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ENTIRE FUNCTIONS AND EQUICONTINUITY OF POWER MAPS IN BAIRE ALGEBRAS

A. EL KINANI

Abstract

We obtain that the power maps are equicontinuous at zero in any Baire locally convex algebra with a continuous product in which all entire functions operate; whence is m-convex in the commutative case. As a consequence, we get the same result of Mityagin, Rolewicz and Zelazko for commutative B_0-algebras.

B. S. Mityagin, S. Rolewicz and W. Zelazko showed ([5]) that a unitary commutative B_0-algebra in which all entire functions operate is necessarily m-convex. Their proof is long and technical. In [3], using a Baire argument and the polarization formula, we obtain that a unitary commutative Baire locally convex algebra with a continuous product in which all entire functions operate is actually m-convex. Our proof is direct and self contained. In the non commutative case, W. Zelazko exhibits in [8] an example of a non m-convex non-commutative B_0-algebra on which all entire functions operate. In his example, the power maps are equicontinuous at zero. Using a Baire argument and the Mazur-Orlicz formula, we obtain the same result in a more general context. We show that the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero in any unitary Baire l. c. a. with a continuous product in which all entire functions operate. Whence, as a consequence, our result of [3], in the commutative case and hence the result of Mityagin, Rolewicz and Zelazko in commutative B_0-algebras ([5]).

A locally convex algebra (*l. c. a.* for short) is a Hausdorff locally convex space which is an algebra over a field K ($K = R$ or $K = C$) with separately continuous product. If the product is continuous in two variables, it is said to be with continuous product. Let (A, τ) be a locally

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convex algebra the topology of which is given by a family of $(p_i)_{i\in I}$ of seminorms. It is said to be multiplicatively m-convex (m-convex for short) if

$$p_i(xy) \leq p_i(x)p_i(y), \text{ for all } x, y \in A, i \in I.$$

A B_0-algebra is a *l. c. a.* whose underlying locally convex space is a completely metrisable space. An entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $a_n \in K$, operates in a unitary *l. c. a.* (A, τ) if, for every x in A, $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, converges in (A, τ). A topological algebra A is said to be Q-algebra if and only if the set of all invertible elements of A is open. For a detailed account of basic properties of general locally m-convex algebras and B_0-algebras, we refer the reader to [4] and [7].

Here is the main result

Theorem 1. *Let (A, τ) be a unitary Baire l. c. a. with a continuous product. If entire functions operate in A, then the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero. In particular, if A is commutative, then it is an m-convex algebra.*

Proof. Let U be an absolutely convex and closed neighbourhood of zero in A, and $\| \cdot \|_U$ its gauge. The product being continuous, there is another continuous seminorm $| \cdot |$ such that

$$\| xy \|_U \leq | x | | y |, \quad x, y \in A.$$

Since entire functions operate in A, one has $\sup_n | x^n |^{\frac{1}{n}} < +\infty$, for every in x in A. Let $r : A \longrightarrow R_+$, be the map given by $r(x) = \sup_n | x^n |^{\frac{1}{n}}$. It is clear that r is lower semicontinuous. For every, integer p, set $A_p = \{a \in A : r(a) \leq p\}$. It is a closed subset of A. By Baire's argument, there is an integer k such that A_k is of non void interior. Hence, there is an $x_0 \in A_k$ and an absolutely convex neighbourhood V of zero such that $x_0 + V \subset A_k$. So for every x in V, we have

$$| (x_0 + x)^n | \leq k^n, \, n = 1, 2, \cdots$$

Hence,

$$\| (x_0 + x)^n \|_U \leq k^n, \, n = 1, 2, \cdots$$

By Mazur-Orlicz formula ([2]), we have

$$\left(\frac{x}{kn}\right)^2 = \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}C_n^j\left(\frac{x_0}{k}+\frac{j}{kn}x\right)^n, \ x \in V,\ n \in N^*.$$

Then

$$\left(\frac{x}{k}\right)^n \in \frac{n^n}{n!}\sum_{j=0}^{n} C_n^j U, \text{ for } U \text{ is balanced }.$$

But there exists $c > 0$ such that $\frac{(2n)^n}{n!} \leq c^n$, for every integer n. Thus

$$\left(\frac{x}{k}\right)^n \in c^n U,\ x \in V,\ n \in N^*, \text{ for } U \text{ is balanced }.$$

Whence

$$x^n \in U; \text{ for every } x \in \frac{1}{ck}V,\ n \in N^*.$$

Now if A is commutative, consider the polarization formula

$$x_1 x_2 \cdots x_n = \frac{1}{n!}\sum_{I}(-1)^{n-c(I)}\left(\sum_{i\in I} x_i\right)^n,$$

where I runs over the collection of all finite subsets of $\{1, 2, \cdots, n\}, c(I)$ the cardinal of I and $x_1, x_2, \cdots, x_n$ elements of A. For $t > 0$, if $x_i \in \frac{1}{ck}tV, 1 \leq i \leq n$, we have

$$x_1 x_2 \cdots x_n \in \frac{(2nt)^n}{n!}U.$$

Then, for t small enough, U contains an idempotent neighbouhood of zero.

As a consequence, we obtain the following results

Theorem 2. *The sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero on any unitary pseudo complete l. c. a. (A, τ) which is a Q-algebra and with continuous inverse. In particular, if A is commutative, then (A, τ) is an m-convex algebra.*

Proof. Since (A, τ) is pseudo complete *l. c. a.* with continuous inverse, the Q-algebra property implies the boundedness of every element. Hence entire functions operate on A. The commutative case follows from [6].

Theorem 3. *Let (A, τ) be a l. c. a. with continuous inverse. If $RadA$ is closed, then, in this radical, the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero. In particular, if $RadA$ is commutative, then $RadA$ is an m-convex algebra.*

Proof. The unitary subalgebra $(RadA)^1 = RadA \oplus Ce$, of A, is closed such that $Rad\left[(RadA)^1\right] = RadA$. It follows that $(RadA)^1$ is a Q-algebra. But, by hypothesis, it is with continuous inverse. Then, by Theorem 2, the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero on $(RadA)^1$ and so on $RadA$. If A is commutative, the conclusion follows also from Theorem 2.

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Ecole Normale Supérieure
B.P. 5118
Takaddoum
10105 Rabat (Morocco)

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