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METRIC THEORY OF SEMIALGEBRAIC CURVES

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Abstract

We present a complete bi-Lipschitz classification of germs of semialgebraic curves (semialgebraic sets of the dimension one). For this purpose we introduce the so-called Hölder Semicomplex, a bi-Lipschitz invariant. Hölder Semicomplex is the collection of all first exponents of Newton-Puiseux expansions, for all pairs of branches of a curve. We prove that two germs of curves are bi-Lipschitz equivalent if and only if the corresponding Hölder Semicomplexes are isomorphic. We also prove that any Hölder Semicomplex can be realized as a germ of some plane semialgebraic curve. Finally, we compare these Hölder Semicomplexes with Hölder Complexes-complete bi-Lipschitz invariant of two-dimensional semialgebraic sets.

1 Introduction

In this paper, we are going to study singular semialgebraic curves (one-dimensional semialgebraic sets) from the metric viewpoint. Some recent papers on the metric theory of semialgebraic sets were devoted to intrinsic metric properties of semialgebraic surfaces (see [2],[1],[6]). One of the main questions of this theory is the question of bi-Lipschitz classification. This question was investigated in [1], for germs of semialgebraic surfaces, with respect to the intrinsic (length) metric. The problem of bi-Lipschitz classification with respect to the induced (euclidian) metric is more complicated. Namely, if two semialgebraic sets are bi-Lipschitz

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equivalent with respect to the euclidian metric they are bi-Lipschitz equivalent with respect to the length metric, but the converse is not true.

The main result of the paper is a bi-Lipschitz classification of germs of semialgebraic curves with respect to the euclidian metric. We construct an invariant - the so-called *Hölder Semicomplex*. This invariant is obtained in the following way. Take two branches X_1 and X_2 of a semialgebraic curve X at the point $x_0 \in X$. Let $x_1(r) \in X_1$ and $x_2(r) \in X_2$ be two points on these branches such that $d_{ind}(x_1(r), x_0) = d_{ind}(x_2(r), x_0) = r$. The function $d_{ind}(x_1(r), x_2(r))$ (the induced metric) is semialgebraic and, thus, admits a Newton-Puiseux expansion at 0. Hölder Semicomplex is the collection of all first exponents of Newton-Puiseux expansions, for all pairs of branches. Note, that this procedure is closely related to the calculation of Lojasiewicz exponent [4].

We prove that Hölder Semicomplex is a complete invariant. It means that germs of two semialgebraic curves are bi-Lipschitz equivalent if their Hölder Semicomplexes are combinatorially equivalent (isomorphic). We prove (Theorem 4.1), moreover, that, for every Hölder Semicomplex, there exists a germ of a plane semialgebraic curve realizing it (i.e. the Hölder Semicomplex of this curve coincides with the given one) (Theorem 5.1).

We study in section 6 some relations between Hölder Complexes [1] and Hölder Semicomplexes. Hölder Complexes defined in [1] are complete invariants of germs of semialgebraic surfaces. We prove that if a two-dimensional set X is normally embedded at x_0 (i.e. intrinsic and induced metrics are bi-Lipschitz equivalent [3]) then the Hölder Semicomplex of the germ of Sing(X) at x_0 is completely determined by the Hölder Complex of X at x_0 (Theorem 6.2). Some examples presented in the section 6 show that this result does not remain true if X is not normally embedded.

Note that all results of the paper are hold for semianalytic set. As a consequence of this observation we obtain that every germ of a semianalytic curve is bi-Lipschitz equivalent to a germ of some semialgebraic curve (Corollary 5.2).

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2 Hölder Semicomplexes

Definition 1. *A complete finite graph* Γ *with a rational valued function* $\alpha : E_\Gamma \to \mathbb{Q} \cap [1, \infty)$ *defined on the set of edges* E_Γ *of* Γ *is called a Hölder Semicomplex if* α *satifies the following:*

Isosceles Property. For every three vertices $a_1, a_2, a_3 \in \Gamma$, we have: if $\alpha(a_1, a_2) \leq \alpha(a_2, a_3) \leq \alpha(a_1, a_3)$ then $\alpha(a_1, a_2) = \alpha(a_2, a_3)$. (Note that, since Γ is a complete graph, any edge is completely determined by two vertices).

Remark 1. A Hölder Semicomplex can be defined in the following equivalent way. Let A be a finite set with a symmetric function α : $A \times A - diag(A \times A) \to \mathbb{Q} \cap [1, \infty)$. If α satisfies the isosceles property then the pair (A, α) can be identified with a Hölder Semicomplex.

Remark 2. Consider a function $d : A \times A \to \mathbb{Q}$ such that

$$d(a_1, a_2) = 0 \quad \text{if} \quad a_1 = a_2 \quad \text{and} \quad d(a_1, a_2) = \frac{1}{\alpha(a_1, a_2)} \quad \text{otherwise.}$$

Then (A, d) is an ultrametric space if and only if (A, α) is a Hölder Semicomplex.

Definition 2. *Two Hölder Semicomplexes* (Γ_1, α_1) *and* (Γ_2, α_2) *are called isomorphic (or combinatorially equivalent) if there exists a graph isomorphism* $h : \Gamma_1 \to \Gamma_2$ *such that, for each pair of vertices* a_1, a_2 *of* Γ_1*, we have:*

$$\alpha_2(h(a_1), h(a_2)) = \alpha_1(a_1, a_2).$$

Definition 3. A *morphism of* (A_1, α_1) *to* (A_2, α_2) *is a map* $\mathfrak{m} : A_1 \to A_2$ *such that, for all* $a_1, a_2 \in A_1$,

$$\alpha_2(\mathfrak{m}(a_1), \mathfrak{m}(a_2)) \geq \alpha_1(a_1, a_2) \quad \text{if} \quad \mathfrak{m}(a_1) \neq \mathfrak{m}(a_2).$$

Thus, we obtain that the set of all Hölder Semicomplexes is a category, the isomorphism defined in Definition 2 is an isomorphism in this category and Remark 2 defines a functor from this category to the category of metric spaces.

3 Semialgebraic curves and Hölder Semicomplex

Let $X \subset \mathbb{R}^n$ be a semialgebraic curve and $x_0 \in X$. By the standard conic structure theorem, we have the following statement:

There exists a neighbourhood U_{x_0} of x_0 in $\mathbb{R}^n$ such that $X \cap U_{x_0} = \bigcup_{i=1}^{k} X_i$ where the sets X_i have the following properties:

(1) X_i is a semialgebraic subset of $\mathbb{R}^n$ and homeomorphic to a semisegment $[1,0)$ by a homeomorphism $h_i : [1,0) \to X_i$ such that $h_i(0) = x_0$;

(2) For every $i \neq j$, $X_i \cap X_j = \{x_0\}$;

(3) There exists a number r_0 such that, for every i and $0 \leq r < r_0$, $\#(X_i \cap S_r(x_0)) = 1$, for every i. (Here $S_r(x_0)$ means a sphere centred at x_0 of radius r).

Remark 3. The collection $\{X_i\}_{i=1}^{k}$ is a 1-dimensional version of the so-called *pancake decomposition of* (X, x_0) (see [3], [8], [11]). The elements of this decomposition we call *branches* (in the similar way to the complex algebraic geometry) or *pancakes* (in the similar way to the real algebraic geometry).

Let A be a k-elements set $A = \{a_1, \dots, a_k\}$. We define a Hölder Semicomplex on A in the following way. Consider a map $x_i(r) = X_i \cap S_r(x_0)$ (for sufficiently small r this function is well defined and semialgebraic). Let $f_{ij}(r) = \|x_i(r) - x_j(r)\|$. By the Newton-Puiseux Theorem we obtain, for sufficiently small r,

$$f_{ij}(r) = b_{ij} r^{\alpha_{ij}} + o(r^{\alpha_{ij}}), \quad \alpha_{ij} \in Q, \quad b_{ij} \in \mathbb{R}.$$

Put $\alpha(a_i, a_j) = \alpha_{ij}$.

Proposition 3.1. (A, α) *is a Hölder Semicomplex.*

Proof. Since $x_i(r)$ and $x_j(r)$ belong to $S_r(x_0)$, we obtain $f_{ij}(r) \leq 2r$. Thus, $1 \leq \alpha(a_i, a_j)$. Let us prove the isosceles property. Let X_1, X_2 and X_3 be three different branches of X. Let $\alpha(a_1, a_2) \leq \alpha(a_2, a_3) \leq \alpha(a_1, a_3)$. Consider three points $x_1(r), x_2(r)$ and $x_3(r)$. We have:

$$C_1 r^{\alpha(a_1,a_2)} + o(r^{\alpha(a_1,a_2)}) = \|x_1(r) - x_2(r)\| \leq$$

$$\|x_1(r) - x_3(r)\| + \|x_2(r) - x_3(r)\| =$$
$$C_2 r^{\alpha(a_2,a_3)} + o(r^{\alpha(a_2,a_3)}),$$

for some positive constants C_1, C_2. Thus, $\alpha(a_1, a_2) = \alpha(a_2, a_3)$

■

The Hölder Semicomplex (A, α) constructed above is called a *Hölder Semicomplex associated to* (X, x_0). We denote it by $sh(X, x_0)$.

Let us define another structure associated to X at the point x_0. Let X_i and X_j be two branches of X. Put

$$g_{ij}(r) = dist(X_i - B_r(x_0), X_j - B_r(x_0))$$

(here $B_r(x_0)$ is the ball centred at x_0 of radius r). By the definition of $g_{ij}(r)$ and the quantifier elimination version of the Tarski-Seidenberg Theorem, we obtain that g_{ij} is a semialgebraic function. Thus, by the Newton-Puiseux Theorem,

$$g_{ij}(r) = c_{ij} r^{\beta_{ij}} + o(r^{\beta_{ij}}).$$

Put $\beta(a_i, a_j) = \beta_{ij}$.

Remark 4. Note that $g_{ij}(r)$ is not necessary equal to $f_{ij}(r)$. To see it consider the set $X \subset \mathbb{R}^2$ defined as a union of graphs of functions $y = x^2$ and $y = x^3$ at the point $x_0 = (0, 0)$.

Order Comparison Lemma. *For all* i, j *we have* : $\beta_{ij} = \alpha_{ij}$.

Proof. Without loss of generality we can suppose that $x_0 = 0$. Let $v \in \mathbb{R}^n$ be a vector such that $\|v\| = 1$. Define a cone

$$C_\varepsilon(v) = \{u \in \mathbb{R}^n; \angle(u, v) < \varepsilon\}.$$

Suppose that two branches X_i and X_j have different unit tangent vectors at $\mathbf{0}$. Let v_i be the unit tangent vector to X_i at $\mathbf{0}$ and v_j be the unit tangent vector to X_j at $\mathbf{0}$. Then there exist $r_0 > 0$ and $\varepsilon > 0$ such that, for $r \leq r_0$,

$X_i \cap B_r(\mathbf{0}) \subset C_\varepsilon(v_i) \cap B_r(\mathbf{0})$;

$X_j \cap B_r(\mathbf{0}) \subset C_\varepsilon(v_j) \cap B_r(\mathbf{0})$;

$C_\varepsilon(v_i) \cap C_\varepsilon(v_j) = \{\mathbf{0}\}$.

Thus, $g_{ij} \geq dist(C_\varepsilon(v_i) - B_r(\mathbf{0}),\ C_\varepsilon(v_j) - B_r(\mathbf{0})) \geq cr$, for some $c > 0$. It means that $\beta_{ij} = \alpha_{ij} = 1$.

Now suppose that X_i and X_j have the same unit tangent vector v_0 at $\mathbf{0}$. Clearly, $\alpha_{ij} \leq \beta_{ij}$. Suppose that $\alpha_{ij} < \beta_{ij}$. Observe that, for sufficiently small r, we can choose the points $y_i(r) \in X_i$ and $y_j(r) \in X_j$ such that $g_{ij}(r) = \|y_i(r) - y_j(r)\|$ depends semialgebraically and continuously on r. Observe that either $y_i(r) = x_i(r)$, or $y_j(r) = x_j(r)$, for sufficiently small r (otherwise the function $g_{ij}(r)$ has to be locally constant what contradicts to semialgebraicity). Suppose that $y_i(r) = x_i(r)$. Consider a triangle with vertices $x_i(r), y_j(r)$ and $x_j(r)$. Since $\beta_{ij} > \alpha_{ij}$, we have:

$$\|x_i(r) - y_j(r)\| \ll \|x_i(r) - x_j(r)\|,$$

for small r. Thus, the angle between $x_i(r) - x_j(r)$ and $y_j(r) - x_j(r)$ tends to zero as r tends to zero. But, since

$$\frac{y_j(r) - x_j(r)}{\|y_j(r) - x_j(r)\|} \to v_0, \quad \text{as} \quad r \to v_0,$$

we obtain that $\angle(x_i(r) - x_j(r), v_0) \to 0$ as $r \to 0$.

On the other hand, X_i and X_j have the same tangent vector v_0 at $\mathbf{0}$. Thus, for every sufficiently small $\varepsilon > 0$, there exists $r_0 > 0$ such that, for every $r < r_0$, we have $x_i(r) \in C_\varepsilon(v_0) \cap S_r(v_0)$ and $x_j(r) \in C_\varepsilon(v_0) \cap S_r(v_0)$. It means that

$$\angle(x_i(r) - x_j(r), v_0) \geq \frac{\pi}{2} - \delta(\varepsilon)$$

where $\delta(\varepsilon) \to 0$, as $\varepsilon \to 0$. This is a contradiction.

■

4 Hölder Semicomplexes as bi-Lipschitz invariants

Theorem 4.1. *Germs of closed semialgebraic curves (X_1, x_1) and (X_2, x_2) are bi-Lipschitz equivalent if and only if the corresponding Hölder Semicomplexes $(A_1, \alpha_1) = sh(X_1, x_1)$ and $(A_2, \alpha_2) = sh(X_2, x_2)$ are combinatorially equivalent.*

Proof. $(\Rightarrow)$Let $\Phi : (X_1, x_1) \to (X_2, x_2)$ be a given bi-Lipschitz map. Let $\{X_i^1\}_{i=1}^{k_1}$ and $\{X_j^2\}_{j=1}^{k_2}$ be pancake decompositions of (X_1, x_1) and (X_2, x_2). Since Φ is a homeomorphism, we have $k_1 = k_2 = k$ and, for each i, $\Phi(X_i^1) = X_i^2$ (we can choose another renumeration if necessary). Let (A_1, α_1) and (A_2, α_2) be the Hölder Semicomplexes corresponding to (X_1, x_1) and (X_2, x_2). Let $A_1 = \{a_i^1\}_{i=1}^k$, let $A_2 = \{a_i^2\}_{i=1}^k$ and put $h(a_i^1) = a_i^2$. Let us prove that $\alpha_1(a_i^1, a_j^1) = \alpha_2(a_i^2, a_j^2)$, for each i, j. Since Φ is a bi-Lipschitz map, there exists $K > 0$ such that, for sufficiently small r, we have:

$$\Phi(X_i^1 - B_r(x_1)) \subset X_i^2 - B_{Kr}(x_2), \qquad (1)$$
$$\Phi(X_j^1 - B_r(x_1)) \subset X_j^2 - B_{Kr}(x_2). \qquad (2)$$

Let $y_i^1(r) \in X_i^1 - B_r(x_1)$ and $y_j^1(r) \in X_j^1 - B_r(x_1)$ be points such that

$$\|y_i^1(r) - y_j^1(r)\| = dist(X_i^1 - B_r(x_1), X_j^1 - B_r(x_1))$$

Thus, by (1),(2) we obtain:

$$\|\Phi(y_i^1(r)) - \Phi(y_j^1(r))\| \geq g_{ij}^2(Kr)$$

(here g_{ij}^1, g_{ij}^2 mean the function g_{ij} defined for X_1 and X_2 correspondingly, see section 3). Since Φ is a bi-Lipschitz map, there exists a constant $L > 0$ such that

$$g_{ij}^1(r) = \|y_i^1(r) - y_j^1(r)\| \geq L\|\Phi(y_i^1(r)) - \Phi(y_j^1(r)\|.$$

Thus,

$$g_{ij}^1(r) \geq L g_{ij}^2(Kr).$$

Using the Newton-Puiseux Theorem and the Order Comparison Lemma, we obtain that $\alpha_1(a_i^1, a_j^1) \le \alpha_2(a_i^2, a_j^2)$. Using the fact that Φ^{-1} is also a bi-Lipschitz map we obtain that $\alpha_1(a_i^1, a_j^1) \ge \alpha_2(a_i^2, a_j^2)$. The first part of the theorem is proved.

($\Leftarrow$) Let (A_1, α_1) and (A_2, α_2) be combinatorially equivalent. We suppose that the isomorphism preserves the enumeration. Let $\{X_i^1\}_{i=1}^{k_1}$ and $\{X_i^2\}_{i=1}^{k_2}$ be pancake decompositions of (X_1, x_1) and (X_2, x_2). Let $r_0 > 0$ be a number such that $x_i^1(r)$ and $x_i^2(r)$ are well defined semialgebraic functions, for every $r < r_0$ and for every i. Consider the semialgebraic function $r : X_1 \cup X_2 \to \mathbb{R}$ defined by

$$r(x) = \begin{cases} \|x - x_1\|, & \text{for} \quad x \in X_1; \\ \|x - x_2\|, & \text{for} \quad x \in X_2. \end{cases}$$

Define $\Phi : X_1 \cap B_{r_0}(x_1) \to X_2 \cap B_{r_0}(x_2)$ by $\Phi(x) = x_i^2(r(x))$, for $x \in X_i^1$.

Claim. There exists $\delta > 0$ such that $\Phi|_{X_1 \cap B_\delta(x_1)}$ is a bi-Lipschitz map.

Proof of the claim. Each X_i^1 and X_i^2 has a tangent vector at x_1 and x_2, respectively, hence, for some $\delta_0 > 0$, we have: $\Phi|_{X_i^1 \cap B_{\delta_0}(x_1)}$ is a bi-Lipschitz map, for all i. It is enough to prove that, for each pair (X_i^1, X_j^1), the map $\Phi|_{(X_i^1 \cup X_j^1) \cap B_{\delta_1}(x_1)}$ is bi-Lipschitz, for some $\delta_1 > 0$.

Let $x \in X_i^1$ and $y \in X_j^1$ be two points sufficiently close to x_1. We can suppose that $r(x) \le r(y)$. Let $z \in X_j^1$ such that $r(z) = r(x)$. Consider the triangles (x, y, z) and $(\Phi(x), \Phi(y), \Phi(z))$. Since the curves $X_i^1, X_j^1, X_i^2, X_j^2$ are semialgebraic sets, for sufficiently small r, they are close enough to their tangent vectors. Thus, the angle at the vertex z in the triangle (x, y, z) and the angle at the vertex $\Phi(z)$ in the triangle $(\Phi(x), \Phi(y), \Phi(z))$ are bounded away from zero. Using this fact we obtain that there exist $K_1, K_2 > 0$ such that

$$K_1 \max(r(y) - r(x), f_{ij}^1(r(x))) \le \|y - x\| \tag{3}$$

and

$$\|y - x\| \le K_2 \max(r(y) - r(x), f_{ij}^1(r(x))) \tag{4}$$

(here f_{ij}^1, f_{ij}^2 mean the function f_{ij} defined for X_1 and X_2 correspondingly, see section 3). By the same way, we obtain that there exist

$L_1, L_2 > 0$ such that

$$L_1\max(r(\Phi(y)) - r(\Phi(x)), f^2_{ij}(r(\Phi(x)))) \leq \|\Phi(y) - \Phi(x)| \qquad (5)$$

and

$$\|\Phi(y) - \Phi(x)\| \leq L_2\max(r(\Phi(y)) - r(\Phi(x)), f^2_{ij}(r(\Phi(x))). \qquad (6)$$

Since $\alpha_1(a^1_i, a^1_j) = \alpha_2(a^2_i, a^2_j)$, there exist constants $M_1, M_2 > 0$ such that

$$M_1 f^1_{ij}(r) \leq f^2_{ij}(r) \leq M_2 f^1_{ij}(r). \qquad (7)$$

Using the inequalities (3),(4),(5),(7) and the facts that $r(y) = r(\Phi(y))$ and $r(x) = r(\Phi(x))$ we obtain that Φ is a bi-Lipschitz map. The claim is proved. And, thus, Theorem 4.1 is proved.

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5 The Realization Theorem

Theorem 5.1. *Let (A, α) be a Hölder Semicomplex. Let $\#A = k$. Then there exists a semialgebraic subset $X \subset \mathbb{R}^2$ with $dim X = 1$ satisfying the following conditions:*

(1) (A, α) is a Hölder Semicomplex corresponding to $(X, \mathbf{0})$;

(2) $(X, 0)$ has a pancake decomposition $\{X_i\}_{i=1}^k$ such that X_i is a graph of an algebraic function $\psi_i : [0, \varepsilon] \to \mathbb{R}$ with $\psi_i(0) = 0$.

Remark 5. This theorem means that each Hölder Semicomplex can be realized as a plane semialgebraic curve. X is called a *realization* of (A, α).

Proof of the theorem 5.1. We use induction on k. For $k = 1$, the statement is obvious. Suppose that the statement is proved, for some k. Let (A, α) be a Hölder Semicomplex such that $\#A = k + 1$. Let $\alpha_0 = \max_{i,j} \alpha(a_i, a_j)$. We can suppose that $\alpha_0 = \alpha(a_k, a_{k+1})$. Put $\tilde{A} = A - \{a_{k+1}\}$ and $\tilde{\alpha} = \alpha|_{\tilde{A}\times\tilde{A}}$. Let $\tilde{X}$ be a realization of $(\tilde{A}, \tilde{\alpha})$. Let $\tilde{X}_i$ be a pancake corresponding to a_i $(i = 1, \ldots, k)$. By the induction hypothesis, we have that $\tilde{X}_i$ is a graph of some algebraic function $\tilde{\psi}_i : [0, \varepsilon_0] \to \mathbb{R}$ such that $\tilde{\psi}_i(0) = 0$. Put

$$\psi_{k+1}(x) = sx^{\alpha_0} + \tilde{\psi}_k(x).$$

Since $\tilde{X}$ contains a finite number of branches, there exist s and $\varepsilon < \varepsilon_0$ such that $\psi_{k+1}(x) \neq \tilde{\psi}_i(x)$, for every $i = 1, \ldots, k$, if $x \neq 0$.

Put $\psi_i = \tilde{\psi}_i|_{[0,\varepsilon]}$ and let X_i be the graph of ψ_i. Let $X = \bigcup_{i=1}^{k+1} X_i$. By straightforward calculations we obtain that (A, α) is a Hölder Semicomplex corresponding to $(X, 0)$.

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Corollary 5.2. *Every semianalytic subset $X \subset \mathbb{R}^n$ such that $dim X = 1$ is locally bi-Lipschitz equivalent to some semialgebraic subset $\tilde{X} \subset \mathbb{R}^2$.*

6 Hölder Semicomplexes and Hölder Complexes

Here we are going to compare the bi-Lipschitz invariants of germs of semialgebraic curves defined in this paper with bi-Lipschitz invariants of germs of semialgebraic surfaces (see [1]). A complete intrinsic bi-Lipschitz invariant of two-dimensional semialgebraic sets is called *Hölder Complex*. Hölder Complex is a pair (Γ, β) where Γ is a finite graph and β is a rational valued function defined on the set of edges of Γ. One can find all the definitions and results related to this subject in [1] and [6].

Let (Γ, β) be a Hölder Complex. A vertex a of Γ is called *essential* if, for every neighbourhood U_a of a, we have: U_a is not a topological 1-manifold (in other words, a is neither artificial nor a loop vertex of Γ, see [1]). We are going to define a Hölder Semicomplex $(\widetilde{A}, \tilde{\alpha})$ corresponding to (Γ, β) in the following way:

(1) The set $V_{\widetilde{A}}$ of vertices of $\widetilde{A}$ be the set of all essential vertices of Γ.

(2) Let a_1, a_2 be vertices of $\widetilde{A}$. Let P be the set of all finite paths $\gamma = \{g_1, \ldots, g_s\}$ (here $g_1, \ldots, g_s \in E_\Gamma$) connecting a_1 and a_2. We define

$$\tilde{\alpha}(a_1, a_2) = \begin{cases} 1, & \text{if } a_1, a_2 \text{ belong to different connected components of } \Gamma; \\ \max_{\gamma \in P} \min_{g_k \in \gamma} \beta(g_k). \end{cases}$$

Proposition 6.1. *$(\widetilde{A}, \tilde{\alpha})$ is a Hölder Semicomplex.*

Proof. Let us prove the isosceles property. Let a_1, a_2, a_3 be three vertices of $\tilde{A}$ such that $\tilde{\alpha}(a_1, a_2) \leq \tilde{\alpha}(a_2, a_3) \leq \tilde{\alpha}(a_1, a_3)$. We can suppose that a_1, a_2, a_3 belong to the same connected component of Γ. (Otherwise the proposition is trivial). Let $\{g_1, \ldots, g_k\}$ be a path in Γ connecting a_1 and a_2. Let $\{g'_1, \ldots, g'_s\}$ be a path connecting a_1 and a_3. Clearly, $\{g_k, \ldots, g_1, g'_1, \ldots, g'_s\}$ is a path connecting a_2 and a_3. Thus, $\tilde{\alpha}(a_2, a_3) \leq \min \{\tilde{\alpha}(a_1, a_2), \tilde{\alpha}(a_1, a_3)\}$. Since $\tilde{\alpha}(a_1, a_2) \leq \tilde{\alpha}(a_1, a_3)$ we obtain $\tilde{\alpha}(a_2, a_3) = \tilde{\alpha}(a_1, a_2)$.

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Let $Y \subset \mathbb{R}^n$ be a two-dimensional semialgebraic set. Let $Y = Y^0 \cup Y^1 \cup Y^2$ be the canonical topological stratification of Y. This stratification can be obtained in the following way. Let Y^2 be the set of all points $y \in Y$ such that there exists a neighbourhood $U_y \subset \mathbb{R}^n$ such that $U_y \cap Y$ is a topological two-dimensional manifold. Let Y^1 be the set of points of $Y - Y^2$ such that, for each $\tilde{y} \in Y^1$, there exists a neighbourhood $U_{\tilde{y}}$ such that $U_{\tilde{y}} \cap (Y - Y^2)$ is a topological one-dimensional manifold. Set $Y^0 = Y - Y^2 - Y^1$. By [5], Y^0, Y^1 and Y^2 are semialgebraic sets and, thus, Y^0 is finite (see also [9]).

Let $y_0 \in Y^0$ and $y_0 \in Cl(Y^1)$ (here $Cl(Y^1)$ means a closure of Y^1). Then the germ of $Cl(Y^1)$ at y_0 is a germ of one-dimensional semialgebraic set. Let (A, α) be a Hölder Semicomplex associated to $(Cl(Y^1), y_0)$ (i.e. $(A, \alpha) = sh(Cl(Y^1), y_0))$. Let (Γ, β) be a canonical Hölder Complex of Y at y_0 (see[1]). Let $(\widetilde{A}, \tilde{\alpha})$ be the Hölder Semicomplex corresponding to (Γ, β) defined above.

Theorem 6.2.

(1) *There exists a morphism $\mathfrak{m} : (\widetilde{A}, \tilde{\alpha}) \to (A, \alpha)$.*

(2) *If Y is locally normally embedded at y_0 (see [3]) then $(\widetilde{A}, \tilde{\alpha})$ and (A, α) are isomorphic.*

Proof. (2) Let Y_i^1 and Y_j^1 be two branches (pancakes) of $Cl(Y^1)$. Let a_i and a_j be the corresponding vertices in (Γ, β). Suppose that a_i and a_j belong to the same connected component of Γ. Let $\gamma = \{g_1, \ldots, g_s\}$ be a "maximal" path in Γ connecting a_i and a_j (it means that $\tilde{\alpha}(a_i, a_j) = \min_{g_k \in \gamma} \beta(g_k))$. By results of [1], the union of the curvilinear triangles corresponding to the edges $g_1, \ldots, g_s$ is bi-Lipschitz equivalent to the standard $\tilde{\alpha}(a_i, a_j)$-Hölder triangle. Let $a_i(r) \in Y_i^1$ and $a_j(r) \in Y_j^1$ be points such that

$$d_{ind}(a_i(r), y_0) = d_{ind}(a_j(r), y_0) = r.$$

Using results of [1] or [10] we obtain that there exist two constants K_1 and K_2 such that

$$K_1 d_l(a_i(r), y_0) \leq r^{\tilde{\alpha}(a_i,a_j)} \leq K_2 d_l(a_i(r), y_0)$$

(here d_l is the length metric). Since Y is locally normally embedded at y_0 we obtain (using results of sections 2-4) that $\alpha(a_i, a_j) = \tilde{\alpha}(a_i, a_j)$. If a_i and a_j belong to different connected components of Γ then $\alpha(a_i, a_j) = 1$ (because Y is locally normally embedded).

(1) Let Y be not locally normally embedded at y_0. Let $\widetilde{Y}$ be the"normalization" obtained in [3]. Let $\widetilde{Y}^0 \cup \widetilde{Y}^1 \cup \widetilde{Y}^2$ be a topological canonical stratification of $\widetilde{Y}$. Let $\Phi : \widetilde{Y} \to Y$ be a semialgebraic map satisfying the following conditions:

(a) Φ is a bi-Lipschitz map with respect to the length metric. **(b)** Φ is a Lipschitz map with respect to the induced metric. Existence of this map is shown in [3]. Clearly, $\Phi(\widetilde{Y}^1 \cup \widetilde{Y}^0) = Y^1 \cup Y^0$. Let $\widetilde{Y}_i^1$ and $\widetilde{Y}_j^1$ be two branches of $\widetilde{Y}^1$ and let $\tilde{y}_0 = \Phi^{-1}(y_0)$. Let $\tilde{a}_i(r) \in \widetilde{Y}_i^1$ and $\tilde{a}_j(r) \in \widetilde{Y}_j^1$ be two points such that

$$d_{ind}(\tilde{a}_i(r), \tilde{y}_0) = d_{ind}(\tilde{a}_j(r), \tilde{y}_0) = r.$$

By [3], we have

$$d_{ind}(\tilde{a}_i(r), \tilde{a}_j(r)) \geq d_{ind}(a_i(r), a_j(r)),$$

for sufficiently small r. Since $\widetilde{Y}$ is normally embedded, we obtain that $(\widetilde{A}, \tilde{\alpha})$ is a Hölder Semicomplex associated to $(Cl(\widetilde{Y}^1), \tilde{y}_0)$. Since

$$d_{ind}(\tilde{a}_i(r), \tilde{a}_j(r)) = r^{\tilde{\alpha}(a_i,a_j)} + o(r^{\tilde{\alpha}(a_i,a_j)})$$

and

$$d_{ind}(a_i(r), a_j(r)) = r^{\alpha(a_i,a_j)} + o(r^{\alpha(a_i,a_j)}),$$

we have that

$$\tilde{\alpha}(a_i, a_j) \leq \alpha(a_i, a_j). \tag{8}$$

Set $\mathfrak{m}(a_i) = a_i$, for all i. By (8), $\mathfrak{m}$ is a morphism of Hölder Semicomplexes. ∎

Remark 6. If Y is not normally embedded (A, α) and $(\widetilde{A}, \tilde{\alpha})$ are not necessary isomorphic. To see it consider the following example. Let (Γ, β) be a Hölder Complex such that Γ has more than one essential vertices. Let a_1 and a_2 be two essential vertices. Let Γ' be a new graph obtained from Γ by adding a new edge h connecting a_1 and a_2. Set $\beta'(h) > \max_{g \in E_\Gamma} \beta(g)$. Then (Γ', β') is a Hölder Complex. Using the algorithm from [6] we construct a set Y and a point y_0 such that (Y, y_0) is a Geometric Hölder Complex corresponding to (Γ, β). We can suppose that Y is normally embedded (otherwise we can obtain it using [3]). Consider now a semialgebraic set Y' obtained from Y by cutting the triangle corresponding to the edge h. For this set the Hölder Semicomplexes considered above are not isomorphic.

Proposition 6.3. *Let (Y_1, y_1) and (Y_2, y_2) be two germs of 2-dimensional semialgebraic sets such that (Y_1, y_1) and (Y_2, y_2) are bi-Lipschitz equivalent with respect to the induced metric. Then the corresponding Hölder Semicomplexes (A_1, α_1) and (A_2, α_2) are isomorphic.*

The proof is straightforward.

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