REVISTA MATEMÁTICA COMPLUTENSE ISSN 1139-1138
(2000) vol. XIII, num. 2, 325-336
http://dx.doi.org/10.5209/rev_REMA.2000.v13.n2.17075

REMARKS ON THE POWERS OF ELLIPTIC OPERATORS *

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Abstract

Under natural regularity assumptions on the data the powers of regular elliptic boundary value problems (e.b.v.p.) are shown to be higher order regular e.b.v.p.. This result is used in description of the domains of fractional powers of elliptic operators which information is in order important in regularity considerations for solutions of semilinear parabolic equations. Presented approach allows to avoid C^∞-smoothness assumption on the data that is typical in many references.

1 Introduction

In the studies of the evolutionary equation

$$\dot{u} + Au = F(u),\ t > 0, \tag{1}$$

with A being a sectorial operator in a Banach space $X = X^0$ and the nonlinear term F subordinated to some power of A, we need often consider fractional power spaces X^α, $\alpha \geq 0$. The knowledge of the linear operator is then crucial for the discussion of the solutions to (1) (cf. [A-C]). In applications to semilinear parabolic equations the operator A usually corresponds to some elliptic boundary value problem given by the triple $(A, \{B_j\}, \Omega)$ with A and B_j as in (2) and (3) respectively. For simplicity of notation we will not distinguish between the abstract operator in (1) and its elliptic counterpart.

*Supported by Polish State Committee for Scientific Research (KBN) under grant # 2 PO3A 035 18

1991 Mathematics Subject Classification: 35J30, 35K35.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2000

The advantage of studying problems of this latter type is that the estimates concerning the resolvent of A are well known in literature. Such results come back to a sequence of papers [A-D-N 1], [A-D-N 2] as well as to a number of later monographs [FR], [L-M], [TR] and [TA]. Roughly speaking, they are a consequence of the regularity assumptions imposed on the triple $(A, \{B_j\}, \Omega)$ like the *smoothness condition*, *strong ellipticity condition*, and *θ-strong complementary condition* ($\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$). Indeed, if this is the case, then $-A$ is the generator of a strongly continuous analytic semigroup on $L^p(\Omega)$, $p > 1$ (cf. [FR], [TA]). Furthermore, assuming A to be positive, $[(X_p^\alpha, A_\alpha), \alpha \geq 0]$ is a compactly injected one-sided fractional power scale generated by $(L^p(\Omega), A)$ (cf. [AM 1]).

The description of the Banach scale $[(X_p^\alpha, A_\alpha), \alpha \geq 0]$ is important in applications. Of course, if X is a Hilbert space, e.g. $X = L^2(\Omega)$, and A is selfadjoint and positive definite on X, then the characterisation of the mentioned scale is quite complete. In this case the imaginary powers of A are bounded and X^α may be described as the intermediate spaces between $L^2(\Omega)$ and $D(A) = H^{2m}_{\{B_j\}}(\Omega)$ based on the complex interpolation method (cf. [TR]). For $p \neq 2$ similar results are known in the whole generality if the coefficients of A, B_j and the boundary $\partial\Omega$ are of class C^∞ (cf. [SEE], [TR]). However, mention should be made of the publications [A-H-S], [P-S], [S-T], [GU] where recent developments in this field can be found.

The embeddings of X^α-spaces in Sobolev and Hölder classes of functions are also of particular importance. Under the natural assumptions on $(A, \{B_j\}, \Omega)$ such inclusions were proved e.g. in [HE] (cf. formula (7)). They are however restricted to $\alpha \in [0, 1]$ whereas it would be convenient to have them for the whole range of $\alpha \geq 0$.

In this note we extend the results of [HE] to all $\alpha \in [0, +\infty)$ under naturally strengthened regularity assumptions imposed on $\partial\Omega$ and the coefficients of A, B_j, $j = 1, \ldots, m$.

2 Strongly regular elliptic boundary value problems

For $m = 1, 2, \ldots$, consider the partial differential operators:

$$A = \sum_{|\sigma| \leq 2m} a_\sigma(x) D^\sigma, \tag{2}$$

$$B_j = \sum_{|\sigma| \leq m_j} b^j_\sigma(x) D^\sigma, \; j = 1, 2, \ldots, m, \tag{3}$$

where $0 \leq m_j \leq 2m - 1$ and $m_i \neq m_j$ for $i \neq j$, $i, j = 1, 2, \ldots, m$. The principal symbols of A and B_j will be denoted by A_0, $(B_j)_0$ respectively; i.e.

$$A_0(x, \xi) = \sum_{|\sigma| = 2m} a_\sigma(x) \xi^\sigma, \; (B_j)_0 = \sum_{|\sigma| = m_j} b^j_\sigma(x) \xi^\sigma, \; j = 1, 2, \ldots, m.$$

We require that the coefficients of the operators A, B_j satisfy the following condition.

- **Smoothness condition**
  a_σ, $|\sigma| \leq 2m$, are uniformly continuous complex valued functions defined on a bounded subdomain $\Omega \subset R^n$, $n \geq 2$, with boundary $\partial\Omega$ of the class C^{2m} whereas $b^j_\sigma : \partial\Omega \to C$, $b^j_\sigma \in C^{2m-m_j}(\partial\Omega)$, for $|\sigma| \leq m_j$, $j = 1, 2, \ldots, m$.

Recall the formulation of the *uniform strong ellipticity condition* (cf. [FR, p. 2]) and the *θ-strong complementary condition* (cf. [FR, p. 77]).

- **Uniform strong ellipticity condition**

$$\exists_{c>0} \, \forall_{x \in \Omega} \, \forall_{\xi \in R^n} \; (-1)^m Re A_0(x, \xi) \geq c|\xi|^{2m}; \tag{4}$$

- **θ-strong complementary condition**
  Let $\theta \in (0, 2\pi)$ be fixed, whereas H_x and $N(x)$ denote, respectively, the tangent hyperplane and the outward normal unit vector to $\partial\Omega$ at $x \in \partial\Omega$. Take any $x \in \partial\Omega$, any $\xi \in H_x$ and arbitrary complex number λ from the ray $arg\lambda = \theta$ such that $(\xi, \lambda) \neq (0, 0)$. Then, the polynomial $p(z) = A_0(x, \xi + zN(x)) - (-1)^m \lambda$ should possess

exactly m roots $z_i^+(x,\xi,\lambda)$, $i=1,\dots,m$, with positive imaginary parts and the polynomials $P_j(z)$ $(j=1,\dots,m)$, where $P_j(z) = (B_j)_0(x,\xi+zN(x))$, should be linearly independent modulo $Q(z) = (z-z_1^+(x,\xi,\lambda))\dots(z-z_m^+(x,\xi,\lambda))$.

Remark 1. Let us make few remarks on the above conditions. Inequality (4) means equivalently that $(-1)^m ReA_0(x,\xi)>0$ for each $x\in\overline{\Omega}$ and $\xi\neq 0$. In particular, for any $x\in\overline{\Omega}$ and any pair of linearly independent vectors $\xi,\eta\in R^n$, the polynomial $A_0(x,\xi+z\eta)$ has equal number m of roots with positive and with negative imaginary parts. The latter property may not be true if A is merely an *elliptic operator* in Ω, i.e. when $A_0(x,\xi)\neq 0$ for $x\in\Omega$, $0\neq\xi\in R^n$, unless the space dimension n is strictly greater than 2 or, if $n=2$, the coefficients of A are real. In other words, validity of (4) is equivalent to *ellipticity* of the following operator

$$A_\theta = \sum_{|\sigma|=2m} a_\sigma(x)D^\sigma - (-1)^m e^{i\theta}\frac{\partial^{2m}}{\partial t^{2m}} \tag{5}$$

in $\overline{\Omega}\times R$ for all values of $\theta\in[\frac{\pi}{2},\frac{3\pi}{2}]$. By what was said above, it is clear that, whenever $\theta\in[\frac{\pi}{2},\frac{3\pi}{2}]$ and $arg\lambda=\theta$, the polynomial $p(z)=A_0(x,\xi+zN(x))-(-1)^m\lambda$ introduced in the θ-*strong complementary condition* possesses exactly m roots with positive imaginary parts. The latter condition remains true for all λ with $arg\lambda\neq 0$ provided the coefficients a_σ, $|\sigma|=2m$, of A are real valued functions (cf. Proposition 1 below).

Proposition 1. *Let the operator A in (2) have real coefficients a_σ, $|\sigma|=2m$, and satisfy the strong ellipticity condition (4). Then, A_θ defined in (5) is an elliptic operator in $\overline{\Omega}\times R$ for each $\theta\in(0,2\pi)$.*

Proof. For $\pi\neq\theta\in(0,2\pi)$, $\xi\in R^n$, $0\neq\eta\in R$ we have; $ImA_0(x,\xi)=0$ and $Im((-1)^m e^{i\theta}\eta^{2m})\neq 0$, which ensures that:

$$A_0(x,\xi)-(-1)^m e^{i\theta}\eta^{2m}\neq 0,\ x\in\overline{\Omega}. \tag{6}$$

For $\theta=\pi$, $\eta\neq 0$, $\xi=0$ the above condition is trivial. Validity of (6) in the remaining two cases (i) $\theta\in(0,2\pi)$, $\eta=0$, and $\xi\neq 0$ as well as (ii) $\theta=\pi$, $0\neq\xi\in R^n$, and $0\neq\eta\in R$, is a direct consequence of (4). The proof is complete. ∎

Definition 1. *Consider a triple $(A, \{B_j\}, \Omega)$, where A and B_j are given by (2), (3) and suppose that the following conditions hold:*

- *the smoothness condition,*
- *the uniform strong ellipticity condition,*
- *the θ-strong complementary condition for each $\theta \in (0, 2\pi)$.*

Then the triple $(A, \{B_j\}, \Omega)$ is called a $2m$-th order strongly regular elliptic boundary value problem ($2m$-th order strongly regular e.b.v.p.).

Remark 2. The simplest possible examples of the strongly regular e.b.v.p. are connected with the three basic problems of potential theory given by the triples $(-\Delta, Id, \Omega)$, $(-\Delta, \frac{\partial}{\partial N}, \Omega)$ and $(-\Delta, \frac{\partial}{\partial N} + aId, \Omega)$ with real a. Of course other triples can be considered, like e.g. $(-\Delta + \frac{\partial}{\partial x_1}, \frac{\partial}{\partial N}, \Omega)$. More complicated examples of higher order e.b.v.p.'s may be easily obtained from the just mentioned triples with the use of Theorem 1 stated below.

Our main concern here is to prove the following result:

Theorem 1. *Let A, B_j, $j = 1, \dots, m$, be given by (2), (3) respectively and the coefficients a_σ, $|\sigma| = 2m$, of A be real valued functions. Suppose further that a triple $(A, \{B_j\}, \Omega)$ forms a $2m$-th order strongly regular e.b.v.p. Then, $(A^2, \{B_j, A \circ B_j\}, \Omega)$ and $(A^2, \{B_j, B_j \circ A\}, \Omega)$ form $4m$-th order strongly regular e.b.v.p. provided the following additional smoothness requirements are satisfied:*

- *$\partial\Omega$ is of class C^{4m},*
- *$a_\sigma \in C^{2m}(\overline{\Omega})$, $|\sigma| \leq 2m$,*
- *$b^j_\beta \in C^{4m-m_j}(\partial\Omega)$, $|\beta| \leq m_j$, $j = 1, 2, \dots, m$.*

Proof. Since $(A^2)_0(x, \xi) = A_0(x, \xi)^2$, it is immediate that A^2 is uniformly strongly elliptic.

Fix $x \in \partial\Omega$, $\xi \in H_x$ (ξ is tangent to $\partial\Omega$ at x), $\theta \in (0, 2\pi)$, $arg\lambda = \theta$ and let $N(x)$ be the outward normal unit vector to $\partial\Omega$ at x. Note that the polynomial

$$\begin{aligned} p(z) &= (A^2)_0(x, \xi + zN(x)) - \lambda = A_0(x, \xi + zN(x))^2 - \lambda \\ &= (A_0(x, \xi + zN(x)) - \sqrt{\lambda})(A_0(x, \xi + zN(x)) + \sqrt{\lambda}) \end{aligned}$$

has exactly $2m$ roots with positive imaginary part. In fact, by Proposition 1, the polynomials $A_0(x, \xi + zN(x)) \pm \sqrt{\lambda}$ have each exactly m zeros with positive imaginary part since $\pm(-1)^m\sqrt{\lambda} \notin R$. Denote by z_i^+, $i = 1, \dots, m$, the roots with positive part of $A_0(x, \xi + zN(x)) - \sqrt{\lambda}$, by z_{m+i}^+, $i = 1, \dots, m$, those of $A_0(x, \xi + zN(x)) + \sqrt{\lambda}$ and let $q_1(z) = (z - z_1^+) \dots (z - z_m^+)$, $q_2(z) = (z - z_{m+1}^+) \dots (z - z_{2m}^+)$. Proving the θ-strong complementary condition now consists in showing that the polynomials $P_j(z)$, $j = 1, \dots, 2m$, given by

$$\begin{aligned} P_j(z) &= (B_j)_0(x, \xi + zN(x)),\ j = 1, \dots, m; \\ P_{m+j}(z) &= (A \circ B_j)_0(x, \xi + zN(x)) \\ &= P_j(z) A_0(x, \xi + zN(x)),\ j = 1, \dots, m; \end{aligned}$$

are linearly independent modulo $q(z) = q_1(z)q_2(z)$. To this effect, consider a linear combination $w(z) := \sum_{j=1}^{2m} c_j P_j(z)$ and suppose $w(z)$ is divisible by $q(z)$. Since $q_1(z)$ divides $A_0(x, \xi + zN(x)) - \sqrt{\lambda}$ and $w(z)$ may be written as

$$\begin{aligned} w(z) &= \sum_{j=1}^{m} (c_j + \sqrt{\lambda} c_{m+j}) P_j(z) \\ &+ \left[\sum_{j=1}^{m} c_{m+j} P_j(z) \right] (A_0(x, \xi + zN(x)) - \sqrt{\lambda}), \end{aligned}$$

then $q_1(z)$ divides $\sum_{j=1}^{m} (c_j + \sqrt{\lambda} c_{m+j}) P_j(z)$. The strong complementary condition for the original system, valid for $arg\ \sqrt{\lambda}$ since $\sqrt{\lambda}$ is not real, implies $c_j + \sqrt{\lambda} c_{m+j} = 0$ for $j = 1, \dots, m$. Similarly we see that $c_j - \sqrt{\lambda} c_{m+j} = 0$ for $j = 1, \dots, m$, hence $c_j = 0$ for $j = 1, \dots, 2m$. The proof of Theorem 1 is complete. ■

The result of Theorem 1 may be immediately generalised by an induction argument to the form:

Corollary 1. *Let $(A, \{B_j\}, \Omega)$ be a $2m$-th order strongly regular e.b.v.p. and the coefficients of the main part of A be real valued functions. If $k \in N$ and*

- *$\partial\Omega$ is of class $C^{2^{k+1}m}$,*
- *$a_\sigma \in C^{2^{k+1}m - 2m}(\overline{\Omega})$, $|\sigma| \le 2m$,*

- $b^j_\beta \in C^{2^{k+1}m-m_j}(\partial\Omega)$, $|\beta| \le m_j$, $j = 1, 2, \dots, m$,

then $(A^{2^l}, \{B_j, A \circ B_j, A^2 \circ B_j, \dots, A^{2^l-1} \circ B_j\}, \Omega)$ *and* $(A^{2^l}, \{B_j, B_j \circ A, B_j \circ A^2, \dots, B_j \circ A^{2^l-1}\}, \Omega)$ *are* $2^{l+1}m$*-th order strongly regular e.b.v.p. for each* $l = 0, \dots, k$.

3 Applications

Let $\Omega \subset R^n$ be a bounded domain with $\partial\Omega$ of class C^{2m}, $1 < p < +\infty$ and Λ a sectorial operator in $L^p(\Omega)$ with the domain $D(\Lambda) \subset W^{2m,p}(\Omega)$ and such that $Re\sigma(\Lambda) > 0$. It is easy to see that the estimates of [HE, Theorem 1.6.1] extend to the *real order* Sobolev spaces $W^{s,r}(\Omega)$ with $s \ge 0$ (cf. [AM 2]) and the following *Sobolev type embeddings* hold:

$$D(\Lambda^\alpha) \subset W^{s,r}(\Omega) \text{ for } s - \frac{n}{r} < 2m\alpha - \frac{n}{p},\ r \ge p,\ \alpha \in [0,1],\ s \ge 0,$$
$$D(\Lambda^\alpha) \subset C^\mu(\overline{\Omega}) \text{ for } 0 \le \mu < 2m\alpha - \frac{n}{p},\ \alpha \in [0,1]. \tag{7}$$

As shown in [FR], if $(A, \{B_j\}, \Omega)$ is a strongly regular e.b.v.p., then $A_\omega = A + \omega Id$ with the domain $D(A_\omega) = W^{2m,p}_{\{B_j\}}(\Omega)$ is sectorial and positive in $L^p(\Omega)$, $p \in (1, +\infty)$, whenever $\omega > 0$ is chosen sufficiently large. Therefore (7) is true for $\Lambda := A_\omega$.

It is possible to extend (7) to cover all values $\alpha \in [0, +\infty)$ using the elliptic regularity theory of [TR] (cf. [C-D, Chapter 1]). Based on Theorem 1 another proof of such type embeddings, under weaker regularity assumptions, can also be given.

Corollary 2. *Let* $(A, \{B_j\}, \Omega)$ *be a* $2m$*-th order strongly regular e.b.v.p. and the coefficients of the main part of* A *(cf. (2)) be real valued functions. Fix* $k \in N$ *and assume that*

- $\partial\Omega$ *is of class* $C^{2^{k+1}m}$,
- $a_\sigma \in C^{2^{k+1}m-2m}(\overline{\Omega})$, $|\sigma| \le 2m$,
- $b^j_\beta \in C^{2^{k+1}m-m_j}(\partial\Omega)$, $|\beta| \le m_j$, $j = 1, 2, \dots, m$.

Then, for $\alpha \in [0, 2^k]$, *the following inclusions hold:*

$$\begin{array}{l} D(A^\alpha_\omega) \subset W^{s,r}(\Omega) \ \text{for} \ s - \frac{n}{r} < 2m\alpha - \frac{n}{p},\ r \ge p,\ s \ge 0, \\ D(A^\alpha_\omega) \subset C^\mu(\overline{\Omega}) \ \text{for} \ 0 \le \mu < 2m\alpha - \frac{n}{p}, \end{array} \tag{8}$$

where a number $\omega > 0$ is assumed to be sufficiently large.

Proof. As a consequence of the Corollary 1, $A_\omega^{2^k}$ is a sectorial operator. Therefore, the embeddings (8) follow immediately from Henry's estimates (7) and [KO, Theorems 6.4, 10.3, 10.6]. ∎

Remark 3. As a consequence of [KO, Th. 10.6], if Λ is a sectorial, positive operator acting in a Banach space X, then

$$(\Lambda^2)^\beta = \Lambda^{2\beta}, \quad \text{for each } \beta > 0. \tag{9}$$

As is well known, an operator A is sectorial if and only if $-A$ is the infinitesimal generator of an analytic semigroup. Note that (9) may not be true if $-\Lambda$ is merely the generator of a C^0-semigroup (cf. [YO]). However, for a sectorial operator, the resolvent set of Λ contains the complement of a sector of half angle ϕ satisfying $0 < \phi < \frac{\pi}{2}$, which is sufficient for the validity of (9) (cf. [KO] for details).

Remark 4. The question whether the embeddings (7) hold also on the boundary of the parameter set; i.e., if the strict inequality in the first line of (7) is replaced by $s - \frac{n}{r} \leq 2m\alpha - \frac{n}{p}$, is not completely settled. However, when the operator A has bounded imaginary powers then it is possible to obtain

$$\begin{aligned} X^\alpha &= [L^p(\Omega), W^{2m,p}_{\{B_j\}}(\Omega)]_\alpha \subset [L^p(\Omega), W^{2m,p}(\Omega)]_\alpha \\ &= H_p^{2m\alpha}(\Omega) \subset W^{2m\alpha,p}(\Omega), \ \alpha \in (0,1), \end{aligned} \tag{10}$$

which shows that for $p = r$, the strict inequality of (7) can be improved to a non-strict one (cf. [SEE], [TR]). But besides the cases considered in [A-H-S], [P-S], [S-T], the boundedness of A^{it} is not known without the C^∞ regularity assumptions introduced in the original paper [SEE].

Remark 5. For particular operators, like e.g. $A = \Delta^2$ with

$$D(A) = W^{4,p}_{\{Id,\Delta\}}(\Omega)$$

the result of [P-S] is sufficient to obtain (10). Indeed, combining (9) with the results of [P-S, Theorem C] we obtain, under C^4-regularity of the boundary, that

$$\begin{aligned} D(A^\alpha) &= D((-\Delta_D)^{2\alpha}) = [L^p(\Omega), W^{4,p}_{\{Id,\Delta\}}(\Omega)]_\alpha \\ &\subset [L^p(\Omega), W^{4,p}(\Omega)]_\alpha \subset W^{4\alpha,p}(\Omega), \ \alpha \in [0,1]. \end{aligned}$$

Therefore we can also get a sharp version of (7) in this case.

There are two further reasons for the technical considerations presented earlier. The first is that one often needs to prove sectoriality of higher order elliptic operators. The existing literature does not give sufficient support for such considerations. For example, when studying higher order problems $(\Delta^2, \{Id, \Delta\}, \Omega)$ or $(\Delta^2, \{\frac{\partial}{\partial N}, \frac{\partial \Delta}{\partial N}\}, \Omega)$ in $L^p(\Omega)$ with $p \neq 2$ (e.g. in connection with the *Cahn-Hilliard equation*) one cannot find the references to ensure that these problems are strongly regular e.b.v.p.. The result of our note at least partially fills this gap. Such information follows immediately from Theorem 1 and Remark 2.

A further motivation for these studies is provided by the regularity theory of higher order semilinear parabolic equations. Regularity considerations for such equations are known in literature. In particular A. Friedman studied such questions in [FR, Part 3]. Various regular solutions are also considered in [L-S-U]. However, often the assumptions imposed on the data are too restrictive, in part because the existing references do not allow anything else.

Consider the initial-boundary value problem

$$\begin{cases} u_t = -Au + f(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \ldots, \frac{\partial^{m_0} u}{\partial x_n^{m_0}}), \ (t,x) \in R^+ \times \Omega, \\ B_1 u = \ldots = B_m u = 0 \ \text{ on } \ \partial\Omega, \\ u(0,x) = u_0(x), \ x \in \Omega. \end{cases} \tag{11}$$

$(m_0 \leq 2m - 1)$ and assume that:

(i) The corresponding triple $(A, \{B_j\}, \Omega)$ with A and B_j given in (2), (3) satisfies the assumption of Theorem 1. Hence, A is a sectorial operator in $X = L^p(\Omega)$, $p \in (1, +\infty)$, with $D(A) = W^{2m,p}_{\{B_j\}}(\Omega)$.

(ii) The Nemytskii operator F, $F : X^{\frac{1}{2m}+\gamma} \to X^{\frac{1}{2m}}$, connected with f is Lipschitz continuous on bounded sets for some $\gamma \in (0,1)$.

As shown in [C-D, Chapter VII] condition (ii) holds with $p > n$ and $\gamma \in (\frac{m_0}{2m}, 1)$ provided that f has all first order partial derivatives locally Lipschitz continuous with respect to functional arguments uniformly for $x \in \Omega$. Moreover, when $u = 0$ on $\partial\Omega$, we need to assume that $f_{|_{u=0}} = 0$.

Consider now the problem

$$\dot{u} + Pu = F(u), \ t > 0, \ u(0) = u_0, \tag{12}$$

on the base space Y. Recall the following existence result of [HE]:

Proposition 2. *Let $P : D(P) \supset Y \to Y$ be sectorial and positive operator in a Banach space Y and, for some $\alpha \in [0,1)$, $F : Y^\alpha \to Y$ be Lipschitz continuous on bounded subsets of Y^α. Then, for each $u_0 \in Y^\alpha$, there exists a unique Y^α-solution $u = u(t, u_0)$ of (12) defined on its maximal interval of existence $[0, \tau_{u_0})$ and such that*

$$u \in C([0,\tau), Y^\alpha) \cap C^1((0,\tau), Y^\beta) \cap C((0,\tau), Y^1), \ \beta \in [0,1).$$

The problem (11) will be considered as an abstract equation (12) on the space $Y = X^{\frac{1}{2m}}$ with sectorial operator $P = A_{|_{X^{\frac{1}{2m}}}}$.

As shown in [AM 1, p. 260]:

$$D(P^\gamma) = D((A_{|_{X^{\frac{1}{2m}}}})^\gamma) = X^{\frac{1}{2m}+\gamma}.$$

Consequently, by Proposition 2 (cf. [HE, Chapter 3]), there exists a unique $X^{\frac{1}{2m}+\gamma}$-solution to that problem such that:

$$u \in C([0,\tau), X^{\frac{1}{2m}+\gamma}) \cap C^1((0,\tau), X^{\frac{1}{2m}+\beta}) \cap C((0,\tau), X^{\frac{1}{2m}+1}), \ \beta \in [0,1). \tag{13}$$

If $p > n$, it is a consequence of (8) that

$$u \in C([0,\tau), C^{2m\gamma+\mu}(\overline{\Omega})) \cap C^1((0,\tau), C^{2m\beta+\mu}(\overline{\Omega})) \cap C((0,\tau), C^{2m+\mu}(\overline{\Omega})), \tag{14}$$

whenever $0 \le \mu < 1 - \frac{n}{p}$. Hence, all derivatives appearing in the main equation and boundary conditions of (11) can be understood in the classical sense. In particular, the time derivative $\dot{u}$ (strong derivative in $X^{\frac{1}{2m}}$) coincides with a pointwise classical time derivative of $u(t,x)$ and we thus deal with the *classical solution* to (11).

Remark 6. Restricting the operator A to a fractional power space of higher order than $X^{\frac{1}{2m}}$ one can similarly obtain smoother solutions to (11).

Acknowledgment. The authors would like to express their deep thanks to the Referee for valuable comments allowing to improve the original version of the paper.

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Recibido: 4 de Octubre de 1999
Revisado: 10 de Abril de 2000