tilde{S}}^2 + \gamma$, which is impossible since $K_{\tilde{S}}^2 = 2$ and $\gamma > 0$.

Suppose that (Y, L) is as in case v). Reasoning as in ([4], Theorem 4.3, Case v)) it follows that

$$2g(L) - 2 - d = f(p_g(S) + q(S) - 1) \tag{21}$$

where $f = K_F \cdot K_F$ and F is a fibre of ψ. Since $\chi(\mathcal{O}_S) = 4, g(L) = 10$ and $d = 11 + \gamma$ we get

$$7 - \gamma = 2f(q(S) + 1) \geq 6(q(S) + 1)$$

Thus $\gamma \leq 1$.

If $\gamma = 1$ then $3 = f(q(S) + 1) \geq 3(q(S) + 1)$, thus $q = 0$ and hence $f = 3$. By ([4], (4.2)) it follows that $(Y, L) = (\tilde{X}, \tilde{L})$, a contradiction since $\gamma = 1$.

If $\gamma = 0$ then $7 = 2f(q(S) + 1)$, a contradiction.

Suppose that (Y, L) is as in case vi). Note that in this case $(K+L)^3 = 0$. On the other hand, since

$$\begin{aligned} &K^3 = \tilde{K}^3 - 8\gamma, \quad K^2L = \tilde{K}^2\tilde{L} + 4\gamma, \\ &KL^2 = \tilde{K}\tilde{L}^2 - 2\gamma, L^3 = 11 + \gamma \end{aligned} \tag{22}$$

we have that $(K + L)^3 = (\tilde{K} + \tilde{L})^3 - \gamma = -\gamma$ since $(\tilde{K} + \tilde{L})^3 = (f^*H - E)^3(2f^*H - E)^2(3f^*H - E) = 0$. Thus $\gamma = 0$ and hence $(\tilde{X}, \tilde{L}) = (Y, L)$. Note that $h^0(\tilde{X}, \tilde{K} + \tilde{L}) = \chi(\mathcal{O}_{\tilde{S}}) - \chi(\mathcal{O}_{\tilde{X}}) = 3$.

Let $\psi : \tilde{X} \longrightarrow Z$ be the structural morphism and let $\tilde{K} + \tilde{L} = \psi^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Z.

Since $2 = K_{\tilde{S}}^2 = 2\mathcal{H}^2$ it follows that $\mathcal{H}^2 = 1$. Moreover, since $h^0(Z, \mathcal{H}) = h^0(\tilde{X}, \tilde{K} + \tilde{L}) = 3$ it follows that $\Delta(Z, \mathcal{H}) = 0$. Hence by [14] it follows that $(Z, \mathcal{H}) = (P^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

Note that $K+L$ is not big since $(K+L)^3 = (\tilde{K}+\tilde{L})^3 - \gamma = -\gamma \leq 0$.

■

Proof. (*of Prop. 3.1*) Combining 3.2, 3.3, 3.4, 3.5 it follows that $\tilde{X}$ and hence X' is a conic bundle over $\mathbb{P}^2$. The latter assertion being true since X' is isomorphic to $\tilde{X}$.

■

We consider the case in which Q^3 and $X' \subset \mathbb{P}^6$ are linked via a complete intersection (2,2,3). The following proposition holds.

Proposition 3.6. *Let $Q^3, X' \subset \mathbb{P}^6$ be linked via a complete intersection $(2,2,3)$. Then X' is a Del Pezzo fibration over $\mathbb{P}^1$ with $d = 10, g = 8, \chi(\mathcal{O}_{X'}) = 1, \chi(\mathcal{O}_{S'}) = 3, K^2_{S'} = 0, e(X') = -36$.*

Proof. Since the proof is similar to that of 3.1 we will only sketch it. We have:

$$c_1(\tilde{X}) = E_{|_{\tilde{X}}} \tag{23}$$

$$c_2(\tilde{X}) = (5f^*H^2 - 3(f^*H)\cdot E)_{|_{\tilde{X}}} \tag{24}$$

$$c_3(\tilde{X}) = (-12f^*H^3 + (f^*H^2)\cdot E - 6(f^*H)\cdot E^2)_{|_{\tilde{X}}} \tag{25}$$

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}}) \;=\; & \frac{1}{24}c_1(\tilde{X})c_2(\tilde{X}) = \frac{5}{2}(f^*H^5)\cdot E - \frac{29}{6}(f^*H^4)\cdot E^2 \\ & + \frac{83}{24}(f^*H^3)\cdot E^3 - \frac{13}{12}(f^*H^2)\cdot E^4 + \frac{1}{8}(f^*H)\cdot E^5 \end{aligned} \tag{26}$$

Computing the multiplication table in $E \cong \mathbb{P}(N^*)$ and using the fact that $\xi = -E_{|E}$ we get the following relations:

$$\begin{aligned} g^*i^*H^5 = 0, \quad g^*i^*H^4\cdot E_{|E} = 0, \quad g^*i^*H^3\cdot E^2_{|E} = 1 \\ g^*i^*H^2\cdot E^3_{|E} = 8, \quad g^*i^*H\cdot E^4_{|E} = 22, \quad E^5_{|E} = 52 \end{aligned} \tag{27}$$

Substituting these values in (26) we get that $\chi(\mathcal{O}_{\tilde{X}}) = 1$. Note that $deg\tilde{X} = (f^*H)^3\cdot(2f^*H - E)^2\cdot(3f^*H - E)$ and $e(\tilde{X}) = c_3(\tilde{X})$. Thus using (27) it follows that $deg\tilde{X} = 10$ and $e(\tilde{X}) = -36$.

Using again (27), as well as 2.2, we get that $g = 8, K^2_{\tilde{S}} = 0, \chi(\mathcal{O}_{\tilde{S}}) = 3$, where $\tilde{S}$ is a smooth member of $|\tilde{L}|$.

We will now determine the structure of $(\tilde{X}, \tilde{L})$. Our aim is to show that $(Y, L) = (\tilde{X}, \tilde{L})$ and that it is a Del Pezzo fibration over $\mathbb{P}^1$ of fibre degree 4.

As in 3.3, 3.4 we see that $\tilde{K} + 2\tilde{L}$ is spanned and big. Let (Y, L) be the first reduction of $(\tilde{X}, \tilde{L})$. We will show that the adjoint bundle $K + L$ is nef and not big and that case vi) is the only possible one in 2.1.

The cases i), ii), iii) in 2.1 are ruled out as the corresponding ones in 3.5.

Let (Y, L) be a Fano variety of coindex 3. Then $K^2_S = 0$, with $S \in |L|$. On the other hand $K^2_S = K^2_{\tilde{S}} + \gamma$. Thus $\gamma = 0$ being $K^2_{\tilde{S}} = 0$ and therefore $(Y, L) = (\tilde{X}, \tilde{L})$. By the adjunction formula it follows that $2g - 2 = (K + 2L)L^2 = L^3 = 11$, which is clearly impossible.

Let (Y, L) be a conic bundle over a surface Z. Then $(K + L)^3 = 0$. On the other hand, by direct computation we see that $(K + L)^3 = -\gamma$. Hence $\gamma = 0$. Recall that, (see [2], (0.7.2)), $K^2_S \geq 2$. We also know that $K^2_S = K^2_{\tilde{S}} + \gamma$ and this is impossible since $K^2_{\tilde{S}} = \gamma = 0$.

Let (Y, L) be a Del Pezzo fibration over a smooth curve Z and $K + L \approx \psi^* \mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z. Reasoning as in ([4], Theorem 4.3, Case v)) it follows that

$$2g(L) - 2 - d = f(p_g(S) + q(S) - 1) \tag{28}$$

where $f = K_F \cdot K_F$ and F is a fibre of ψ. Since $\chi(\mathcal{O}_S) = 3, g(L) = 8$ and $d = 10 + \gamma$ we get

$$4 - \gamma = f(2q(S) + 1) \geq 3(2q(S) + 1)$$

Thus $\gamma \leq 1$.

If $\gamma = 1$ then $3 = f(2q(S) + 1) \geq 3(2q(S) + 1)$, thus $q = 0$ and hence $f = 3$. By ([4], (4.2)) it follows that $(Y, L) = (\tilde{X}, \tilde{L})$, a contradiction since $\gamma = 1$.

If $\gamma = 0$ then $4 = f(2q(S) + 1)$, and thus $f = 4, q = 0$.

In order to complete our proof we need to rule out the case in which $K+L$ is big. A direct computation shows that $(K+L)^3 = -\gamma \leq 0$. Hence $K+L$ is not big.

■

We now consider the case in which $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^6$ and $X' \subset \mathbb{P}^6$ are linked via a complete intersection $(2,2,3)$.

Proposition 3.7. *Let $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^6$ and $X' \subset \mathbb{P}^6$ be linked via a complete intersection (2,2,3). Then X' is gotten by blowing up a simple point on a Fano manifold $X \subset \mathbb{P}^7$ which is the section of the Grassmannian $Gr(1,4) \subset \mathbb{P}^9$ by a linear subspace of a codimension two and a quadric, $d = 9, g = 6, \chi(\mathcal{O}_{X'}) = 1, \chi(\mathcal{O}_{S'}) = 2, K^2_{S'} = -1, e(X') = -14$.*

Proof. The proof is similar to that of 3.1. We will sketch it emphasizing the parts in which they differ.

Let $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^6$ and let i be the inclusion map. Let $i^*H = h_1 + h_2 = p_1^*\mathcal{O}_{\mathbb{P}^1}(1) + p_2^*\mathcal{O}_{\mathbb{P}^2}(1)$, where p_1 and p_2 are the projections on the first and second factor, respectively. We have:

$$c_1(\tilde{X}) = E_{|\tilde{X}} \tag{29}$$

$$c_2(\tilde{X}) = (5f^*H^2 - 2f^*H \cdot E - b_2)_{|\tilde{X}} \tag{30}$$

$$\begin{aligned} c_3(\tilde{X}) \;=\; & \big[-12f^*H^3 + 11(f^*H^2) \cdot E - 3(f^*H) \cdot E^2 \\ & +b_1 \cdot (f^*H - E) + b_2 \cdot (7f^*H - 3E)\big]_{|\tilde{X}} \end{aligned} \tag{31}$$

where $b_2 = j_*(g^*h_2)$ and $b_1 = j_*(g^*h_1)$

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}}) \;=\; & \frac{1}{24}c_1(\tilde{X})c_2(\tilde{X}) = \frac{5}{2}(f^*H^5) \cdot E - \frac{13}{3}(f^*H^4) \cdot E^2 \\ & +\frac{67}{24}(f^*H^3) \cdot E^3 - \frac{1}{2}b_2(f^*H^3) \cdot E + \frac{2}{3}b_2(f^*H^2) \cdot E^2 \\ & -\frac{19}{24}(f^*H^2) \cdot E^4 - \frac{7}{24}b_2(f^*H) \cdot E^3 + \frac{1}{12}(f^*H) \cdot E^5 \\ & -\frac{1}{24}b_2E^4 \end{aligned} \tag{32}$$

The multiplication table in $E \cong \mathbb{P}(N^*)$ is computed as in the previous cases. Moreover using the fact that $\xi = -E_{|E}$ we get the following relations:

$$\begin{array}{rrl} g^*i^*H^5 = 0, & g^*i^*H^4 \cdot E_{|E} = 0, & g^*i^*H^3 \cdot E^2_{|E} = 3 \\ g^*i^*H^2 \cdot E^3_{|E} = 13, & g^*i^*H \cdot E^4_{|E} = 37, & E^5_{|E} = 85 \\ b_2 \cdot g^*i^*H^4 = 0, & b_2 \cdot E_{|E} \cdot g^*i^*H^3 = 0, & b_2 \cdot E^2_{|E} \cdot g^*i^*H^2 = 2 \\ b_2 \cdot E^3_{|E} \cdot g^*i^*H = 9, & b_2 \cdot E^4_{|E} = 27, & b_1 \cdot E_{|E} \cdot g^*i^*H^3 = 0 \\ b_1 \cdot E^2_{|E} \cdot g^*i^*H^2 = 1, & b_1 \cdot E^3_{|E} \cdot g^*i^*H = 4, & b_1 \cdot E^4_{|E} = 10 \end{array} \tag{33}$$

Substituting these values in (32) we get that $\chi(\mathcal{O}_{\tilde{X}}) = 1$. We also have that $deg\tilde{X} = (f^*H)^3 \cdot (2f^*H - E)^2 \cdot (3f^*H - E) = 9$ and $e(\tilde{X}) = -14$.

Using (33) and (2.1) we get that $g = 6, K^2_{\tilde{S}} = -1, \chi(\mathcal{O}_{\tilde{S}}) = 2$, where $\tilde{S}$ is a smooth member of $|\tilde{L}|$.

We will now determine the structure of $(\tilde{X}, \tilde{L})$.

As in 3.3, 3.4 we see that $\tilde{K} + 2\tilde{L}$ is spanned and big. Let (Y, L) be the first reduction of $(\tilde{X}, \tilde{L})$.

Reasoning as in 3.5 we will show that the adjoint bundle $K + L$ is nef and not big. The cases i), ii), iii) in 2.1 are ruled out as the corresponding ones in 3.5.

Let (Y, L) be a Del Pezzo fibration over a smooth curve Z and $K + L \approx \psi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Z. By ([20], (0.5.1)) it follows that $deg\mathcal{L} = \chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_Y) = 0$, which is impossible since $\mathcal{L}$ is ample.

Let (Y, L) be a conic bundle over a surface Z. Then $(K + L)^3 = 0$. On the other hand, by direct computation we see that $(K + L)^3 = 1 - \gamma$. Hence $\gamma = 1$. Recall that, (see [2], (0.7.2)), $K^2_S \geq 2$. We also know that $K^2_S = K^2_{\tilde{S}} + \gamma$ and this is impossible since $K^2_{\tilde{S}} = -1, \gamma = 1$.

Note that $K + L$ cannot be big. If it was then, since $(K + L)^3 = 1 - \gamma$, it would follow that $\gamma = 0$ and hence $(Y, L) = (\tilde{X}, \tilde{L})$. By ([3], (1.2)) we have $d_2 \geq 3$. On the other hand $d_2 = K^2_{\tilde{S}} = -1$, whence a contradiction.

Hence (Y, L) is a Fano variety of coindex 3 and thus $K^2_S = 0$. Moreover since $K^2_S = K^2_{\tilde{S}} + \gamma = -1 + \gamma$ it follows that $\gamma = 1$. Hence our claim.

∎

We consider next the case in which $X = \mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6$ and $X, X' \subset \mathbb{P}^6$ are linked via a complete intersection $(2, 2, 4)$.

Proposition 3.8. *Let $\mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6$ and $X' \subset \mathbb{P}^6$ be linked via a complete intersection (2,2,4). Then X' is a conic bundle over $Q^2, d = 12, g = 12, \chi(\mathcal{O}_{X'}) = 1, \chi(\mathcal{O}_{S'}) = 5, K_{S'}^2 = 4, e(X') = -62$.*

Proof. We will sketch the proof emphasizing the parts in which it differs from that of 3.1.

Let $\mathbb{P}^1 \times \mathbb{P}^3 \cap \mathbb{P}^6 \subset \mathbb{P}^6$ and let i be the inclusion map. Let $i^*H = h_1 + h_2 = p_1^*\mathcal{O}_{\mathbb{P}^1}(1) + p_2^*\mathcal{O}_{\mathbb{P}^3}(1)$, where p_1 and p_2 are the projections on the first and second factor, respectively. We have:

$$c_1(\tilde{X}) = (-f^*H + E)_{|_{\tilde{X}}} \tag{34}$$

$$c_2(\tilde{X}) = (9f^*H^2 - 3(f^*H) \cdot E - b_2)_{|_{\tilde{X}}} \tag{35}$$

$$\begin{aligned} c_3(\tilde{X}) \quad = \quad & \big[-33f^*H^3 + 29(f^*H^2) \cdot E - 2(f^*H) \cdot E^2 \\ & +b_1 \cdot (8f^*H - 3E) - 2b_2 \cdot E - 18a_1 - 12a_2 \big]_{|\tilde{X}} \end{aligned} \tag{36}$$

where $a_1 = j_*((g^*h_1) \cdot h_2), a_2 = j_*(g^*h_2^2), b_1 = j_*(g^*h_1)$ and $b_2 = j_*(g^*h_2)$

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}}) \quad = \quad & \frac{1}{24} c_1(\tilde{X}) c_2(\tilde{X}) = -6f^*H^6 + \frac{31}{2}(f^*H^5) \cdot E \\ & + \frac{2}{3} b_2 \cdot (f^*H^4) - \frac{3}{2} b_2 \cdot (f^*H^3) \cdot E + \frac{55}{8}(f^*H^3) \cdot E^3 \\ & + \frac{7}{6} b_2 (f^*H^2) \cdot E^2 - \frac{3}{2}(f^*H^2) \cdot E^4 - \frac{3}{8} b_2 \cdot (f^*H) \cdot E^3 \\ & -15(f^*H^4) \cdot E^2 + \frac{1}{8}(f^*H) \cdot E^5 + \frac{1}{24} b_2 \cdot E^4 \end{aligned} \tag{37}$$

Computing the multiplication table in $E \cong \mathbb{P}(N^*)$ and using the fact that $\xi = -E_{|E}$ we get the following relations :

$$\begin{aligned}
&g^*i^*H^5 = 0, \quad g^*i^*H^4 \cdot E_{|E} = 0, \quad g^*i^*H^3 \cdot E^2_{|E} = 4 \\
&g^*i^*H^2 \cdot E^3_{|E} = 18, \quad g^*i^*H \cdot E^4_{|E} = 52, \quad E^5_{|E} = 120 \\
&b_2 \cdot g^*i^*H^4 = 0, \quad b_2 \cdot E_{|E} \cdot g^*i^*H^3 = 0, \quad b_2 \cdot E^2_{|E} \cdot g^*i^*H^2 = 6 \\
&b_2 \cdot E^3_{|E} \cdot g^*i^*H = 28, \quad b_2 \cdot E^4_{|E} = 84, \quad b_1 \cdot g^*i^*H^4 = 0 \\
&b_1 \cdot E_{|E} \cdot g^*i^*H^3 = 0, \quad b_1 \cdot E^2_{|E} \cdot g^*i^*H^2 = 1, \quad b_1 \cdot E^3_{|E} \cdot g^*i^*H = 4 \\
&b_1 \cdot E^4_{|E} = 10, \quad a_1 \cdot g^*i^*H^3 = 0, \quad a_1 \cdot E^3_{|E} = 4 \\
&a_1 \cdot E^2_{|E} \cdot g^*i^*H = 1, \quad a_1 \cdot E_{|E} \cdot g^*i^*H^2 = 0, \quad a_2 \cdot g^*i^*H^3 = 0 \\
&a_2 \cdot E^3_{|E} = 10, \quad a_2 \cdot E^2_{|E} \cdot g^*i^*H = 2, \quad a_2 \cdot E_{|E} \cdot g^*i^*H^2 = 0,
\end{aligned} \tag{38}$$

Substituting these values in (37) we see that $\chi(\mathcal{O}_{\tilde{X}}) = 1$. We also have that $deg\tilde{X} = (f^*H)^3 \cdot (2f^*H - E)^2 \cdot (4f^*H - E) = 12$ and $e(\tilde{X}) = -62$.

Using (38) and (2.2) we get that $g = 12, K^2_{\tilde{S}} = 4, \chi(\mathcal{O}_{\tilde{S}}) = 5$, where $\tilde{S}$ is a smooth member of $|\tilde{L}|$.

We now determine the structure of $(\tilde{X}, \tilde{L})$.

As in 3.3, 3.4 we see that $\tilde{K} + 2\tilde{L}$ is spanned and big. Let (Y, L) be the first reduction of $(\tilde{X}, \tilde{L})$. We will show that $K + L$ is nef and not big and that in 2.1 the only possible case is vi), that is (Y, L) is a conic bundle over a smooth surface. If the pair (Y, L) is either one of the cases i), ii), iii), iv) in 2.1 then the same reasoning as the corresponding ones in 3.5 rules it out.

The case in which (Y, L) is a Del Pezzo fibration over a smooth curve Z is ruled out as the corresponding one in 3.6.

Let (Y, L) be a conic fibration over a surface Z. By [7], Z is smooth. Note that in this case $(K + L)^3 = 0$. On the other hand, since

$$\begin{aligned}
&K^3 = \tilde{K}^3 - 8\gamma, \quad K^2L = \tilde{K}^2\tilde{L} + 4\gamma, \\
&KL^2 = \tilde{K}\tilde{L}^2 - 2\gamma, L^3 = 12 + \gamma
\end{aligned} \tag{39}$$

we have that $(K + L)^3 = (\tilde{K} + \tilde{L})^3 - \gamma = -\gamma$ since $(\tilde{K} + \tilde{L})^3 = (f^*H - E)^3(2f^*H - E)^2(4f^*H - E) = 0$. Thus γ=0 and hence $(\tilde{X}, \tilde{L}) = (Y, L)$. Note that $h^0(\tilde{X}, \tilde{K} + \tilde{L}) = \chi(\mathcal{O}_{\tilde{S}}) - \chi(\mathcal{O}_{\tilde{X}}) = 4$, see ([2], (1.2)).

Let $\psi : \tilde{X} \longrightarrow Z$ the structural morphism and let $\tilde{K} + \tilde{L} = \psi^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Z. Since $4 = K_{\tilde{S}}^2 = 2\mathcal{H}^2$ it follows that $\mathcal{H}^2 = 2$. Moreover, since $h^0(Z, \mathcal{H}) = h^0(\tilde{X}, \tilde{K} + \tilde{L}) = 4$ it follows that $\Delta(Z, \mathcal{H}) = 0$. Hence by [14] it follows that $(Z, \mathcal{H}) = (Q^2, \mathcal{O}_{Q^2}(1))$.

In order to complete our proof we need to rule out the case in which $K + L$ is big. A direct computation shows that $(K + L)^3 = -\gamma \leq 0$. Hence our claim follows.

■

Before considering the next case we recall that a polarized pair $(\tilde{X}, \tilde{L})$, where $\tilde{X}$ is a 3-dimensional manifold, is said to be of *log-general type* if $K_{\tilde{X}} + \tilde{L}$ is nef and big.

We consider now the case in which $X = G(1,4) \cap \mathbb{P}^6$ and $X, X' \subset \mathbb{P}^6$ are linked via a complete intersection $(2, 3, 3)$. The following proposition holds.

Proposition 3.9. *Let $G(1,4) \cap P^6$ and $X' \subset P^6$ be linked via a complete intersection (2,3,3). Then X' is of log-general type, $d = g = 13, \chi(\mathcal{O}_{X'}) = 1, \chi(\mathcal{O}_{S'}) = 5, K_{S'}^2 = 5, e(X') = -38$.*

Proof. Recall that on $G = G(1,4)$ we have the following canonical sequence

$$0 \longrightarrow U \longrightarrow \mathcal{O}_{|G}^{\oplus 5} \longrightarrow Q \longrightarrow 0 \tag{40}$$

where U is the universal fibre bundle of rank 2 and Q is the quotient bundle of rank 3. Hence $T_G \cong U^* \otimes Q$.

Let $G \cap \mathbb{P}^6 \subset \mathbb{P}^6$ and let i be the inclusion map. We have:

$$c_1(\tilde{X}) = (-f^*H + E)_{|_{\tilde{X}}} \tag{41}$$

$$c_2(\tilde{X}) = (8f^*H^2 - 4(f^*H) \cdot E)_{|_{\tilde{X}}} \tag{42}$$

$$c_3(\tilde{X}) = \big[- 26f^*H^3 + 26(f^*H^2) \cdot E - 6(f^*H) \cdot E^2 - 2a_1\big]_{|\tilde{X}} \tag{43}$$

where $a_1 = j_*(g^* c_2(U^*))$

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}}) &= \frac{1}{24} c_1(\tilde{X}) c_2(\tilde{X}) = -6 f^* H^6 + 16 (f^* H^5) \cdot E \\ &-\frac{97}{6}(f^* H^4) \cdot E^2 + \frac{47}{6}(f^* H^3) E^3 - \frac{11}{6}(f^* H^2) \cdot E^4 \\ &+\frac{1}{6}(f^* H) \cdot E^5 \end{aligned} \tag{44}$$

In this case the multiplication table in $E \cong \mathbb{P}(N^*)$ is:

$$\begin{aligned} g^*i^*H^5 = 0, &\quad g^*i^*H^4 \cdot E_{|E} = 0, \quad g^*i^*H^3 \cdot E^2_{|E} = 5 \\ g^*i^*H^2 \cdot E^3_{|E} = 25, &\quad g^*i^*H \cdot E^4_{|E} = 82, \quad E^5_{|E} = 220 \\ a_1 \cdot g^*i^*H^3 = 0, &\quad a_1 \cdot E_{|E} \cdot g^*i^*H^2 = 0, \quad a_1 \cdot E^2_{|E} \cdot g^*i^*H = 2 \\ a_1 \cdot E^3_{|E} = 10 & \end{aligned} \tag{45}$$

Thus we get that $\chi(\mathcal{O}_{\tilde{X}}) = 1, deg\tilde{X} = (f^*H)^3 \cdot (2f^*H - E) \cdot (3f^*H - E)^2 = 13$ and $e(\tilde{X}) = -38$.

Again (45) and (2.2) give $g = 13, K^2_{\tilde{S}} = 5, \chi(\mathcal{O}_{\tilde{S}}) = 5$, where $\tilde{S}$ is a smooth member of $|\tilde{L}|$.

We will now determine the structure of $(\tilde{X}, \tilde{L})$. Our aim is to show that $(Y, L) = (\tilde{X}, \tilde{L})$ and that $(\tilde{X}, \tilde{L})$ is of log-general type.

As in 3.3, 3.4 we see that $\tilde{K} + 2\tilde{L}$ is spanned and big. Let (Y, L) be the first reduction of $(\tilde{X}, \tilde{L})$.

We will show that $K + L$ is nef and big. The cases i), ii), iv), v) in 2.1 are ruled out as the corresponding ones in 3.5.

The case iii) in 2.1 is ruled out as follows. From ([4], (4.3.1) we see that

$$4K^2_{\tilde{S}} + 8(g-1) - 3d + \gamma = 0 \tag{46}$$

Plugging the values of $K^2_{\tilde{S}}, g, d$ we get that $\gamma = -77$, a contradiction.

Suppose that (Y, L) is as in case vi), that is (Y, L) is a conic bundle over a smooth surface Z. Note that in this case $(K + L)^3 = 0$. On the other hand $(K + L)^3 = (\tilde{K} + \tilde{L})^3 - \gamma = 1 - \gamma$, the latter equality follows from the fact that $(\tilde{K}+\tilde{L})^3 = (f^*H-E)^3(2f^*H-E)(3f^*H-E)^2 = 1$. Thus γ=1. Note that $h^0(Y, K+L) = \chi(\mathcal{O}_{\tilde{S}}) - \chi(\mathcal{O}_{\tilde{X}}) = 4$.

Let $\psi : Y \longrightarrow Z$ the structural morphism and let $K + L = \psi^*\mathcal{H}$ for an ample line bundle $\mathcal{H}$ on Z. Since $6 = K_{\tilde{S}}^2 = 2\mathcal{H}^2$ it follows that $\mathcal{H}^2 = 3$. Moreover, since $h^0(Z,\mathcal{H}) = h^0(Y, K+L) = 4$ it follows that $\Delta(Z,\mathcal{H}) = 1$. Hence by [14] it follows that $(Z,\mathcal{H})$ is a del Pezzo surface of degree 3.

If $K+L$ is big then since $(K+L)^3 = 1-\gamma$ it follows that $\gamma = 0$. Hence $(\tilde{X},\tilde{L}) = (Y, L)$ and by ([3], (2.8), (2.3)) there exists a proper modification $r : \tilde{X} \longrightarrow P^3$ such that $\tilde{K}+\tilde{L} = r^*\mathcal{O}_{\mathbb{P}^3}(1)$, $d_1 = 11, d_2 = 5, d_3 = 1, \chi(\mathcal{O}_{\tilde{S}}) = 5$.

Thus we have the following two cases:

A) (Y, L) is a conic bundle over a del Pezzo surface of degree 3, or

B) $(Y, L) = (\tilde{X},\tilde{L})$ and $(\tilde{X},\tilde{L})$ is of log-general type.

To finish off the proof of the proposition we need to rule out case A).

The following two lemmas are needed.

Lemma 3.10 *Let (Y,L) be as in case A). Then Y has no divisorial fibres.*

Proof. Let (Y, L) be as in A). We know that $\tilde{X}$ is the blow up of Y at a point $y \in Y$. Let $r : \tilde{X} \longrightarrow Y$ be the blow up map and let D be the exceptional divisor.

We will show that $\psi : Y \longrightarrow Z$ has no divisorial fibres. Assume otherwise and let F be a divisorial fibre of ψ. By ([5], Theorem (2.3)) we have the following possibilities for F:

- $F = F_0$ with $L_{F_0} = \mathcal{O}_{F_0}(1,2)$;
- $F = F_0 \cup F_1$ with $L_{F_0} = \mathcal{O}_{F_0}(1,1)$ and $L_{F_1} = E + 2f$.

We also know that $y \notin F$, see ([5], (3.1.2)). Thus $r^{-1}(F) \cong F$. Note that $\tilde{L}_{F_0} = L_F$ since $\tilde{L} = r^*L - D$ and $D \cap F = \emptyset$. The adjunction bundle $\tilde{K}+\tilde{L}$ is trivial on $r^{-1}(F) \cong F$. In fact $\tilde{K}+\tilde{L} = r^*(K+L)+D$ and since D and F don't meet it follows that $(\tilde{K}+\tilde{L})_F = r^*(K+L)_F = \mathcal{O}_F$.

Since $\tilde{X}$ is isomorphic to X' we now work on $X' \subset \mathbb{P}^6$. From now on we will denote with F either F itself or $r^{-1}(F) \subset \tilde{X}$ or $f(r^{-1}(F)) \subset X'$ this because $r_{|r^{-1}(F)}$ and f are isomorphisms. Being $f^*(K'+L') = \tilde{K}+\tilde{L}$

it follows that $K' + L'$ is trivial on F and hence $N_{F/X'} = K_F - K_{X'|F} = K_F + L'_F$.

From the sequence

$$0 \longrightarrow T_F \longrightarrow T_{\mathbb{P}^6|F} \longrightarrow N_{F/\mathbb{P}^6} \longrightarrow 0 \qquad (47)$$

we get that

$$21{L'_F}^2 \quad = \quad c_2(F) - K_F \cdot (K_F + 7L'_F) + c_2(N_{F/\mathbb{P}^6}) \qquad (48)$$

Also from the sequence

$$0 \longrightarrow N_{F/X'} \longrightarrow N_{F/\mathbb{P}^6} \longrightarrow N_{X'/\mathbb{P}^6,F} \longrightarrow 0 \qquad (49)$$

we get

$$c_1(N_{X'/\mathbb{P}^6,F}) = 6L'_F$$

and

$$c_2(N_{F/\mathbb{P}^6}) \quad = \quad (K_F + L'_F) \cdot 6L'_F + c_2(N_{X'/\mathbb{P}^6,F}) \qquad (50)$$

Claim 3.11. $c_2(N_{X'/\mathbb{P}^6,F}) = c_2(N_{\tilde{X}/\tilde{P},F})$

Proof. (of Claim) Since $f_{|\tilde{X}} : \tilde{X} \longrightarrow X'$ is an isomorphism we have that $c_i(\tilde{X}) = f^* c_i(X')$ and thus

$$f^* c_1(P)_{|X'} = f^* c_1(X') + f^* c_1(N_{X'/\mathbb{P}^6}) = c_1(\tilde{X}) + f^* c_1(N_{X'/\mathbb{P}^6})$$

and similarly,

$$f^* c_2(P)_{|X'} = c_2(\tilde{X}) - c_1(\tilde{X})^2 + c_1(\tilde{X}) f^* c_1(P)_{|X'} + f^* c_2(N_{X'/\mathbb{P}^6})$$

i.e

$$f^* c_2 N_{X'/\mathbb{P}^6} = f^* c_2(P)_{|X'} - c_2(\tilde{X}) + c_1(\tilde{X})^2 - c_1(\tilde{X}) f^* c_1(P)_{|X'} \qquad (51)$$

Since $N_{\tilde{X}/\tilde{P}} = (2(f^*H) \otimes [-E]) \oplus (3(f^*H) \otimes [-E]) \oplus (3(f^*H) \otimes [-E])$ it follows that

$$c_2(N_{\tilde{X}/\tilde{P},F}) = (21 f^* H^2 - 16(f^*H) \cdot E + 3E^2)_F \qquad (52)$$

From (8) we know that

$$c_2(N_{\tilde{X}/\tilde{P}}) = c_2(\tilde{P}) - c_2(\tilde{X}) - c_1(\tilde{P})c_1(\tilde{X}) + c_1(\tilde{X})^2. \tag{53}$$

Subtracting (53) from (51) we get

$$\begin{aligned} c_2(N_{\tilde{X}/\tilde{P}}) - f^*c_2(N_{X'/\mathbb{P}^6}) &= c_2(\tilde{P}) - f^*(c_2(P)_{|X'}) \\ &\quad -c_1(\tilde{P})c_1(\tilde{X}) + c_1(\tilde{X})f^*c_1(P)_{|X'} \\ &= -2((f^*H)\cdot E)_{|\tilde{X}} - c_1(\tilde{X})(-2E)_{|\tilde{X}} \\ &= -2((f^*H)\cdot E)_{|\tilde{X}} - ((f^*H - E)\cdot E)_{|\tilde{X}} \\ &= -2(\tilde{L} + \tilde{K})\cdot E \end{aligned} \tag{54}$$

Since $\tilde{K} + \tilde{L}$ is trivial on F, (54) gives

$$c_2(N_{\tilde{X}/\tilde{P},F}) = f^*c_2(N_{X'/\mathbb{P}^6,F}) = c_2(N_{X'/\mathbb{P}^6,F})$$

hence our claim is proven.

■

On the other hand, since $\tilde{K} + \tilde{L} = 2f^*H - E$ it follows that $(2f^*H - E)_F = \mathcal{O}_F$ or equivalently, $2\tilde{L}_F = E_F$. Hence (52) becomes

$$\begin{aligned} c_2(N_{\tilde{X}/\tilde{P},F}) &= (21f^*H^2 - 16(f^*H)\cdot E + 3E^2)_F \\ &= 21\tilde{L}_F^2 - 32\tilde{L}_F^2 + 12\tilde{L}_F^2 = \tilde{L}_F^2. \end{aligned} \tag{55}$$

Assume now that there is a divisorial fibre F_0 with $\tilde{L}_{F_0} = \mathcal{O}_{F_0}(1,2)$. We have:

$$\tilde{L}_{F_0}^2 = 4, {K_{F_0}}^2 = 8, K_{F_0}\cdot\tilde{L}_{F_0} = -6.$$

Then from (48) $c_2(N_{F_0/\mathbb{P}^6}) = 46$ while from (50) $c_2(N_{F_0/\mathbb{P}^6}) = -8$, a contradiction. In the same way we exclude F_0 with $\tilde{L}_{F_0} = \mathcal{O}_{F_0}(1,1)$. In fact in this case $\tilde{L}_{F_0}^2 = 2, K_{F_0}\cdot\tilde{L}_{F_0} = -4$. Thus (48) gives $c_2(N_{F_0/\mathbb{P}^6}) = 18$ while (50) gives $c_2(N_{F_0/\mathbb{P}^6}) = -10$, which is again impossible.

■

The above lemma implies that (Y, L) is a geometric conic bundle over a smooth surface.

Let us fix some notation and recall few facts about 3-dimensional geometric conic bundles over smooth surfaces, see [9] for details.

Notation. Let $p : Y \longrightarrow B$ be a 3-dimensional geometric conic bundle over a smooth surface B. Let S be a generic surface section of Y. Then $p : S \longrightarrow B$ is finite $2 : 1$. Let $2R \subset B$ be the ramification divisor of $p : S \longrightarrow B$.

We set $p_*\mathcal{O}_X(1) =: E$, a rank 3 vector bundle over B. Let $W := \mathbb{P}(E)$ be a $\mathbb{P}^2$-bundle over B.

We denote the natural projection of W onto B also by p and by H the divisor on W corresponding to $\mathcal{O}_W(1)$. Hence $Y = 2H - p^*\mathcal{M}$ for some divisor $\mathcal{M}$ on B.

The divisor $D \subset B$ corresponding to points whose fibres are singular conics, is called the discriminant divisor. Moreover it can be easily seen that $D = 2c_1(E) - 3\mathcal{M}$. Furthermore we have $c_1(E) = 3R - D, \mathcal{M} = 2R - D$.

As for the Chern classes of Y we have the following:

$$\begin{aligned}
c_1(Y) &= H - p^*K_B - p^*R \\
c_2(Y) &= H^2 + H \cdot [-p^*K_B - 2p^*R + p^*D] \\
&\quad +(-2p^*R^2 + p^*K_B \cdot p^*R + p^*D \cdot p^*R + p^*c_2(B) + p^*c_2(E)) \\
c_3(Y) &= 2c_2(B) - D^2 - D \cdot K_B
\end{aligned}$$

see ([9], Prop. 4.12). Indeed such relations hold true for any 3-dimensional geometric conic bundle over a smooth surface.

Lemma 3.12. *There is no conic bundle (Y,L) as in case A).*

Proof. Recall that the base surface B of our conic bundle is a del Pezzo surface of degree 3. Moreover by ([9], Prop. 4.10) for any two divisors G and G' on B we have

i) $H \cdot p^*G \cdot p^*G' = 2G \cdot G'$;

ii) $H^2 \cdot p^*G = (4R - D) \cdot G$.

We also know that $K^3 = -2, c_1(Y) \cdot H^2 = 4, c_2(Y) \cdot H = 44, c_1(Y)^2 \cdot H = 0, c_2(B) = 9$. Using these and the relations i) and ii) we get a system of 5 equations on the unknown $R^2, K_B \cdot R, D \cdot R, K_B \cdot D, D^2$ which does not have integer solutions. Hence there is no conic bundle (Y, L) as in case A).

■

The proof of Proposition 3.9 is now complete.

■

4 New examples of smooth 3-folds in P^6 of degree 11

The aim of this section is to give three new examples of smooth 3-folds in $\mathbb{P}^6$ of degree 11 and genus 9. They are obtained via liaison. The first two examples are linked to 3-folds in $\mathbb{P}^6$ of degree 7: the hyperquadric fibration over $\mathbb{P}^1$ and the scroll over $\mathbb{P}^2$.

The third one is a 3-fold which is Pfaffian linked to a 3-dimensional quadric in $\mathbb{P}^6$.

For these examples the Bertini-type criterion does not apply hence we will use the computer algebra system Macaulay to show their smoothness.

We recall the following proposition which is a test for non-singularity.

Proposition 4.1. *(Test for non singularity) [16] Let $X \subset \mathbb{P}^N$ be a projective algebraic set of dimension n, with defining ideal $I = \{f_1, ..., f_r\}$. Let $Df = \left(\partial f_i/\partial x_j\right)$ be the (N+1) by r Jacobian matrix. Let J be the ideal consisting of I together with all the N-n by N-n determinants of the matrix Df.*

i) The singular locus of X is exactly the zero locus Z(J).

ii) X is non singular if and only if codim J = N+1.

Example 4.2. Let $X, X' \subset \mathbb{P}^6$ be linked via a complete intersection $(2, 3, 3)$ where X is a hyperquadric fibration over $\mathbb{P}^1$ of degree 7. Then X' is smooth, deg $X' = 11, g = 9$, with resolution

$$
\begin{aligned}
&0 \to \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^6}(-5)^{\oplus 3} \to \\
&\mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-5)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-6) \to \\
&\mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^6}(-2) \to I_{X'} \to 0
\end{aligned}
$$

Proof. Let $X \subset \mathbb{P}^6$ be a smooth 3-fold of degree 7 which is a hyperquadric fibration over $\mathbb{P}^1$. The existence of such 3-fold has been shown by [15]. We observe that such threefold can also be seen as the 1st degeneracy locus of a general morphism $u \in Hom(F, E)$ where $F = \mathcal{O}_{\mathbb{P}^6}^{\oplus 2}$ and $E = \mathcal{O}_{\mathbb{P}^6}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^6}(2)$. Let $X = D_1(u)$ be the 1st degeneracy locus, i.e. $D_1(u) = \{x \in \mathbb{P}^6 | rku(x) \leq 1\}$. Since $F^\vee \otimes E$ is ample and spanned, $f = rkF < rkE = e$ and $N < (f-1+1)(e-1+1)$, by ([17], Prop. §2), it follows that $D_1(u)$ is non empty, smooth and of codimension $(f-1)(e-1) = 3$. Thus $X \subset \mathbb{P}^6$ is defined by the 2 by 2 minors of the following matrix

$$\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \\ x_4 & x_5 \\ q_1 & q_2 \end{pmatrix} \tag{56}$$

where $x_0, ..., x_6$ are homogenous coordinates in $\mathbb{P}^6$ and q_1, q_2 are quadrics in $\mathbb{P}^6$, $q_1 = x_3^2 + x_4^2 + x_5^2 + x_6^2, q_2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$.

The Eagon-Northcott complex associated with $u : F \longrightarrow E$ yields explicit locally free resolution of the ideal of X, see ([17], pg.6).

In the specific case we have

$$0 \to \wedge^4 E^\vee \otimes S^2 F \to \wedge^3 E^\vee \otimes F \to \wedge^2 E^\vee \to I_X \otimes detF^\vee \to 0 \tag{57}$$

i.e.,

$$\begin{aligned} 0 &\to \mathcal{O}_{\mathbb{P}^6}(-5)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 2} \\ &\to \mathcal{O}_{\mathbb{P}^6}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 3} \to I_X \to 0 \end{aligned} \tag{58}$$

We'd like to construct a smooth 3-fold $X' \subset P^6$ which is linked to the above degree 7 threefold X via a complete intersection $(2, 3, 3)$.

Note that after twisting (58) with $\mathcal{O}_{P^6}(3)$ we see that $I_X \otimes \mathcal{O}_{P^6}(3)$ is globally generated, but we can't say the same thing about $I_X \otimes \mathcal{O}_{P^6}(2)$. Thus we can't use 2.4 to get the smoothness of X'.

We'll overcome such obstacle by using the computer algebra system Macaulay. We know the generators of I_X, they are the 2 by 2 minors of the matrix (56). We will choose some ad hoc hypersurfaces containing X, precisely two cubics C_1, C_2 and a quadric Q.

Let $f_1 = x_6(x_0x_3 - x_1x_2) + x_3(x_0x_5 - x_1x_4) + x_1(x_2x_5 - x_3x_4) + x_4(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2) - x_5(x_3^2 + x_4^2 + x_5^2 + x_6^2)$ and $f_2 = x_5(x_0x_3 -$

$x_1x_2)+x_2(x_0x_5-x_1x_4)+x_0(x_0^2+x_1^2+x_2^2+x_3^2+x_4^2)-x_1(x_3^2+x_4^2+x_5^2+x_6^2)$ be the equtions of the cubics C_1, C_2 respectively and let $f_3 = x_0x_3 - x_1x_2 + x_0x_5 - x_1x_4 + x_2x_5 - x_3x_4$ be the equation of the quadric Q. Let $I_A = (f_1, f_2, f_3)$. The quotient of I_A with I_X gives the ideal of $I_{X'}$. Using Macaulay and 4.1 we see that $sing(X') = \emptyset$. Again with the help of Macaulay we get that $degX' = 11$ and $g = 9$.

■

Example 4.3. Let $X, X' \subset \mathbb{P}^6$ be linked via a complete intersection $(2,3,3)$ where X is a scroll over $\mathbb{P}^2$ of degree 7. Then X' is smooth, $degX' = 11, g = 9$, with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^6}(-5) \to \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-5)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^6}(-6)$$
$$\to \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^6}(-2) \to I_{X'} \to 0$$

Proof. Let $X \subset \mathbb{P}^6$ be a smooth 3-fold of degree 7 which is a scroll over $\mathbb{P}^2$. As Okonek observed, such 3-fold is generally linked $(2,2,2)$ to $\mathbb{P}^3$. Thus using Macaulay one can compute the generators of the ideal I_X, they are:

$f_1 = x_0^2+x_0x_1+2x_1x_3+x_2x_3-x_1x_4+x_2x_4+x_1x_5-x_0x_6$, $f_2 = x_0x_3+x_1x_4+x_2x_5$, $f_3 = x_0x_2+x_1x_2-x_1x_3-x_2x_3+x_1x_4+x_0x_6+x_2x_6$, $f_4 = x_3^2(x_2+x_3-x_4+x_5-2x_6)+x_4^2(x_1-x_2+x_3-x_6)+x_5^2x_6-x_4x_6^2+x_3(x_2x_5-x_1x_4+x_4x_6)+x_4(x_0x_6-x_1x_6-x_1x_5-x_0x_5)+x_5x_6(x_0+2x_2)$

We'd like to construct a smooth 3-fold $X' \subset \mathbb{P}^6$ by linking it to the above degree 7 threefold X via a complete intersection $(2,3,3)$.

As in the previous example, we'll choose two cubics and a quadric containing X. We use Macaulay to get the smoothness of X'.

■

The next example is obtained via Pfaffian liaison. A good reference on this subject is [1].

We recall that two codimension three subvarieties X and X' of $\mathbb{P}^6$ are *Pfaffian linked* if their union is a Pfaffian subvariety Y of $\mathbb{P}^6$.

For our purpose the Pfaffian subvariety Y we consider is the first non-trivial degeneracy locus of a skew-symmetric morphism $f : \mathcal{E} \longrightarrow \mathcal{E}^* \otimes L$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^6}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-1)$ is a rank 5 vector bundle over $\mathbb{P}^6$ and

$L = \mathcal{O}_{\mathbb{P}^6}(1)$ is a line bundle on $\mathbb{P}^6$. Following Arrondo's notation in [1] we call such Y an $(\mathcal{E}, L)$-Pfaffian subvariety.

The equations of Y are the 5 principal Pfaffians associated to the following skew-symmetric matrix

$$\begin{pmatrix} 0 & x_0 & x_1 & x_2 & q_1 \\ -x_0 & 0 & x_3 & x_4 & q_2 \\ -x_1 & -x_3 & 0 & x_5 & q_3 \\ -x_2 & -x_4 & -x_5 & 0 & q_4 \\ -q_1 & -q_2 & -q_3 & -q_4 & 0 \end{pmatrix} \tag{59}$$

where $x_0, ..., x_6$ are homogenous coordinates in $\mathbb{P}^6$ and $q_1, ..., q_4$ are quadrics in $\mathbb{P}^6$.

From general results on Pfaffian subvarieties, see [17], [1], it is easy to see that Y is a smooth 3-fold of degree 13 and genus 14.

Example 4.4. Let $Q^3 \subset \mathbb{P}^6$ be a smooth quadric, let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^6}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-1)$ be a rank 5 vector bundle over $\mathbb{P}^6$ and let $L = \mathcal{O}_{\mathbb{P}^6}(1)$ a line bundle on $\mathbb{P}^6$. Let Y be the $(\mathcal{E}, L)$-Pfaffian subvariety containing Q^3. Then Y yields a residual X' which is smooth, $degX' = 11, g = 9$ and with resolution

$$\begin{aligned} 0 &\to \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^6}(-5) \to \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-5)^{\oplus 2} \\ &\to \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^6}(-2) \to I_{X'} \to 0 \end{aligned}$$

Proof. We can assume that $Q^3 \subset \mathbb{P}^6$ is defined by

$$\{(x_0, ..., x_6) \in P^6 | x_0x_5 - x_1x_4 + x_2x_3 = x_6 = x_0 + x_1 + x_2 + x_3 + x_5 = 0\}$$

Since we want the Pfaffian Y to contain $Q^3 \subset P^6$ it is enough to choose $q_1, ..., q_4$ in (59) so that they contain the chosen Q^3.

It's easy to see that the Bertini-type criterion for Pfaffian liaison, see ([1], Prop. 1.8), does not apply in this case. Hence the smoothness of the residual X' is proved using the computer algebra system Macaulay.

Knowing the resolution of the ideal I_{Q^3} in $\mathbb{P}^6$

$$\begin{aligned} 0 &\to \mathcal{O}_{\mathbb{P}^6}(-4) \to \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^6}(-2) \to \mathcal{O}_{\mathbb{P}^6}(-2) \oplus \mathcal{O}_{\mathbb{P}^6}(-1)^{\oplus 2} \\ &\to I_{Q^3} \to 0 \end{aligned}$$

we get that of $I_{X'}$ in $\mathbb{P}^6$ by using Prop 1.6 in [1].

■

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