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GENERAL METHODS OF CONSTRUCTING EQUIVALENT MINIMAL PAIRS NOT UNIQUE UP TO TRANSLATION

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Abstract

In this paper we give general methods of construction of various equivalent minimal pairs of compact convex sets that are not translates of one another.

1 Introduction and preliminary remarks

Let X be a topological vector space; $\mathcal{K}(X)$ be the family of all non-empty compact convex subsets of X, and $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$. Let

$$A + B := \{a + b|\, a \in A, b \in B\}$$

be the *Minkowski sum* of A and B in $\mathcal{K}(X)$. The equivalence relation between pairs of compact convex sets is given by the relation $(A, B) \sim (C, D)$ if and only if $A + D = B + C$. A partial order is given by the relation: $(A, B) \leq (C, D)$ if and only if $(A, B) \sim (C, D)$ and $A \subseteq C, B \subseteq D$. By $[A, B]$, we denote the equivalence class of (A, B) in $\mathcal{K}^2(X)/_\sim$. The pair $(A, B) \in \mathcal{K}^2(X)$ is called *minimal*, if for all pairs $(A', B') \sim (A, B)$ the inclusions $A' \subset A$, and $B' \subset B$ imply $A' = A$ and $B' = B$. For every $(A, B) \in \mathcal{K}^2(X)$ there exists a minimal element (A', B') that is equivalent to (A, B) and $A' \subset A$, $B' \subset B$ [4]. If $dim X \leq 2$, *the minimal element is unique up to translation* [1],[10] i.e. if (A, B) is minimal and (A', B'), equivalent to (A, B), is also minimal, then there exists $x \in X$ such that $A = A + x$ and $B = B + x$. For $dim X \geq 3$ there was given an example of equivalent minimal pairs not unique up to translation [1]. In

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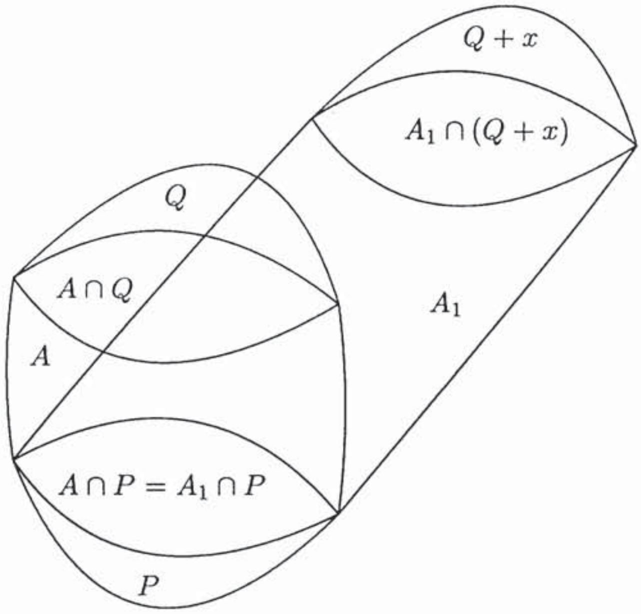
[6] for the 3-dimensional case a continuum of equivalent minimal pairs wich are not connected by translation was constructed.
Now, in this paper, we give more general methods of constructing various (not necessarily polytops) equivalent minimal pairs not unique up to translation.
Let X be a real topological vector space, $f \in X^*$ be a continuous linear functional, and $K \subseteq X$ a nonempty compact convex set. Then we denote by $H_f(K) := \{z \in K | f(z) = max_{y \in K} f(y)\}$ the face of K with respect to f. For the sum of two nonempty compact convex sets $A, B \subseteq X$ and $f \in X^*$ the following identity holds: $H_f(A+B) = H_f(A) + H_f(B)$. We will use the notation: $A \vee B := conv(A \cup B)$, $A \dot{-} B := \{x | x+B \subset A\}$. In [7], by Pinsker, it was shown that for a locally convex topological vector space X if $A, B, C \subseteq \mathcal{K}(X)$ then $(A \vee B)+C = (A+C) \vee (B+C)$. In [9], we can find for $A, B \subseteq \mathcal{K}(X)$. If $A \cup B$ is convex then $A+B = A \vee B + A \cap B$.

2 First manner of constructing of equivalent minimal pairs not unique up to translation

Theorem 2.1. *Let $A, A_1, P, Q \in \mathcal{K}(X)$, $x \in X$ and assume that $A \cup P$, $A \cup Q$, $A \cup P \cup Q$, $A_1 \cup P$, $A_1 \cup (Q+x)$, $A_1 \cup P \cup (Q+x)$ are convex sets. Moreover, let $A \cap P = A_1 \cap P$, $(A \cap Q)+x = A_1 \cap (Q+x)$. Then (A, B) is equivalent to (A_1, B_1) where $B = A \cup P \cup Q$ and $B_1 = A_1 \cup P \cup (Q+x)$.*

Proof. The sets $A \cup P$, $A \cup Q$ are convex. Hence, we have

$$A + P = A \cup P + A \cap P,$$



$$A + Q = A \cup Q + A \cap Q.$$

Analogously for $A_1 \cup P$ and $A_1 \cup (Q + x)$

$$A_1 + P = A_1 \cup P + A_1 \cap P,$$

$$A_1 + (Q + x) = A_1 \cup (Q + x) + A_1 \cap (Q + x).$$

From this it follows that

$$A + P + A_1 \cup P + A_1 \cap P = A \cup P + A \cap P + A_1 + P,$$

$$A + Q + A_1 \cup (Q + x) + A_1 \cap (Q + x) = A \cup Q + A \cap Q + A_1 + Q + x.$$

Now, from the ordered law of cancellation we obtain

$$A + (A_1 \cup P) = A \cup P + A_1,$$

$$A + A_1 \cup (Q + x) = A \cup Q + A_1.$$

Therefore from the Pinsker rule

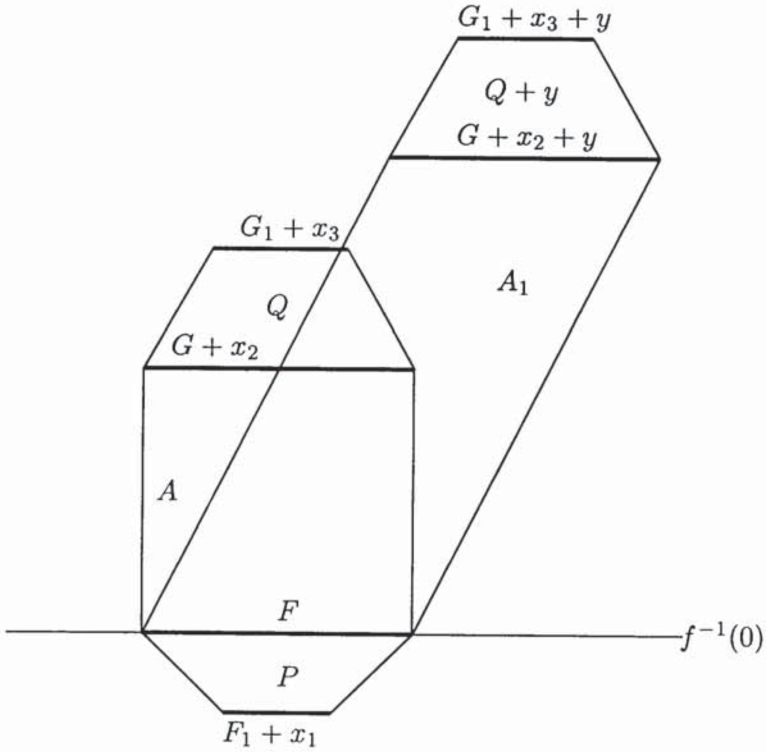
$$A + (A_1 \cup P) \vee [A_1 \cup (Q + x)] = A_1 + A \cup P \cup Q.$$

Hence

$$A + B_1 = A_1 + B.$$

■

Suppose that $F, F_1, G, G_1 \subset \mathcal{K}(X)$ and $F, F_1, G, G_1 \subset f^{-1}(0)$ for some $f \in X^*$. Denote $P = F \vee (F_1 + x_1)$, $Q = (G + x_2) \vee (G_1 + x_3)$, where $x_1, x_2, x_3 \in X$ with $f(x_1) < 0 < f(x_2) < f(x_3)$.



Corollary 2.2. *Let* $A = F \vee (G + x_2)$, $B = P \vee Q$, $A_1 = F \vee (G + y + x_2)$, $B_1 = P \vee (Q + y)$, $y \in X$ *and*

$$f(x_2)(F_1 + x_1) - f(x_1)(G + x_2) \subset f(x_2 - x_1)F,$$

$$f(x_3 - x_2)F + f(x_2)(G_1 + x_3) \subset f(x_3)(G + x_2),$$

$$f(x_2 + y)(F_1 + x_1) - f(x_1)(G + x_2 + y) \subset f(x_2 + y - x_1)F, \quad (2.1)$$

$$f(x_3 - x_2)F + f(x_2 + y)(G_1 + x_3 + y) \subset f(x_3 + y)(G + x_2 + y). \quad (2.2)$$

Then (A, B) *is equivalent to* (A_1, B_1).

Corollary 2.2 follows immediately from Theorem 2.1.
Now, we define:

$$\mathcal{M} := \{(A, B) \in \mathcal{K}^2(X) |\ (A, B) \text{ is minimal}\},$$

$$\mathcal{M}_t := \{(A, B) \in \mathcal{M} |\ \forall (A', B') \in \mathcal{M}, (A', B') \sim (A, B) \Rightarrow \exists x \in X,$$
$$A' = A + x \text{ and } B' = B + x\},$$

$$\mathcal{M}_{nt} := \mathcal{M} \setminus \mathcal{M}_t.$$

Theorem 2.3. *Let* $F, F_1, G, G_1 \in \mathcal{K}(X)$, $F, F_1, G, G_1 \subset f^{-1}(0)$ *for some* $f \in X^*$, $A = F \vee (G + x_2)$, $B = A \vee (F_1 + x_1) \vee (G_1 + x_3)$, *where* $f(x_1) < 0 < f(x_2) < f(x_3)$ *and*

$$f(x_2)(F_1 + x_1) - f(x_1)(G + x_2) \subset f(x_2 - x_1)F,$$

$$f(x_3 - x_2)F + f(x_2)(G_1 + x_3) \subset f(x_3)(G + x_2).$$

If $(F, F_1), (G, G_1) \in \mathcal{M}_t$ *and* $F \dot{-} G = G \dot{-} F = \emptyset$, *then the pair* (A, B) *is minimal.*

Proof. Suppose that $(A', B') \leq (A, B)$ for some $(A', B') \in \mathcal{K}^2(X)$. Hence

$$A + B' = B + A'.$$

From assumption there exists $f \in X^*$ such that

$$H_f A = G + x_2,\ H_f B = G_1 + x_3,\ H_{-f} A = F,\ H_{-f} B = F_1 + x_1.$$

Since we have

$$H_f A + H_f B' = H_f B + H_f A'$$

and

$$H_{-f} A + H_{-f} B' = H_{-f} B + H_{-f} A'.$$

We get

$$G + x_2 + H_f B' = G_1 + x_3 + H_f A'$$

and

$$F + H_{-f}B' = F_1 + x_1 + H_{-f}A'.$$

Hence

$$(G + x_2, G_1 + x_3) \sim (H_f A', H_f B')$$

and

$$(F, F_1 + x_1) \sim (H_{-f}A', H_{-f}B').$$

Since $(G, G_1) \in \mathcal{M}_t$, then $G + x_2 + z \subset H_f A'$ and $G_1 + x_3 + z \subset H_f B'$ for some $z \in X$. Therefore, $G + z + x_2 \subset A$. Since $F \dot{-} G = \emptyset$ and $A = F \vee (G + x_2)$ then $G + z + x_2 \subset G + x_2$ and $z = 0$. Hence $G + x_2 \subset A'$.
In similar way, taking $-f$ instead of f, we can prove that $F \subset A'$.
Then $A = F \vee (G + x_2) \subset A' \subset A$.
Hence $A' = A$, and by the ordered law of cancellation $B' = B$.

■

Corollary 2.4. *Let us take sets described in Theorem 2.3. If there exists $y \in X$, $y \neq 0$ such that inclusions (2.1) and (2.2) are satisfied then Corollary 2.2 implies that $(A, B) \in \mathcal{M}_{nt}$.*

The pair (A_1, B_2) defined in Corollary 2.2 belongs $[A, B]$. Replacing y by αy ($\alpha \in (0, 1)$) in definition of (A_1, B_1) we obtain a pair in $[A, B]$. In fact, the set of all y satisfying (2.1) and (2.2) is convex. Theorem 2.3 and Corollary 2.2 enable constructing great number of various pairs belonging to $\mathcal{M}_{nt}$. These pairs do not have to be polytops. In section 5 we present specific examples of pairs (A, B) in $\mathcal{M}_{nt}$.

3 The "flat" case

It is interesting that $[A, B]$ containes a pair (C, D) with "flat" C i.e. such that $f(C) = \{0\}$. In this section we investigate that case.

Proposition 3.1. *Let $A, B \in \mathcal{K}(X)$ be defined like in Theorem 2.3, $C = F + G$, $D = C \vee (F + G_1 + x_3 - x_2) \vee (G + F_1 + x_1)$, $f(x_3) - f(x_2) \leq f(x_2)$, $-f(x_1) \leq f(x_2)$,*

$$(f(x_3) - f(x_2))F + f(x_2)(G_1 + x_3) \subset f(x_3)(G + x_2)$$

and

$$-f(x_1)(G+x_2)+f(x_2)(F_1+x_1)\subset f(x_2-x_1)F.$$

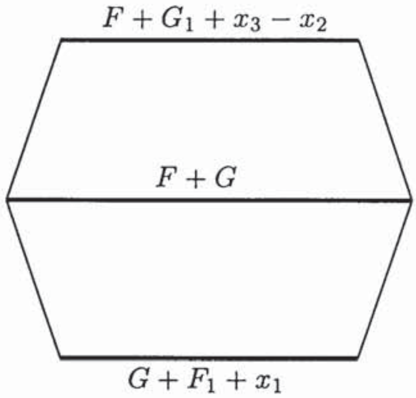
Then (C,D) *is equivalent to* (A,B).

Proof. We know that

$$A=F\vee(G+x_2),$$

and

$$B=F\vee(F_1+x_1)\vee(G+x_2)\vee(G_1+x_3).$$



We have

$$A+D=(2F+G)\vee(2F+G_1+x_3-x_2)\vee(F+G+F_1+x_1)\vee$$

$$\vee(F+2G+x_2)\vee(F+G+G_1+x_3)\vee(2G+F_1+x_1+x_2),$$

$$B+C=(2F+G)\vee(F+G+F_1+x_1)\vee(F+2G+x_2)\vee(F+G+G_1+x_3).$$

Denote

$$\alpha=\frac{f(x_3)-f(x_2)}{f(x_2)},\ \beta=1-\alpha.$$

Then

$$\alpha F+G_1+x_3\subset(1+\alpha)(G+x_2).$$

Therefore,

$$2F+G_1+x_3-x_2\subset(1+\alpha)(G+x_2)+(1+\beta)F-x_2$$

and

$$2F + G_1 + x_3 - x_2 \subset (2\alpha + \beta)(G + x_2) + (\alpha + 2\beta)F - x_2.$$

We have

$$2F + G_1 + x_3 - x_2 \subset \alpha(F + 2G + x_2) + \beta(2F + G). \tag{3.1}$$

Analogously

$$\alpha_1 = \frac{-f(x_1)}{f(x_2)}, \qquad \beta_1 = 1 - \alpha_1.$$

Then

$$\alpha_1(G + x_2) + F_1 + x_1 \subset F(1 + \alpha_1),$$

hence

$$2G + F_1 + x_1 + x_2 \subset (1 + \beta_1)G + \beta_1 x_2 + F(1 + \alpha_1).$$

So

$$2G + F_1 + x_1 + x_2 \subset (\alpha_1 + 2\beta_1)G + \beta_1 x_2 + F(2\alpha_1 + \beta_1),$$

and

$$2G + F_1 + x_1 + x_2 \subset \beta_1(F + 2G + x_2) + \alpha_1(2F + G). \tag{3.2}$$

From (3.1) and (3.2) we have

$$2F + G_1 + x_3 - x_2 \subset (F + 2G + x_2) \vee (2F + G),$$

$$2G + F_1 + x_1 + x_2 \subset (F + 2G + x_2) \vee (2F + G).$$

Hence

$$A + D = B + C.$$

■

Let $A, B \in \mathcal{K}(X)$, then the following result holds:

Theorem 3.2. *If $(F', G') \leq (F, G)$, then (C, D) is equivalent to (C', D'), where $C' = F + G'$, $D' = C' \vee (F' + G_1 + x_3 - x_2) \vee (G' + F_1 + x_1)$, $C = F + G$ and $D = C \vee (F + G_1 + x_3 - x_2) \vee (G + F_1 + x_1)$.*

Proof. We have $C = F + G$,

$$C + D' = (2F + G + G') \vee (F + G + F' + G_1 + x_3 - x_2) \vee (F + G + G' + F_1 + x_1)$$

and

$$D+C' = (F+G+F+G')\vee(F+F+G'+G_1+x_3-x_2)\vee(F+G'+G+F_1+x_1) =$$

$$= (2F+G+G')\vee(F+G+F'+G_1+x_3-x_2)\vee(F+G'+G+F_1+x_1) =$$

$$= C + D'.$$

■

Theorem 3.3. *Let* $(F', G') \leq (F, G)$ *and* $(F' + G_1, G' + F_1) \in \mathcal{M}_t$, *then* (C', D') *from Theorem 3.2 is also minimal pair.*

Proof. Suppose that $(A', B') \leq (C', D')$. We have

$$A' + D' = B' + C',$$

and

$$H_f A' + H_f D' = H_f B' + H_f C',$$

$$H_{-f} A' + H_{-f} D' = H_{-f} B' + H_{-f} C'.$$

Then

$$A' + F' + G_1 + x_3 - x_2 = H_f B' + F' + G,$$

$$A' + G' + F_1 + x_1 = H_{-f} B' + F + G'.$$

Hence

$$A' + G_1 + x_3 - x_2 = H_f B' + G, \tag{3.3}$$

$$A' + F_1 + x_1 = H_{-f} B' + F. \tag{3.4}$$

Therefore

$$A' + G_1 + x_3 - x_2 + H_{-f} B' + F = A' + F_1 + x_1 + H_f B' + G.$$

From the ordered law of cancellation we obtain

$$G_1 + F + H_{-f} B' + x_3 - x_2 = G + F_1 + H_f B' + x_1.$$

Hence

$$(F + G_1, G + F_1) \sim (H_f B' + x_1, H_{-f} B' + x_3 - x_2).$$

By assumption $(F' + G_1, G' + F_1) \in \mathcal{M}_t$.
Hence

$$F' + G_1 + x \subset H_f B' + x_1,$$

$$G' + F_1 + x \subset H_{-f} B' + x_3 - x_2 \text{ for some } x \in X.$$

From (3.3) it follows that

$$A' + G_1 + x_3 - x_2 \supset F' + G_1 + x - x_1 + G.$$

Therefore,

$$A' \supset F' + G + x - x_1 + x_2 - x_3 = C' + x - x_1 + x_2 - x_3.$$

Since $C' \subset A'$ then $x - x_1 + x_2 - x_3 = 0$ and $A' = C'$. From the ordered law of cancellation, we have $B' = D'$.

■

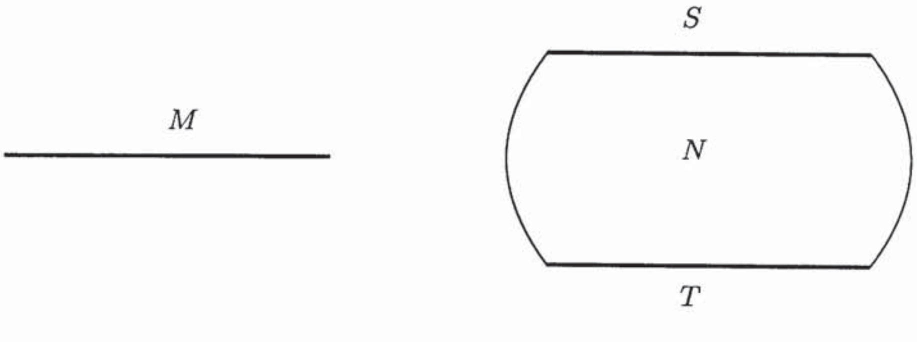
Even if $(F' + G_1, G' + F_1) \notin \mathcal{M}_t$ we know that there exists a minimal pair $(C', D') \leq (C, D)$ [4]. It means that $[A, B]$ contains minimal pair (C', D') with "flat" C'.

Proposition 3.4. *Let $M, N \in \mathcal{K}(X)$, $(S, T) \in \mathcal{M}_t$, $f \in X^*$, $H_f M = M$ and $H_f N = S$, $H_{-f} N = T$. Then for every $(M', N') \sim (M, N)$ such that $H_f M' = M'$ and (M', N') is minimal we have $M' = M + x$, $N' = N + x$ for some $x \in X$.*

Proof. Let $M' + N = N' + M$. Then

$$M' + S = H_f N' + M, \tag{3.5}$$

$$M' + T = H_{-f} N' + M. \tag{3.6}$$



Hence

$$M' + S + H_{-f}N' + M = M' + T + H_fN' + M.$$

Then

$$S + H_{-f}N' = T + H_fN',$$

and we have

$$(S, T) \sim (H_fN', H_{-f}N').$$

But $(S, T) \in \mathcal{M}_t$. Therefore

$$S + x \subset H_fN', \ T + x \subset H_{-f}N' \qquad \text{for some } x \in X.$$

Now, from (3.5) we have

$$M' + S \supset S + M + x.$$

This implies that $M' \supset M + x$. From $M' + N = N' + M$, we obtain $N + x \subset N'$. But (N', M') is minimal.
Hence $N' = N + x$ and $M' = M + x$.

■

Corollary 3.5. *Let (C', D') be defined like in Theorem 3.3. For every $(C_1, D_1) \sim (C', D')$ such that $H_fC_1 = C_1$ and (C_1, D_1) is minimal there exists $x \in X$ such that $C_1 = C' + x$ and $D_1 = D' + x$. Then (C', D') is unique up to translation minimal pair in $[A, B]$ that has "flat" C'.*

4 Second manner of constructing equivalent minimal pairs not unique up to translation

Let A, B be defined like in Theorem 2.1 and $C = A \cup P$, $D = A \cup Q$. The pair (C, D) is not equivalent to (A, B), but the pair (C, D) opens new possibility of construction equivalent minimal pairs not unique up to translation.

Theorem 4.1. *Let A, A_1, P, Q be defined like in Theorem 2.1. Then (C, D) is equivalent to (C_1, D_1) where $C = A \cup P$, $D = A \cup Q$, $C_1 = A_1 \cup P$ and $D_1 = A_1 \cup (Q + x)$.*

Proof. The sets $A \cup P$, $A \cup Q$ are convex. Hence

$$A + P = A \cup P + A \cap P,$$

$$A + Q = A \cup Q + A \cap Q.$$

Analogously for $A_1 \cup P$ and $A_1 \cup (Q + x)$

$$A_1 + P = A_1 \cup P + A_1 \cap P,$$

$$A_1 + (Q + x) = A_1 \cup (Q + x) + A_1 \cap (Q + x).$$

From above it follows

$$A + P + A_1 \cup P + A_1 \cap P = A_1 + P + A \cup P + A \cup P + A \cap P,$$

$$A + Q + A_1 \cup (Q + x) + A_1 \cap (Q + x) = A_1 + Q + x + A \cup Q + A \cap Q.$$

Now from the ordered law of cancellation we obtain

$$A + A_1 \cup P = A_1 + A \cup P,$$

$$A + A_1 \cup (Q + x) = A_1 + A \cup Q.$$

Therefore

$$A + A_1 \cup P + A_1 + A \cup Q = A_1 + A \cup P + A + A_1 \cup (Q + x),$$

$$A_1 \cup P + A \cup Q = A \cup P + A_1 \cup (Q + x).$$

Hence

$$C_1 + D = C + D_1.$$

∎

Corollary 4.2. *Let A, A_1, P, Q be defined like in Corollary 2.2. Then (C, D) is equivalent to (C_1, D_1) where $C = A \vee P$, $D = A \vee Q$, $C_1 = A_1 \vee P$ and $D_1 = A_1 \vee (Q + x)$.*

Theorem 4.3. *Let $F, F_1, G, G_1 \in \mathcal{K}(X)$, $F, F_1, G, G_1 \subset f^{-1}(0)$ for some $f \in X^*$, $C = (F_1 + x_1) \vee F \vee (G + x_2)$, $D = F \vee (G + x_2) \vee (G_1 + x_3)$ where $f(x_1) < 0 < f(x_2) < f(x_3)$ and*

$$f(x_2)(F_1 + x_1) - f(x_1)(G + x_2) \subset (f(x_2) - f(x_1))F,$$

$$f(x_3 - x_2)F + f(x_2)(G_1 + x_3) \subset f(x_3)(G + x_2).$$

If $(F, F_1), (G, G_1) \in \mathcal{M}_t$ and $F \dot{-} G = G \dot{-} F = \emptyset$, then the pair (C, D) is minimal.

Proof. Suppose that $(C', D') \leq (C, D)$ for some $(C', D') \in \mathcal{K}^2(X)$. Hence

$$C + D' = D + C'.$$

From assumption there exists $f \in X^*$ such that $H_f C = G + x_2$, $H_f D = G_1 + x_3$, $H_{-f} C = F$, $H_{-f} D = F_1 + x_1$. Since we have

$$H_f C + H_f D' = H_f D + H_f C'$$

and

$$H_{-f} C + H_{-f} D' = H_{-f} D + H_{-f} C'.$$

Then

$$G + x_2 + H_f D' = G_1 + x_3 + H_f C'$$

and

$$F_1 + x_1 + H_{-f} D' = F + H_{-f} C'.$$

Hence

$$(G + x_2, G_1 + x_3) \sim (H_f C', H_f D')$$

and

$$(F, F_1 + x_1) \sim (H_{-f} D', H_{-f} C').$$

Since $(G, G_1) \in \mathcal{M}_t$, then $G + x_2 + x \subset H_f C'$ and $G_1 + x_3 + x \subset H_f D'$ for some $x \in X$. Because $F \dot{-} G = \emptyset$ and $C = (F_1 + x_1) \vee F \vee (G + x_2)$ then $G + x_2 + x \subset H_f C' \subset G + x_2$ and $x = 0$. Hence $H_f C' = G + x_2$ and from the ordered law of cancellation $H_f D' = G_1 + x_3$. Similarly, taking $-f$ instead of f, we can prove that $H_{-f} C' = F_1 + x_1$ and $H_{-f} D' = F$. Denote $C'' = (F_1 + x_1) \vee (G + x_2)$ and $D'' = F \vee (G_1 + x_1)$. We know that $C'' \subset C' \subset C$ and $D'' \subset D' \subset D$. We have

$$C'' \vee D'' = (F_1 + x_1) \vee F \vee (G + x_2) \vee (G_1 + x_3) = C \vee D$$

then $C' \vee D' = C \vee D$. Moreover $C + D' = D + C'$. This implies that

$$C + C' \vee D' = (C + C') \vee (C + D') = (C + C') \vee (D + C') = C' + C \vee D.$$

But $C' \vee D' = C \vee D$ and from the ordered law of cancellation, we obtain $C = C'$ and $D = D'$.

■

Theorem 4.4. *Let (C, D) be defined like in Theorem 4.3, $M = (F_1 + G + x_1 + x_2) \vee (F + G + x_2)$ and $N = (G + F + x_2) \vee (G_1 + F + x_3)$. Then (M, N) is equivalent to (C, D).*

Proof. We have $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$, where $C_1 = (F_1 + x_1) \vee F$, $C_2 = D_1 = F \vee (G + x_2)$ and $D_2 = (G + x_2) \vee (G_1 + x_3)$.
Pairs (C_1, C_2) and (D_1, D_2) are convex. Hence

$$C_1 + C_2 = C + F$$

and

$$D_1 + D_2 = D + G + x_2.$$

Then

$$C_1 + C_2 + D + G + x_2 = D_1 + D_2 + C + F.$$

Now, from the ordered law of cancellation, we obtain

$$C_1 + D + G + x_2 = C + F + D_2.$$

Therefore

$$D + M = C + N.$$

■

The sets M, N are a special type of sets called *general frustums*. Criteria of minimality of pairs of frustums, can be found [3].

5 Examples

1). Let $G = \{(x, y, z) | z = 0, \ -2 \leq x \leq 2, \ -1 \leq y \leq 1\}$, $F = \{(x, y, z) | z = 0, \ -1 \leq x \leq 1, \ -2 \leq y \leq 2\}$, $G_1 = \{(0,0,0)\}$, $F_1 = \{(0,0,0)\}$. $x_1 = (0,0,-1)$, $x_2 = (0,0,2)$, $x_3 = (0,0,3)$, $y = (1,1,1)$. $P = F \vee (F_1 + (0,0,-1))$, $Q = (G + (0,0,2)) \vee (G_1 + (0,0,3))$, $A = F \vee (G + (0,0,2))$, $B = P \vee Q$, $A_1 = F \vee (G + (1,1,3))$, $B_1 = P \vee (Q + (1,1,1))$, Pairs (A, B) and (A_1, B_1) satisfy Theorem 2.3 i.e., $(A, B) \sim (A_1, B_1)$ and $(A, B), (A_1, B_1) \in \mathcal{M}_{nt}$.
Take $F' = \{0\} \times [-1, 1] \times \{0\}$ and $G' = [-1, 1] \times \{0\} \times \{0\}$, then $(F', G') \leq (F, G)$ and $(F' + G_1, G' + F_1) \in \mathcal{M}_t$. From Theorems 3.2 and 3.3 we have $C' = F + G'$, $D' = C' \vee (F' + G_1 + (0,0,1)) \vee (G' + F_1 + (0,0,-1))$, $(C', D') \sim (A, B)$ and $(A, B), (C', D') \in \mathcal{M}_{nt}$.

2). Let $F = \{(x,y,z)|z = 0,\ (x+2)^2 + y^2 \leq 8,\ (x-2)^2 + y^2 \leq 8\}$, $G = \{(x,y,z)|z = 0,\ x^2 + (y+2)^2 \leq 8,\ x^2 + (y-2)^2 \leq 8\}$, $F_1 = \{0\} \times [-1,1] \times \{0\}$, $G_1 = [-1,1] \times \{0\} \times \{0\}$. $x_1 = (0,0,-1)$, $x_2 = (0,0,2)$, $x_3 = (0,0,3)$, $y = (1,1,1)$. $P = F \vee (F_1 + (0,0,-1))$, $Q = (G + (0,0,2)) \vee (G_1 + (0,0,3))$, $A = F \vee (G + (0,0,2))$, $B = P \vee Q$, $A_1 = F \vee (G + (1,1,3))$, $B_1 = P \vee (Q + (1,1,1))$. We have $(A,B) \sim (A_1,B_1)$ and $(A,B),(A_1,B_1) \in \mathcal{M}_{nt}$. In this example (F,G) satisfies Theorem 3.3 namely $(F+G_1, G+F_1) \in \mathcal{M}_t$. $C = F+G$, $D = C \vee (F + G_1 + (0,0,1)) \vee (G + F_1 + (0,0,-1))$, $(A,B) \sim (C,D)$ and $(A,B),(C,D) \in \mathcal{M}_{nt}$.

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